Endogenous Risk-Exposure and Systemic Instability *

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Abstract

Most research on financial systemic stability assumes an economy in which banks are subject to exogenous shocks, but in practice, banks choose their exposure to risk. This paper studies the determinants of this endogenous risk exposure when banks are connected in a financial network. I first show that there exists a network risk-taking externality: connected banks' choices of risk exposure are strategically complementary. Banks in financial networks, particularly densely connected ones, endogenously expose to greater risks. Furthermore, due to this externality, connected banks choose to correlate their risk exposure, aggravating the systemic fragility. The deposit insurance scheme is crucial to this risk-taking externality. For policy implications, I show that (i) a network-adjusted capital regulation, (ii) a transparency policy, or (iii) a government bailout can alleviate the risk-taking externality and hence reduce banks' equilibrium risk exposure.

Keywords: systemic risks, financial networks, capital regulation, government bailout, CCP

JEL Classification: G21, G28, L14

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Introduction

Since the 2008 financial crisis, the relationship between financial networks and systemic stability has been an important subject of research (Glasserman and Young, 2016). Most of the existing literature assumes exogenous shocks and studies how these idiosyncratic shocks are propagated across a financial network.[1] However, banks' exposure to which particular shock is an endogenous choice variable. For example, a bank chooses between safe borrowers and subprime borrowers, or chooses its exposure on asset-backed securities.[2] This paper extends the theory of interbank networks and systemic stability by incorporating endogenous risk exposure. Introduction of a risk exposure choice changes the received intuition about financial stability in an important way, and yields novel policy implications.

Pioneering works by Allen and Gale (2000) and Freixas et al. (2000) show that connected networks are more robust to the contagion of exogenous shocks than unconnected ones due to a co-insurance mechanism. They conclude that a highly connected banking sector promotes financial stability. In contrast to the conclusions of the above papers, I show that although shocks are better co-insured in densely connected networks, banks in those networks initially choose greater risk exposure. Furthermore, they choose correlated risks. In other words, in densely connected networks, bank-specific endogenous losses are more likely and they tend to happen simultaneously. As a result, the banking sector as a whole becomes more fragile.

The basic intuition for this result relies on a network risk-taking externality. Banks in networks, if solvent, partially reimburse failed banks through interbank payments, which I dub as cross-subsidy. The cross-subsidy reduce banks' upside payoffs (the payoffs when they are solvent). On the other hand, banks' downside payoffs are always zero due to limited liability. To compensate for this asymmetric distortion from the cross-subsidy, banks choose riskier projects with greater upside payoffs. This risk-taking distortion is higher when each bank anticipates a higher likelihood of having to bail out other banks, that is, when its counterparties take greater risks.

Moreover, banks in greater connected networks will be more affected by such risk-taking externality. In particular, I show that banks in networks with stronger connections, more complete structure, or more counterparties choose greater risk exposure. The model contributes to the debates on the relationship between a financial network's connectedness and the systemic stability.[3] My result stands in sharp contrast to the "connected-stability" view that relies on financial networks' co-insurance mechanism. I show that losses that are better co-insured, as in Allen and

[1] For example, Allen and Gale (2000), Freixas et al. (2000) and Gai et al. (2011) consider exogenous liquidity shocks. Shin (2009), Elliott et al. (2014) and Acemoglu et al. (2015) considers exogenous economic shocks.

[2] Mian and Sufi (2009) empirically documented an unprecedented growth of subprime credit right before the 2008 financial crisis. They also found a concurrent rapid increase in the securitization of subprime mortgages.

[3] For "connected-stability" view, Allen and Gale (2000) show that a complete network is more robust to the loss contagion due to a co-insurance mechanism. For "connected-fragility" view, Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) argue that the "complete-stability" relationship does not apply to larger shocks due to a propagation mechanism. Elliott, Golub, and Jackson (2014) find similar non-monotonic relationships for equity networks.

Gale (2000)'s complete network, will be more likely to endogenously evolve in the first place for highly connected banks. Furthermore, by studying the core-periphery structure, the model also sheds new insight on the hotly debated central clearing counterparties (CCP). [4] According to Duffie and Zhu (2011), the main efficiency of a CCP comes from multilateral netting. In this paper, instead, I show that the risk-taking equilibrium with a CCP is equivalent to the outcome of a maximum connected complete network. In other words, a CCP may instead increase originally loosely connected banks' risk-taking incentives if they hold un-nettable contracts as in Donaldson and Piacentino (2017) and Acemoglu et al. (2015).

The model builds on a payment equilibrium model by Eisenberg and Noe (2001), which has later been utilized by Shin (2008, 2009) and Acemoglu et al. (2015). My innovation is to allow banks to choose their risk exposure endogenously after anticipating the payment equilibrium and their counterparties' risk exposure. One important contribution of this model is to show that the standard intuition about the stabilizing effect of financial networks reverses with endogenous risk-taking. The theory also yields several novel perspectives on policy debates:

- **Government Bailouts**: conventional wisdom states that a government bailout, or simply anticipation of it, is harmful to the systemic stability since it encourages excessive risk-taking by reducing banks' "skin in the game". I show that a government bailout may instead reduce connected banks' risk-taking incentives because it decreases the network risk-taking distortion. In presence of the possibility of a government bailout, every bank will anticipate a smaller cross-subsidy to its failed counterparties. Hence the network risk-taking distortion is reduced and so does every bank's choices of risk exposure.

- **Deposit Insurance**: deposit insurance is argued to promote financial stability by eliminating bank runs (Diamond and Dybvig, 1983).[5] However, I show that deposit insurance scheme plays a crucial role in the network risk-taking externality and hence destabilizes the banking sector. The reason is that deposit insurance eliminates depositors' *price disciplining* ability, an invisible hand. Absent deposit insurance, depositors fully anticipate the network risk-taking externality. As a result, deposit rates will endogenously adjust to equalize banks' default probabilities regardless of financial networks' topologies.[6] This result is a generalization of the Modigliani-Miller (MM) theorem in the sense that banks' *interbank debt structure* is also irrelevant to their choices of risk exposure. On the other hand, with deposit insurance, depositors are "informative insensitive" to the structure of the banking network.

[4] After the global financial crisis, G20 leaders committed at the 2009 Pittsburgh Summit to clear all OTC derivatives through CCPs (Domanski et al., 2015).

[5] In a World Bank dataset of 189 countries, 59 percent had deposit insurance by year end of 2013 (Demirgüç-Kunt et al., 2014) The United States adopted the deposit insurance scheme after the 1930s Great Depression. For a detailed account of the deposit insurance history in the U.S., see Gorton (2012).

[6] The exact mechanism is as follows. Absent deposit insurance, depositors in densely connected networks will expect greater returns due to better co-insurance. Hence they will demand lower deposit rates. The lowered deposit rates exactly compensate connected banks' anticipated cross-subsidy. As a result, banks' choices of risk exposure are equalized regardless of the network structures.

This is the friction that violates the generalized MM theorem's competitive-market assumption. In this case, there is no *price disciplining* from deposit rates. The network risk-taking externality exists.

- **Network Transparency**: I argue that the interbank connectedness' transparency can reduce banks' endogenous risk exposure. An opaque banking network loses the deposit rates' *pricing* ability to discipline banks' choices of risk exposure: when depositors are unaware of the banking network, they will not compensate connected banks with lower deposit rates for the better co-insurance. The network risk-taking externality exists in opaque networks, and as a result, banks choose greater risk exposure. The interbank network's transparency is particularly important for mitigating the risks of shadow banks. Shadow banks' liabilities (e.g. repo, asset-backed commercial papers, and money markets funds) are not insured and their creditors will have greater incentives to observe the interbank relationships. My result contributes to the debate started by Dang, Gorton, and Holmström (2012), who argues that shadow banks' transparency is sub-optimal because it creates endogenous adverse selection.

- **Network-adjusted Capital Regulation**: motivated by discussions on Basel III capital regulation, I show that each bank's equity buffer has a positive network effect on systemic stability. The equity buffer not only directly reduces a bank's own risk-taking (Jensen and Meckling, 1976) but also reduces the risk-taking of every other bank in the same financial network. The intuition for this multiplier effect is as follows: if a bank's project fails, its equity buffer first induces its own shareholders to absorb part of the loss. As a result, the loss that may be otherwise propagated to other banks will now be curbed at the origin. That implies every bank in the financial network will anticipate a smaller cross-subsidy to failed banks, and will ex-ante choose to expose to fewer risks. From that, I propose a network-adjusted capital regulation: a higher tier-one capital ratio requirement for banks with stronger interbank connections or more counterparties.

In the final part of the paper, I endogenize banks' decisions to correlate their risk exposure. I show that in a financial network, every bank will coordinate to expose to one single systemic risk. The intuition is as follows: in anticipation of the interbank transfers, a correlated portfolio will reduce the possibility of cross-subsidy. Hence it will increase each bank's expected profit. As a result of the endogenous correlation, a financial crisis (or simultaneous failure of several banks) will be more likely to endogenously evolve in connected banking systems. The model explains the empirical findings of the 2008 financial crisis by the Financial Crisis Inquiry Commission (2011): "Some financial institutions failed because of a common shock: they made similar failed bets on housing."

This paper makes several contributions to the topic of systemic stability. In contrast to previous papers that study the ex-post contagion of exogenous shocks, this paper provides a tractable model to study banks' endogenous risk exposure incentives in financial networks. The model can be regarded as a generalization of the asset substitution problem in financial networks. Using the machinery of networks to study the asset substitution is important because it provides tractability to study banks' risk-taking incentives among different financial structures, particularly the widely observed core-periphery structures and the evolving central clearing counterparties. Second, the paper contributes to the "connected-fragility" view by showing that banks in more densely connected networks, although more co-insured as most people believe, endogenously expose to greater risks. The paper also explains the observation that connected banks tend to expose to correlated risks, especially in the recent global financial crisis. Finally, the theory yields several novel perspectives on policy debates.

Related Literature This paper is related a recent and growing literature on the relationship between interconnectedness of modern financial institutions and the systemic stability. Most research focuses on the question do more connections tend to amplify or dampen systemic shocks. Glasserman and Young (2016) provide a survey of this literature and here I will summarize a few that are related to the present paper. One branch of literature conforms to a "connected-stability" view: a connected network provides better liquidity insurance against some exogenous shocks to one individual bank. The view is supported by Allen and Gale (2000), Freixas, Parigi, and Rochet (2000), Leitner (2005). Allen and Gale (2000) argues that the initial loss will be widely divided in a connected complete network. Therefore banks will less likely to default in such a network. In Freixas et al. (2000), depositors face uncertainties about where they will consume. They also show that the interbank connections enhance the resiliency. Leitner (2005) argues that the interbank connection is optimal ex-ante due to the probability of cross-subsidy.

On the other hand, the "connected-fragility" view is supported by Gai, Haldane, and Kapadia (2011), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), and Donaldson and Piacentino (2017). Using numerical simulations, Gai et al. (2011) demonstrate that a more complex and concentrated financial network may amplify the fragility. Acemoglu et al. (2015) use Eisenberg and Noe (2001)'s model to study the shock propagation. They conclude that a highly connected complete network becomes least stable under a large exogenous shock. Donaldson and Piacentino (2017) study the liquidity co-insurance benefits of long-term interbank debts. None of the above papers, nevertheless, studies how those initial shocks evolved in the first place.

Some recent papers study endogenous network formations and interbank liquidities. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2014) study the network externalities of bilateral lending on other third parties in the same financial system. They show that although banks internalize the bilateral counterparty risks through the interest rate, they fail to internalize the externalities on the rest of the network. In this case, banks may "overlend" in equilibrium. The present paper utilizes the same framework to illustrate another financial network externalities: risk-taking externalities. Di Maggio and Tahbaz-Salehi (2014) study the interbank intermediation capacity with moral hazard. They show that the collateral's liquidity may have a huge effect on haircuts and intermediation capacity due to the cumulative nature of the moral hazard.

Brusco and Castiglionesi (2007) study banks' contracting behaviors in financial networks. They utilize the models of Diamond and Dybvig (1983) and Allen and Gale (2000) to study bankers' private benefit from gambling along with their contracting with depositors.[7] They assume negatively correlated liquidity shocks and show that the co-insured depositors in a network may want to increase their long-term investment. As a result, bankers are more able to enjoy the private benefits of gambling from the increased long-term investment. The present paper's prediction is related to theirs; however, the mechanism is different. Our model is neoclassical and assumes no friction from managers' gambling perks. This paper is more interested in banks' rational choices of risk exposure than bank managers' agency problem. The network risk-taking distortion is from the asymmetric interbank payments rather than the increased resources that bankers can gamble with. In contrast to Brusco and Castiglionesi (2007), my model does not assume negatively correlated risks but rather shows that banks' risk exposure is endogenously correlated.

1 Model

The economy consists of $N \in \mathbb{N}^+$ risk-neutral banks that are interconnected through the cross-holdings of unsecured debt contracts $\bar{d}_{ij} > 0$, where $\bar{d}_{ij}$ is the face value of the interbank debt that bank j owes to bank i. Assume that all interbank liabilities have equal seniority. Denote $\bar{d}_j \equiv \sum_i \bar{d}_{ij}$ as bank j's total interbank liabilities. Following Acemoglu et al. (2015), we restrict our analysis to *regular* network structures in which the total interbank liabilities and claims are equal for all banks (i.e., $\sum_j \bar{d}_{ij} = \sum_j \bar{d}_{ji} = \bar{d}$ for all i). In this way, we abstract away the effect of network asymmetry (e.g. the existence of a dominant player). Define $\theta_{ij} \equiv \bar{d}_{ij}/\bar{d}_j$ as bank i's share in j's total interbank liabilities. By the regularity assumption, we have $\sum_j \theta_{ij} = \sum_i \theta_{ij} = 1$. Denote $\mathbf{\Theta} \equiv [\theta_{ij}]$ as an $N \times N$ matrix, which determine the network completeness, which will be further discussed in section 3. Besides the interbank liabilities, each bank also owes a more senior outside debt $v_i = v > 0$ that needs to be paid in full before the interbank debt. One example of such outside debt is a bank's retail deposits. [8] In summary, an economy can be characterized by $(\bar{d}, \mathbf{\Theta}, N, v)$. In the benchmark model, $(\bar{d}, \mathbf{\Theta}, N, v)$ is publicly observable.

In the initial date, each bank i simultaneously chooses one project Z_i among a set of available projects $[\underline{Z}, \overline{Z}]$. This project Z_i will produce a random return of $\tilde{e}_i(Z_i)$ with the following payoff distribution

$$\tilde{e}_i = \begin{cases} Z_i & \text{w.p} \quad P(Z_i) \\ 0 & \text{w.p} \quad 1 - P(Z_i) \end{cases} \tag{1}$$

where $P(Z) \in (0, 1)$ is some deterministic function that denotes the probability of project Z's success. In the benchmark model, I assume each bank's project is independent. This assumption

[7] The authors give examples of private benefits as "on-the-job perks or simply monies illegally diverted to personal accounts."

[8] For non-financial firms, this outside debt can be their unpaid wages or government taxes.

is later relaxed in Section 5. To guarantee a non-trivial banking sector, a bank will be able to pay off its total liabilities whenever its project succeeds. That implies $\underline{Z} \geqslant v + \bar{d}$, and suppose this condition holds throughout the rest of the paper. Let's further impose the following assumption

ASSUMPTION 1. *$P(Z)$ is decreasing in Z, and $P(Z) \cdot Z$ is concave in Z. i.e.*

The first part captures the fact that high-return projects come with high risks. Each bank faces a trade-off between project payoff and project safety. A large Z denotes a project with large return along with high risks. Therefore, we can interpret Z_i as bank i's choice of its risk exposure. It is immediate that the efficient risk exposure for each individual bank is when $\mathbb{E}[\tilde{e}]$ is maximized: $Z^* = \text{argmax}_Z P(Z)Z$. An economy's total surplus will be later formalized in definition 3. The second part of assumption 1 is to ensure an unique interior risk exposure. A sufficient condition is to let $P()$ be concave: the project risk increases at a growing rate in the project return.

After all banks choose their risk exposure $\mathbf{Z} = (Z_1, ..., Z_N)$, the state of nature $\boldsymbol{\omega} = (\omega_1, ..., \omega_N)$ will be independently drawn from the distribution according to equation (1). [9] For each bank, ω_i can take two values: success ($\omega_i = s$) or fail ($\omega_i = f$). Therefore, $\boldsymbol{\omega} \in \boldsymbol{\Omega} = 2^N$. After the realization of the state of nature, the interbank debts' reimbursement will be determined by a payment equilibrium: a bank's interbank payments depend on what it has, which depend on other banks' interbank payments. Hence, the payment equilibrium is solved by a fixed point system. The notion is introduced by Eisenberg and Noe (2001) and then utilized by Shin (2008, 2009) and Acemoglu et al. (2015). The current paper differs from theirs in that the payment vector here is parametrized by a vector of risk exposure $\mathbf{Z}$ and a vector of states $\boldsymbol{\omega}$. Definition 1 formally defines the payment equilibrium.

DEFINITION 1. *For a network structure ($\bar{d}, \boldsymbol{\Theta}, N$) and given a risk vector $\mathbf{Z}$, the payment equilibrium is a vector of functions $\mathbf{d}^*(\boldsymbol{\omega}; \mathbf{Z}) = [d_1^*(\boldsymbol{\omega}; \mathbf{Z}), ..., d_N^*(\boldsymbol{\omega}; \mathbf{Z})]$ that solves*

$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \left\{ \min \Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\omega_i, Z_i) - v, \bar{d} \Big] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \tag{2}$$

where $d_i^*(\boldsymbol{\omega}; \mathbf{Z})$ denotes bank i's total payments to its interbank liabilities in state $\boldsymbol{\omega}$ after it chooses risk exposure $\mathbf{Z}$. On the right hand side, $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\mathbf{Z}, \boldsymbol{\omega})$ is bank i's available resources for payments to its total liabilities (deposits and interbank debts). The function $\min[., \bar{d}]$ captures banks' limited liabilities, so they pay either what they owe or what they have, whichever is smaller. $\{.\}^+ \equiv \max\{., 0\}$ denotes the fact that banks' interbank payments are non-negative. It binds when a bank cannot pay off its senior deposits.

Note that $d_i^*(\boldsymbol{\omega}; \mathbf{Z})$ is a function of $\boldsymbol{\omega}$: for each state of nature $\boldsymbol{\omega}$, we will have a separate fixed point system. Therefore, given a risk vector $\mathbf{Z}$, we need to solve 2^N fixed point systems, one for each state of nature. Before we proceed, one immediate task is to show that the above payment equilibrium exists and is unique.

[9] For the remaining text, I refer a vector as in bold letters. For example, $\boldsymbol{x} = (x_1, ...x_N)$ and $\boldsymbol{x_{-i}} = (x_1, .., x_{i-1}, x_{i+1}, ..x_N)$

LEMMA 1. [Eisenberg-Noe] *For any risk vector $\boldsymbol{Z}$, the payment equilibrium exists and is generic unique.*

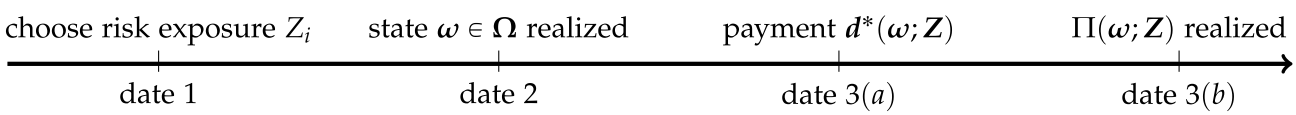
The proof is a simple utilization of Brouwer fixed point theorem. and is identical to Eisenberg and Noe (2000) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). Hence it is omitted here to conserve space. They show that for each $\tilde{\boldsymbol{e}}$, the fixed point exists and is generic unique. It is identical to say that for every combination of $(\boldsymbol{\omega}, \boldsymbol{Z})$, the fixed point exists and is generic unique. Hence lemma 1 naturally follows.

After the realization of the state of nature $\boldsymbol{\omega}$ and the interbank payments $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$, each bank's profit at the final date becomes

$$\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) = \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}, \boldsymbol{\omega}) - v_i - d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) \Big)^+ \tag{3}$$

Note that each banks' profit $\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})$ depends on the all banks' risk exposure and all banks' states of nature. In Equilibrium, each bank choose its own risk exposure Z_i to maximize the expected payoff $\mathbb{E}_{\boldsymbol{\omega}}[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})]$. Figure 1 summarizes the timeline.

Figure 1: Timeline



From equation 3, we can derive each bank's expected profit as

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})\Big] = \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) \cdot \Pr(\boldsymbol{\omega}) = \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) \cdot \prod_j \Pr(\omega_j)\Big]$$

The last equality is due to the current assumption that each bank's project outcome is independent. Each bank chooses its risk exposure to maximize the expected profit. Therefore, the Nash Equilibrium for banks' risk exposure can be derived from the following fixed point:

$$Z_i^* = \operatorname*{argmax}_{Z_i} \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Big[\Pi_i(\boldsymbol{\omega}; Z_i, \boldsymbol{Z}^*_{\boldsymbol{-i}}) \cdot \prod_j \Pr(\omega_j)\Big] \quad \forall i \in \mathcal{N} \tag{4}$$

We observe that other banks' risk exposure $\boldsymbol{Z}_{\boldsymbol{-i}}$ enters bank i's expected profit in two ways: first through the distribution of the state of nature, $Pr(\omega_j = s) = P(Z_j)$, and second through the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega}, \boldsymbol{Z})$. In next section, I will show that the second channel has no effect and bank j's risk choice affects bank i's expected profit only through the distribution of $\boldsymbol{\omega}$.

2 Risk-Taking Equilibrium and Network Distortion

It's immediate that we can define a risk-taking equilibrium as every bank chooses its exposure to risks simultaneously, anticipating other banks' optimal risk exposure and the resulting payment equilibrium.

DEFINITION 2. *The risk-taking equilibrium in a financial network* $(\bar{d}, \boldsymbol{\Theta}, N)$ *is a pair* $(\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z}), \boldsymbol{Z}^*)$ *consisting of a vector of payment functions* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ *and a vector of risk exposure* $\boldsymbol{Z}^*$ *such that:*

1. *The vector of functions* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ *is a payment equilibrium for any* $\boldsymbol{Z}$.
2. *For each* $i \in \mathcal{N}$, Z_i^* *is optimal and solves equation 4, given* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ *and* $\boldsymbol{Z}_{-i}^*$.

We first observe that the above risk-taking equilibrium is the solution of two intertwined systems of equations (equation 2 and 4): when choosing the risk vector Z_i, each bank anticipates the payment equilibrium. When determining the interbank debt payment $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$, banks' chosen risk vector is a parameter.

At first glance, the fixed point solutions to the two intertwined systems look complicated to derive. Thanks to the following lemma 2 and proposition 1, the existence and analytical solutions for the risk-taking equilibrium can be attained.

LEMMA 2. *The payment equilibrium* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ *is constant in the risk exposure vector* $\boldsymbol{Z}$.

Proof. In the Appendix

As a result, we can rewrite $\boldsymbol{d}^*(\boldsymbol{\omega}) = \boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$. The idea is that when a bank's project succeeds, its total interbank payment is the face value $\bar{d}$, independent of any bank's chosen risk exposure. On the other hand, when a bank's project fails, its contribution to the payment system is 0, also independent of any bank's chosen risk exposure.[10] Therefore, the payment equilibrium is independent of the risk exposure vector $\boldsymbol{Z}$.

As a result of lemma 2, we can disentangle definition 2's two intertwined systems of equations: we first solve the fixed point function for the payment equilibrium (equation 2), and then use it to derive the fixed point for the risk-taking Nash Equilibrium (equation 4).

we also observe that a bank will earn a positive profit only if its project succeeds. Suppose a bank's project fails, at most its available resource will be $\max_{\omega} \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) = \bar{d}$, that is when its interbank claims get paid in full. That implies this bank will default on its interbank debts (i.e. $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - v < \bar{d}$). Therefore the bank with a failed project will earn a zero profit at the final date. Hence, we can easily rewrite bank i's expected profit as:

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})\Big] = P(Z_i) \sum_{\boldsymbol{\omega}_{-i}} \Big[Z_i - v - \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s})\Big)\Big] \cdot \Pr(\boldsymbol{\omega}_{-i})$$

[10] Although a failed bank's contribution to the payment system is zero, its interbank payments maybe positive. See figure 3(a) as an example: bank 2's project fails but it has a positive interbank payment to bank 3.

where $\boldsymbol{\omega_{-i}} \in 2^{N-1}$ denotes the vector of states for all banks except bank i. With a slight abuse of notation, I denote $\boldsymbol{\omega^{i=s}} \equiv (\omega_1 ..., \omega_{i-1}, s, \omega_{i+1}, ...\omega_N)$ as the vector that appends bank i's success to other banks' states of nature $\boldsymbol{\omega_{-i}}$.

Define a function $\mathcal{D}(\mathbf{Z_{-i}})$ as

$$\mathcal{D}(\mathbf{Z_{-i}}) \equiv \sum_{\omega_{-i}} \left(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{i=s}}) \right) \cdot \Pr(\boldsymbol{\omega_{-i}}) > 0 \tag{5}$$

Note that $\mathcal{D}(\mathbf{Z_{-i}})$ is parameterized by the network structure $(\bar{d}, \boldsymbol{\Theta}, N)$. We will study their effects in section 3. After plugging $\mathcal{D}(\mathbf{Z_{-i}})$ into the bank's expected profit, we have

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z})\Big] = P(Z_i)(Z_i - v) - P(Z_i)\mathcal{D}(\mathbf{Z_{-i}}) \tag{6}$$

Remember that $P(Z_i)(Z_i - v)$ is the expected profit of a stand-alone bank. From equation 5, $\mathcal{D}(\mathbf{Z_{-i}})$ is bank i's expected net interbank payment (or "cross-subsidy") to other banks if it succeeds. It is worth noting that, when bank i's project fails, its profit is always zero and hence there is no distortion. The asymmetric effect of $\mathcal{D}(\mathbf{Z_{-i}})$ induces a bank to expose to more risks at date 1. Hence this cross-subsidy $\mathcal{D}(\mathbf{Z_{-i}})$ can be interpreted as a network risk-taking distortion.

Anticipating the payment equilibrium (or more importantly the network distortion) at date 3, each bank simultaneously chooses its risk exposure at date 1 to maximize its expected profit (equation 6). Essentially, banks' choices of risk exposure $\mathbf{Z}^*$ are the result of a Nash Equilibrium. The following lemma shows that, in financial networks, banks' risk exposure is strategically complementary.

PROPOSITION 1. *The choice of risk exposure* $\mathbf{Z}$ *is strategically complementary among all banks in the same financial network.*

Proof. In the Appendix

Formally, $\mathrm{d}\widehat{Z}_i / \mathrm{d}Z_m \geqslant 0, \forall i$ and $m \neq i$, where $\widehat{Z}_i = \operatorname{argmax}_{Z_i} \mathbb{E}[\Pi_i(\boldsymbol{\omega}; \mathbf{Z})]$. It means that a bank's optimal risk exposure is increasing in any other bank's risk exposure. The intuition is as follows. if bank m chooses a greater risk exposure, its project will be more likely to fail, i.e. $Pr(\omega_m = f)$ increases in Z_m. When bank m's project fails compared with when it succeeds, bank i's net interbank payments (or cross-subsidy) to other banks will increase. To compensate this increased distortion, bank i will optimally choose a greater risk exposure in response to bank m's increased risk exposure. (note that proposition 1 is not yet an equilibrium result.)

Proposition 1 conveys the first important message of this paper. It assigns a new meaning to the view of the "too connected to fail" in the sense that a connected bank not only affects other banks through an ex-post loss contagion, as in Allen and Gale (2000), Freixas et al. (2000), or Acemoglu et al. (2015). But it also creates an ex-ante moral hazard problem to other banks.

With the supermodular property for banks' choices of risk exposure, we are now able to establish the risk-taking equilibrium's existence.

PROPOSITION 2. *In any network structure $(\bar{d}, \mathbf{\Theta}, N)$, the risk-taking equilibrium exists.*

Proof. In the Appendix

The proof is a simple application of the Tarski (1955) fixed point theorem to a supermodular game. In general, the equilibrium is not unique. For the remaining text, let's focus on the Pareto-dominant equilibrium when $\mathbf{Z}$ is the smallest among of the set of fixed points.[11].

After establishing the existence of the risk-taking equilibrium, we can finally compare a connected bank's endogenous choice of risk exposure with that of a stand-alone bank. The following proposition shows that the interconnectedness indeed encourages banks to expose to more risks.

COROLLARY 1. *A bank in any network structure $(\bar{d}, \mathbf{\Theta}, N)$ will choose a greater exposure to risks than a stand-alone bank.*

Proof. In the Appendix.

With the cross-holding of interbank debts, a solvent bank reimburses a net positive amount of cross-subsidy to failed banks' creditors (i.e. depositors). As argued earlier by proposition 1, every bank in the financial network, anticipating this distortion, will increase its exposure to risks. In equilibrium, no bank will internalize the effect of its choice of risk exposure on other banks. As a result, there exists a risk-taking externality, and connected banks will choose greater risk-exposure than stand-alone banks.

In order to distinguish the friction of proposition 1 from a stand-alone moral hazard problem of Jensen and Meckling (1976), let's first define the total social welfare.

DEFINITION 3. *The social welfare is the sum of the expected returns to all agents in the economy, namely banks and retail depositors. Formally,*

$$u = \mathbb{E}\left\{ \underbrace{\sum_i \left(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\mathbf{Z}, \boldsymbol{\omega}) - v_i - d_i^*(\boldsymbol{\omega}; \mathbf{Z}) \right)^+}_{\textit{expected return to banks}} + \underbrace{\sum_i \min\left[v_i, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\mathbf{Z}, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) \right]}_{\textit{expected return to depositors}} \right\}$$

The return to banks is similar to a call option with a strike price of zero and the return to depositors is similar to a standard debt contract with a face value v_i. It is easy to see that

$$u = \mathbb{E}\left\{ \sum_i \left(e_i(\mathbf{Z}, \boldsymbol{\omega}) + \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}; \mathbf{Z}) \right) \right\} = \mathbb{E}\left\{ \sum_i e_i(\mathbf{Z}, \boldsymbol{\omega}) \right\} = \sum_i P(Z_i) \cdot Z_i$$

Comparing the total surplus with each individual bank's objective function (equation 6), we notice that there exists two frictions in a decentralized banking network: a friction between banks

[11] Focusing on the least exposure equilibrium is to abstract away a self-fulfilling failure. See Elliott et al. (2014) for more details. They also consider the "best-case" equilibrium, in which as few organizations as possible fail. Furthermore, all of following results are robust to any stable equilibrium.

and depositors (Jensen and Meckling, 1976) and a risk-taking externality among connected banks, which is the main focus of this paper.

So far, we have shown that a connected bank will endogenously expose to greater risks due to a network risk-taking distortion. Let's now examine the extent this network distortion for different network structures. In particular, in the next section, we will study how this risk distortion varies with the size of interbank liabilities, the degree of network completeness, and the number of counterparties.

3 Network Structures

To begin with, we first study the effect of the interbank liabilities' size $\bar{d}$ on the network risk-taking distortion $\mathcal{D}(\boldsymbol{Z}_{-\boldsymbol{i}})$ and the subsequent equilibrium risk exposure $\boldsymbol{Z}^*$. We do so by fixing the network topology $\boldsymbol{\Theta}$ and the number of counterparties N. Lemma 3 formalizes the result.

LEMMA 3. *In any network structure ($\bar{d}, \boldsymbol{\Theta}, N$), the network risk-taking distortion $\mathcal{D}(\boldsymbol{Z}_{-\boldsymbol{i}}; \bar{d})$ is increasing and concave in the size of interbank liabilities $\bar{d}$.*

Proof. In the Appendix.

To understand the intuition behind lemma 3, it is helpful to first notice that there are three types of bank outcomes at the final date. The first type contains banks with successful projects. Denote them as $\mathcal{S}_\omega \equiv \{i : \omega_i = s\}$. The second type contains banks that failed its project but can fully fulfill their outside liabilities (i.e. deposits). Denote them $\mathcal{F}^+_\omega \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d^*_j(\boldsymbol{\omega}) \geqslant v\}$. Since those banks can fulfill their deposits, they will contribute back to the interbank payment system. Let's call them "in-the-money" failed banks. The third type contains banks that failed its project and cannot fully fulfill their outside liabilities. Denote them $\mathcal{F}^-_\omega \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d^*_j(\boldsymbol{\omega}) < v\}$ and call them "out-of-money" failed banks. The depositors of those banks will incur losses (or be reimbursed by deposit insurance if existed).

In a network with larger interbank liabilities, successful banks $\mathcal{S}$ will expect larger net interbank payments (cross-subsidy) to failed banks ($\mathcal{F}^- \cup \mathcal{F}^+$). Those cross-subsidy are due to the difference between what a successful bank pays, $\bar{d}$, and what it receives, $\sum_j \theta_{ij} d^*_j(\boldsymbol{\omega})$. As argued earlier, those cross-subsidy are the causes of the network risk-taking distortion. Therefore, the network risk-taking distortion is increasing in the size of interbank liabilities.

On the other hand, the larger cross-subsidy also increase the likelihood for a failed bank to be "in-the-money" ($\mathcal{F}^- \to \mathcal{F}^+$). An "in-the-money" failed bank will contribute back to the payment system, which in turn partially lowers the cross-subsidy that a successful bank need to pay. As a result of the above two countervailing effects, the network risk-taking distortion is increasing (due to larger interbank payment) and concave (due to more "in-the-money" banks) in the size interbank liabilities. We can then immediately apply lemma 3 to obtain the following equilibrium result.

PROPOSITION 3. *In any network structure $(\bar{d}, \mathbf{\Theta}, N)$, each bank's choice of risk exposure Z_i^* is increasing in the size of interbank liabilities $\bar{d}$.*

Proof. In the Appendix.

The proof is a simple application of the monotone selection theorem. From lemma 3, we know that, given a fixed counterparty risk, each bank will experience a larger risk-taking distortion if they are connected through greater interbank liabilities. In equilibrium, this increased distortion will induce every bank to expose to greater risks.

From the concavity result of lemma 3, we know that the interbank liabilities have a diminishing marginal effect on the connected banks' risk-taking distortion. Eventually, they will cease to have additional effect after a certain threshold. The following corollary formalizes this fact.

COROLLARY 2. *There exists a maximum network distortion $\mathcal{D}^{max}(\boldsymbol{Z_{-i}})$, such that $\mathcal{D}(\boldsymbol{Z_{-i}}; \bar{d}) \leqslant \mathcal{D}^{max}(\boldsymbol{Z_{-i}})$ for all $\bar{d}$. We have*

$$\mathcal{D}^{max}(\boldsymbol{Z_{-i}}) = \sum_{f=1}^{N-1} \frac{f}{N-f} \cdot v \cdot \binom{N-1}{f} \Big[P(Z_{-i})\Big]^{N-1-f} \Big[1 - P(Z_{-i})\Big]^{f} \tag{7}$$

Proof. In the Appendix.

We have known that the network risk-taking distortion is the result of the connected bank's expected "cross-subsidy" to other banks' depositors. That implies the distortion will stop increasing when every banks' deposits can be fulfilled in every state of nature (a sufficient condition is $n \cdot v \leqslant \bar{d}$).

Equation 7 has a clean interpretation. Suppose in some state of nature ω, f banks' projects fail and $(N-f)$ banks' projects succeed. The maximum amount of money that needs to be bailed out is $f \cdot v$, the total deposits of failed banks. At date zero, a successful bank's expected interbank payment is $f \cdot v/(N-f)$. The probability with which f banks fail is $\binom{N-1}{f}[P(Z_{-i})]^{N-1-f}[1 - P(Z_{-i})]^f$. Note that $\mathcal{D}^{max}(\boldsymbol{Z_{-i}})$ is independent of the network topology $\mathbf{\Theta}$ conditional on $\boldsymbol{Z_{-i}}$.

Let's now turn our attention to two particular network structures: the complete network and the ring network. The ex-post contagion of those two networks has been studied by Allen and Gale (2000) and Acemoglu et al. (2015) among others. Here we will study their ex-ante network risk-taking distortion. In a ring network, every bank is connected only to its direct neighbors. In a complete network, every bank is connected to every other bank. Definition 4 formalizes the above description.

DEFINITION 4. *In a financial network with N banks, a ring network and a complete network are defined as*

$$\mathbf{\Theta}^{\boldsymbol{R}} = \begin{bmatrix} \mathbf{0}'_{N-1} & 1 \\ \boldsymbol{I}_{N-1} & \mathbf{0}_{N-1} \end{bmatrix} \quad \textit{and} \quad \mathbf{\Theta}^{\boldsymbol{C}} = \frac{1}{N-1}(\mathbb{1}_{N,N} - \boldsymbol{I}_N)$$

where $\mathbf{0}_{N-1}$ is a vector of $N-1$ zeros, $\mathbb{1}_{N,N}$ is a matrix of ones with a dimension (N, N), and $\boldsymbol{I}$ is an identity matrix. In addition, let's defined a λ network as $\boldsymbol{\Theta^{\lambda}} = (1-\lambda)\boldsymbol{\Theta^{R}} + \lambda\boldsymbol{\Theta^{C}}$, a convex combination of a ring and a complete network. Figure 2.(a) - (c) illustrates a complete, a ring network, and a $\lambda = 0.2$ network with 5 banks.

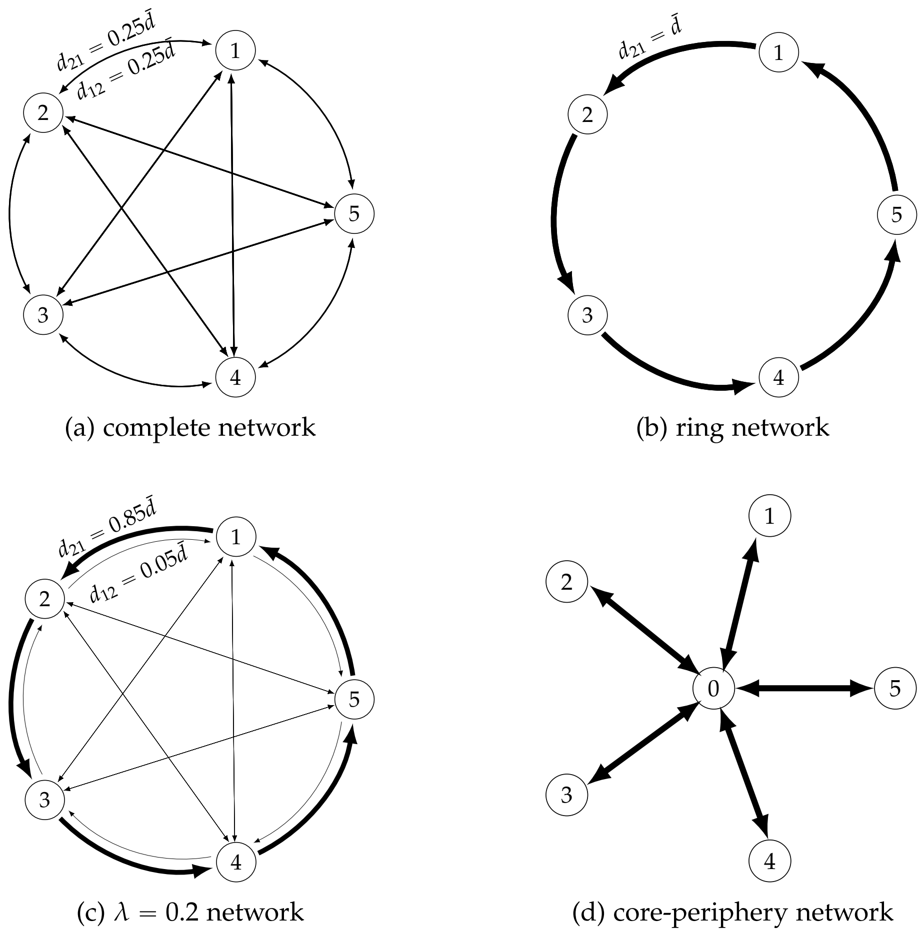


Figure 2: Network Topology

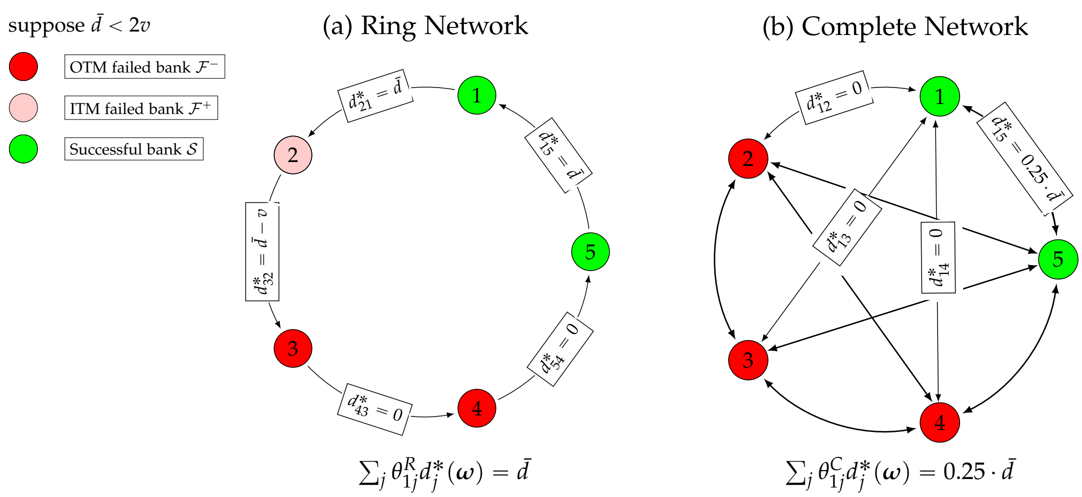
The following proposition studies the equilibrium risk-taking of banks in a complete and a ring network.

PROPOSITION 4. *In any network structure $(\bar{d}, N)$, each bank's choice of risk exposure Z_i^* is larger in a complete network than in a ring network.*

Proof. In the Appendix.

The above proposition states that every bank in a complete network, anticipating the interbank payment and counterparties' risk exposure, will choose a greater risk exposure than banks in a ring network. The result stands in sharp contrast to the view of Allen and Gale (2000). They argue that a complete network is better at co-insurance and hence more robust. However, we show that because of precisely this co-insurance, solvent banks will anticipate a greater amount of "cross-subsidy" to failed banks' depositors. As argued earlier, due to such distortion, every bank will have an ex-ante incentive to expose to greater risks. As a result, in equilibrium, every bank in a complete network will choose a greater risk exposure.

Figure 3: Payment Equilibrium at $\boldsymbol{\omega} = (s, f, f, f, s)$



To illustrate the intuition, figure 3 displays the interbank payment for a ring and complete network with the same interbank liabilities $\bar{d}$. We assume a modest $\bar{d}$, i.e. $\bar{d} \in (v, 2v)$, otherwise the network topology will be irrelevant (corollary 2). Suppose only bank 1 and 5 have successful projects. In the ring network, bank 1 receives $\bar{d}$ from bank 5 and pays $\bar{d}$ to bank 2. There is no distortion. In a complete network, however, bank 1 pays a total of $\bar{d}$ but only receives $\bar{d}/4$ from bank 5, hence a $3\bar{d}/4$ distortion. Bank 1's expected distortion is increasing in all other banks' failed probability. However, in a ring network, bank 1 will not be affected by the counterparty risk from bank 2, 3 and 4 if bank 5 succeeds. To summarize, banks in a ring network will be less affected by the network risk-taking externality than banks in a complete network, and hence will choose less risk exposure in equilibrium.

Similarly, the logic can be applied to networks with greater numbers of banks. Since the dimension of $\boldsymbol{\Theta}$ varies with N, we hence focus on the maximum risk-taking distortion and risk-exposure, which, according to corollary 2, are independent of the network topology $\boldsymbol{\Theta}$.

PROPOSITION 5. *In any network structure $(\bar{d}, \boldsymbol{\Theta}, N)$, each bank's maximum risk exposure Z^*_i is increasing in the number of banks, N, in the network.*

Proof. In the Appendix.

It states that every bank will choose a greater risk exposure if they are connected to more counterparties. A bank in a larger network expects to bail out more failed banks' deposits; hence it will be affected by the risk-taking externalities of more other parties. As a result, each bank will ex-ante choose a greater risk exposure.

The machinery of networks also allows us to study the financial structures that are widely observed in the current financial systems. For example, interbank markets have shown to exhibit a core-periphery structure (Craig and Von Peter, 2014; Afonso et al., 2013). Indeed, a special form of core-periphery structure, called central clearing counterparty (CCP), has becomes popular after the 2009 Pittsburgh G20 summit (Domanski et al., 2015). We first observe that any network structure with a CCP is equivalent to a core-periphery network where the core acts as the clearing party with no asset and no outside liability. Figure 2.(d) illustrates such a network. The next proposition studies the risk-taking incentives for banks in networks with a CCP.

PROPOSITION 6. *In any network structure* $(\bar{d}, \mathbf{\Theta}, N)$ *with a central clearing counterparty, the risk-taking equilibrium is equivalent to that of a complete network with* $(\frac{N-1}{N}\bar{d}, \mathbf{\Theta}^C, N)$.

Proof. In the Appendix.

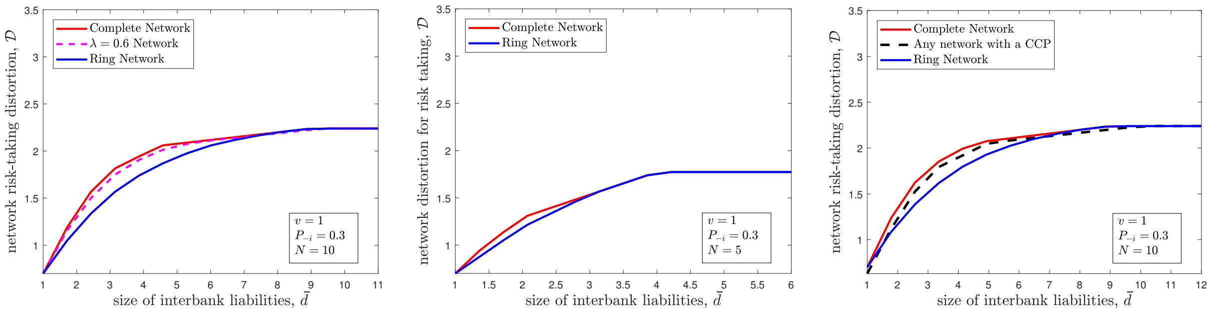
We observe that a CCP has two opposite effects on the risk-taking equilibrium. First, a CCP increases banks' risk-taking incentives by increasing the network's completeness. Through the central clearing, each bank is "forced" to connect to every other bank, and hence becomes exposed to their risk-taking externalities. This "CCP-riskier" effect is greater for a loosely connected ring network than a complete network, on which a CCP has no effect since a complete network is already maximum connected. Second, a CCP reduces banks' risk-taking incentives by making banks' connections indirect; or in other words, it reduces the size of the connection from $\bar{d}$ to $\frac{N-1}{N}\bar{d}$. [12] Hence, a CCP reduces banks' risk-taking externalities (proposition 3). This "CCP-safer" effect is greater for a complete network than for a ring network. To sum up, the net effect of a CCP depends on the original network topology. For an original loosely connected network, the "CCP-riskier" effect will dominate, and hence the CCP will encourage more risk-taking. Figure 4.(c) provides a numerical simulation for the effects of a CCP on a complete and a ring network for different $\bar{d}$. We can see that introduction of a CCP on a ring network with a modest $\bar{d}$ will actually increase banks' risk-taking.

So far, we have studied the comparative statistics of the size interbank liabilities $\bar{d}$, the completeness of the network structure $\mathbf{\Theta}$, the number of counterparties N, and the existence of a CCP on the network risk-taking distortion. The following figure provides a numerical simulation that summarizes the results. Figure 4.(a) displays a benchmark case where $N = 10$, $v = 1$, and $P(Z_{-i}) = 0.3$. It plots the network risk-taking distortion against the size of interbank liabilities for a complete, a $\lambda = 0.6$, and a ring network. It first shows that the network distortion is increasing and concave in the size of interbank liabilities, confirming proposition 3. We also see that the distortion is larger in a complete network (red) than a ring network (blue), confirming proposition 4.. In figure 4.(b), we decrease the number of banks from 10 to 5, and we see that the network risk-taking distortion decreases for both ring and complete networks. It confirms

[12] To illustrate this point, suppose there are four banks. In one state of nature, three succeed and one fails. Suppose the failed bank is "out-of-the-money". In this case, the distortion for a successful bank in a complete network is $\bar{d} - 2 \cdot \frac{1}{3}\bar{d} = \frac{1}{3}\bar{d}$. However, the distortion for a successful bank in a network with a CCP is $\bar{d} - 3 \cdot \frac{1}{4}\bar{d} = \frac{1}{4}\bar{d}$

Figure 4: Numerical Simulation

(a) Benchmark (b) Decrease N (c) With a CCP



proposition 5. In figure 4.(c), we see that the effect of a CCP depends on the original network's topology. For a complete network, a CCP decreases the network risk-taking distortion. However, for a ring network with a modest $\bar{d}$, a CCP actually increases the network risk-taking distortion.

4 Policy Implications

In this section, we will extend the benchmark model to study several widely adopted prudential policies that aim at stabilizing the financial system: deposit insurance, capital ratio requirement, and government bailout. We will also study the effect of a banking network's transparency on systemic stability. Although the effects of those policies on stand-alone banks have been well explored in the literature, their effects in complex financial networks, to the best of my knowledge, are still under-studied. I believe that understanding the network effects of those government policies is particularly important in the current growingly connected financial systems.

4.1 Deposit Insurance

In this section, I study depositors' rational decisions to lend to a bank while being aware of the interbank connections. In the words of equilibrium, I endogenize the deposit rates. To be more specific, each bank in the network $(\bar{d}, \Theta, N)$ needs to borrow $M_i = 1$ (normalized to 1) from atomistic depositors to finance a productive project Z_i. The borrowing takes the form of a standard debt contract with a face value v_i, which will be determined in equilibrium. v_i can be interpreted as the gross interest rate. For expository purposes, we assume depositors are risk neutral and have time discount rate β. After each bank receives the deposits, they simultaneously choose their project choices. The subsequent timeline follows figure 1.

To summarize, a financial system consists of atomistic depositors and a fixed number of connected banks. Depositors, in anticipation of banks' optimal risk exposure, choose whether to deposit their money. Banks, in anticipation of the payment equilibrium, simultaneously choose their risk exposure. A competitive market results in a zero profit for atomistic depositors. [13] The

[13] A competitive market condition is identical to let banks be residual claimants, a common practice in the literature.

equilibrium $(d_i^*(\boldsymbol{\omega}), \boldsymbol{v}^*, \boldsymbol{Z}^*)$ is hence characterized by

$$
\begin{aligned}
d_i^*(\boldsymbol{\omega}) &= \left\{ \min \Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \boldsymbol{Z}) + e_i(\omega_i, Z_i) - v_i^*, \bar{d}, \Big] \right\}^+ && \forall i \in \mathcal{N}, \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \\
Z_i^* &= \underset{Z_i}{\operatorname{argmax}}\, \mathbb{E}\Big[\Pi_i^B(\boldsymbol{\omega}; \boldsymbol{Z}, v_i^*)\Big] && \forall i \in \mathcal{N} \\
0 &= -M_i + \beta \cdot \mathbb{E}\Big[\Pi_i^D(\boldsymbol{\omega}; v_i^*, \boldsymbol{Z}^*)\Big] && \forall i \in \mathcal{N}
\end{aligned}
$$

where $\Pi_i^B(\boldsymbol{\omega})$ is bank i's payoff in state $\boldsymbol{\omega}$, which is the same as equation 3, and $\Pi_i^D(\boldsymbol{\omega})$ is bank i's depositors' payoff in state $\boldsymbol{\omega}$. With deposit insurance, $\Pi_i^D(\boldsymbol{\omega}) = v_i^*$ for all $\boldsymbol{\omega}$. In this case, the return to depositors are guaranteed by the government. Without deposit insurance, $\Pi_i^D(\boldsymbol{\omega}) = \min[v^*, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega})]$ as a debt contract. It's worth noting that, in this case, $\Pi_i^D(\boldsymbol{\omega})$ is a function of $\boldsymbol{\Theta}$ and $\boldsymbol{Z}^*$: depositors perfectly observe the network structure and perfectly anticipate banks' optimal risk exposure. The following proposition shows the equilibrium result for the two cases.

PROPOSITION 7. *Consider the following two cases:*

(a) Without deposit insurance, (i) banks' equilibrium risk exposure is identical in any network structure: $Z^{C*} = Z^{R*} = Z^{S*}$ (ii) banks' equilibrium deposit rates are higher in a ring network than in a complete network: $v^{S*} > v^{R*} \geqslant v^{C*}$.

(b) With deposit insurance, (i) banks' equilibrium risk exposure is higher in a complete network than in a ring network: $Z^{C*} \geqslant Z^{R*} > Z^{S*}$. (ii) banks' equilibrium deposit rates are identical in any network structure: $v^{C*} = v^{R*} = v^{S*}$.

(where the superscript C denotes complete network, R denotes ring network, and S denotes stand-alone)

Proof. In the Appendix.

Proposition 7.(a) states that without deposit insurance, banks' choices of risk exposure are identical in any network structure, in contrast to proposition 3 of the benchmark model. The benchmark model assumes fixed deposit rates and shows that banks in highly connected networks expose to greater risks due to a network risk-taking distortion. To understand the difference, let's first recall that this network risk-taking distortion is the result of "cross-subsidy" from successful banks to failed banks' depositors. Without deposit insurance, depositors in highly connected networks will feel more co-insured from the interbank connections and will demand lower interest rates. Both the lowered deposit rates and the "cross-subsidy" affect connected banks' upside payoffs. Their countervailing effects will equalize banks' choices of risk exposure in any network structure. The result is a generalization of the Modigliani-Miller (MM) theorem in the sense that banks' *interbank* debt structure is irrelevant to their choices of risk exposure.

With that, depositors' participation constraints bind.

It is worth noting that Proposition 7.(a) is related to a general equilibrium model with *rational competitive price perceptions (RCPP)* by Magill and Quinzii (2002). They show that the capital markets with informed participants can act as a disciplining device for the ex-post moral hazard. In my setup, the deposit rate is a *price disciplining* device for banks' choice of risk exposure.

Proposition 7.(b) considers financial systems where depositors are fully insured by the government. The result is identical to the benchmark model with a fixed deposit rate. With a government's guarantee, depositors are "informative insensitive" to banks' financial structures. As a result, the deposit rates are constant across all network structures and equal to depositors' time cost ($1/\beta$). Without deposit rates' *price disciplining*, banks in highly connected networks will face greater network risk-taking distortion and choose greater exposure to risks (proposition 3 to 5).

4.2 Network Transparency

In the benchmark model, we assumed that all banks can fully observe the network structure $(\bar{d}, \mathbf{\Theta}, N)$. While this is a reasonable assumption, atomistic retail depositors' observability of the network structure is not immediately guaranteed. This section will study the effect of retail depositors' observability of the interbank network on banks' risk exposure choices.

We begin by noticing that the logic of proposition 7.(b) can again be applied to banking sectors where depositors are unaware of the interbank relationships. Due to the unobservability of the interbank network, deposit rates are again identical among all network structures. Losing the deposit rates' *pricing disciplining* ability, the network risk-taking distortion will not be compensated. And as a result, banks in opaque networks, especially connected ones, will choose to expose to greater risks than banks whose interbank network is fully observable to their depositors. The following proposition formalizes the result.

PROPOSITION 8. *In any network structure $(\bar{d}, \mathbf{\Theta}, N)$, each bank's choice of risk exposure Z_i^* is larger if the interbank network is not observable to depositors.*

Proof. In the Appendix.

Without being able to observe the network structure, depositors are unaware of their banks' co-insurance mechanism. In equilibrium, the deposit rates will not adjust to reflect such co-insurance. Connected banks still face risk-taking distortion from interbank payments. In other words, there will be no *price disciplining* from deposit rates to endogenously align connected banks' risk exposure.

The transparency of the interconnectedness is particularly important for the shadow banking sector. Without a government's guarantee, shadow banks' creditors will have greater incentives to observe the interbank relationships, and the transparency of them are hence essential.

4.3 Capital Requirement

So far, we have been studying banks' risk-taking equilibrium in financial networks where banks do not hold any equity. Since the 1980s, regulators began using capital adequacy requirement to ensure that banks do not take excessive risks. (Gorton, 2012). With more "skin in the game", banks are less willing to gamble with their own equity (Jensen and Meckling, 1976). Nevertheless, the effect of each individual bank's capital on the entire systemic stability is less understood so far. In this section, I will extend the benchmark model to study the network effects of the bank capital.

Now suppose that each bank is required to have equity of size $r_i = r$. As a result, the outside debt a bank needs to borrow decreased to $v - r$. When a bank's total cash flow is smaller than its total liabilities, the equity holders will be incur a loss. The payment equilibrium now becomes

$$d_i^*(\boldsymbol{\omega}; r) = \left\{ \min\left[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; r) + e_i(\mathbf{Z}, \boldsymbol{\omega}) - v + r, \bar{d} \right] \right\}^+ \quad \forall i \tag{8}$$

The expected profit becomes

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z})\Big] = P(Z_i)(Z_i - v + r) - P(Z_i)\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r) \tag{9}$$

where

$$\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r) \equiv \sum_{\boldsymbol{\omega}_{-i}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; r) \Big) \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})$$

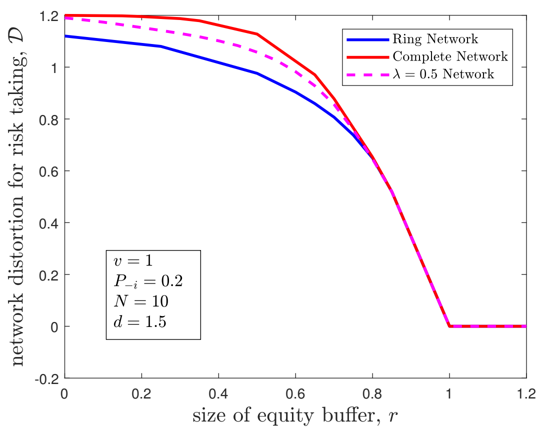
We first notice that equity enters a bank's expected payoff in two ways: its upside payoff $(Z_i - v + r)$ and its network risk-taking distortion $\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r)$. The next proposition studies how an equity buffer will affect the network risk-taking distortion.

LEMMA 4. *In any network structure $(\bar{d}, \boldsymbol{\Theta}, N; r)$, the network risk-taking distortion $\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r)$ is decreasing and concave in the size of equity buffers r.*

Proof. In the Appendix.

If a bank fails, its equity holders will first incur the loss before its depositors or interbank counterparties. The loss that may be otherwise propagated to other banks will now be absorbed by this equity buffer. As a result, any bank's equity will decrease the cross-subsidy that solvent banks need to pay. The network risk-taking distortion is hence reduced. Moreover, the equity makes more failed banks "in-the-money", and hence be able to contribute back to the payment system. This further reduces a solvent bank' "cross-subsidy" to other failed banks. Therefore, the network risk-taking distortion is decreasing at a growing rate in the size of an equity buffer. Figure 5 plots the network risk-taking distortion against the size of the equity buffer.

Figure 5: Equity Buffer



Lemma 4 immediately implies that banks' equilibrium risk exposure will be reduced by an equity buffer in any network structure.

PROPOSITION 9. *In any network structure* $(\bar{d}, \boldsymbol{\Theta}, N; r)$, *each bank's choice of risk exposure* Z_i^* *is decreasing in the size of equity* r.

Proof. In the Appendix.

There are two effects of equity buffers on banks' choices of risk exposure. First, an equity buffer has a direct effect on a bank's own risk-taking. A bank will choose to expose to fewer risks if it has a higher equity ratio. The argument is identical to the asset substitution problem of debt financing (Jensen and Meckling, 1976). In other words, banks are unwilling to gamble with more "skin in the game". More interestingly, there is a second network effect from the equity buffers. Equity buffers will curb a failed bank's loss at the origin, and hence they will reduce the network risk-taking distortion. Banks in the network with abundant equities will choose lower exposure to risks. In other words, a capital ratio requirement will have a multiplier effect on systemic stability.

4.4 Government Bailout

The 2008 bailout of Bear Stearns and the subsequent Troubled Asset Relief Program (TARP) have sparked continuing debates among both policy-makers and academics. Government bailouts have been widely argued to incentivizes harmful ex-ante behaviors (Gale and Vives, 2002; Farhi and Tirole, 2012; Erol, 2018). In this subsection, we study the effect of a government bailout on banks' ex-ante risk-taking incentives when they are connected in financial networks.

I define a government bailout (n, t) as a transfer t from the government to each failed bank if and only if the number of failed banks exceeds n. Similar to Erol (2018), I assume the government bailout only happens in crisis times when a large number of banks have failed.[14] With the

[14] This definition is consistent with section 101 of the 2008 Emergency Economic Stabilization Act (EESA), which

government bailout in place, the payment equilibrium becomes

$$d_i^*(\boldsymbol{\omega};\mathbf{Z}) = \left\{ \min \Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\mathbf{Z}) + e_i(\omega_i, Z_i) + t_i(\boldsymbol{\omega}) - v, \bar{d}, \Big] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \tag{10}$$

where the state-contingent transfer is defined as

$$t_i(\boldsymbol{\omega}) \equiv t \cdot \mathbb{1}(\omega_i = f) \cdot \mathbb{1}(\text{\# failed banks} \geqslant n)$$

The rest of the definition for the network risk-taking equilibrium remains unchanged from equation 4. As we see from the following proposition, a government bailout contributes to the systemic stability by reducing the network risk-taking distortion.

PROPOSITION 10. *In any financial network ($\bar{d}, \boldsymbol{\Theta}, N$), each bank's network risk-taking distortion and equilibrium risk exposure is reduced if there exists a government bailout.*

Proof. In the Appendix.

In contrast to the conventional wisdom, proposition 10 states that a credible government bailout will discourage the ex-ante risk-taking. With a government bailout, the loss will be curbed before spreading to successful banks in crisis times. Ex-ante, each bank will anticipate a smaller "cross-subsidy" if it succeeds. By the argument before, the reduction of the cross-subsidy will encourage connected banks to reduce their choices of risk exposure.

Proposition 10 also stands in contrast to Erol (2018), who show that a government bailout creates the systemic instability by encouraging excessive network formation. In their model, with a government bailout, banks will not be worried about contagion during network formation. In contrast, this paper shows that exactly because banks will not worry about the contagion, they will be subject to less network risk-taking externalities. While both effects (excessive network formation and less risk-taking) are reasonable, the net effect of a government bailout is an empirical question.

5 Correlated Risk Exposure

In previous sections, we have assumed that banks' project outcomes are independent. While this is a reasonable assumption for local banks serving idiosyncratic mortgages across different regions, the portfolio of large national banks may be well correlated. They may also endogenously choose similar portfolios. In this section, I model banks' decision whether to expose to correlated risks, and explain why a systemic crisis can endogenously evolve as a result of the interbank connectedness.

states "TARP was only part of the government's response to the crisis" and "to restore liquidity and stability to the financial system". See https://www.treasury.gov/initiatives/financial-stability/about-tarp.

Now suppose bank i, besides choosing the marginal distribution of its project outcome $P(Z_i)$, also chooses the conditional distribution $\boldsymbol{\lambda}_i = [\lambda_1, ..., \lambda_N]$ on the project outcomes of other connected banks in the same network.

$$\lambda_{ij} \equiv Pr(\omega_i = s | \omega_j = s)$$

where $\mathbf{0} \leqslant \boldsymbol{\lambda}_i \leqslant \mathbf{1}$. We can interpret λ_{ij} as bank i's endogenous correlation with bank j. This notion of pairwise conditional probabilities matrix was first proposed in the IMF's *Global Financial Stability Review* (2009) and later utilized by Bisias et al. (2012).

As a result of the above definition, the pairwise correlation between bank i and j's projects is

$$\rho_{ij} \equiv \frac{\lambda_{ij}P(Z_j) - P(Z_i)P(Z_j)}{P(Z_i)^{1/2}P(Z_j)^{1/2}[1-P(Z_i)]^{1/2}[1-P(Z_j)]^{1/2}}$$

In contrast to the benchmark model, each banks' project outcomes are no longer independent. Bank i's expected profit hence becomes

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; Z_i, \lambda_i)\Big] = P(Z_i)(Z_i - v) - \sum_{\boldsymbol{\omega}_{-i}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s})\Big) \cdot \Pr(\boldsymbol{\omega}_{-i}) \cdot \Pr(\omega_i = s | \boldsymbol{\omega}_{-i})$$

The above equation uses the property $\Pr(\boldsymbol{\omega}^{i=s}) = \Pr(\boldsymbol{\omega}_{-i}) \cdot \Pr(\omega_i = s|\boldsymbol{\omega}_{-i})$. Note that the dependence vector $\boldsymbol{\lambda_i}$ enters the last term.

DEFINITION 5. *The correlated risk-taking equilibrium in a financial network $(\bar{d}, \boldsymbol{\Theta}, N)$ is a triplet $(\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z}), \mathbf{Z}^*, \boldsymbol{\lambda}^*)$ consisting of a vector of payment functions $\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z})$, a vector of risk exposure $\mathbf{Z}^*$, and a matrix of conditional distribution $\boldsymbol{\Lambda}^* \equiv [\lambda_{ij}^*]$ such that:*

1. *The vector of functions $\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z})$ is a payment equilibrium for any $\mathbf{Z}$.*

$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \left\{ \min\Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\omega_i, Z_i) - v, \bar{d}, \Big] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega}$$

2. *For each bank $i \in \mathcal{N}$, $(Z_i^*, \boldsymbol{\lambda_i^*})$ is optimal and solves the following equation, given $\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z})$, $\mathbf{Z}_{-i}^*$ and $\boldsymbol{\Lambda}_{-i}^*$*

$$(Z_i^*, \boldsymbol{\lambda_i^*}) = \underset{\substack{\underline{Z} \leqslant Z_i \leqslant \overline{Z} \\ \mathbf{0} \leqslant \boldsymbol{\lambda}_i \leqslant \mathbf{1}}}{\operatorname{argmax}} \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z}_{-\boldsymbol{i}}^*, \boldsymbol{\lambda}_{-\boldsymbol{i}}^*)\Big] \quad \forall i \in \mathcal{N}$$

3. *The pairwise correlations are compatible among all banks. i.e. $\boldsymbol{\rho} = [\rho_{ij}]$ is symmetric and positive semi-definite.*

Part 2 of the above definition implies that banks are unrestricted in choosing their conditional dependence with their counterparties. Any bank can choose a project that is arbitrarily correlated with any other bank: a notion similar to Denti (2018). However, part 3 of the above definition states that the conditional dependence has to be mutually and jointly compatible in equilibrium.[15]

Note that part 3 implies $\lambda^*_{ij}/\lambda^*_{ji} = P(Z^*_i)/P(Z^*_j)$ for all i, j. There exists a dependence between $\boldsymbol{\lambda}$ and $\boldsymbol{Z}$ as well. That means, in equilibrium, the marginal and conditional distribution should also be compatible.

PROPOSITION 11. *In any network structure $(\bar{d}, \boldsymbol{\Theta}, N)$, the correlated risk-taking equilibrium exists and every bank's risk exposure is perfectly correlated: $\lambda^*_{ij} = 1$ for all $i, j \in \mathcal{N}$.*

Proof. In the Appendix.

Proposition 11 implies that connected banks will coordinate to expose to one single systemic risk. The intuition is as follows. In anticipation of the interbank transfers (cross-subsidy) to failed banks, each bank will endogenously align their project outcomes with other connected banks, for any chosen risk exposure. By doing so, there will be no downward distortion when the bank's project succeeds, and hence each bank will enjoy a higher expected profit. The perfect correlation, however, will be harmful to economy as a whole. Since every bank chooses to exposure to one single systemic risk, there is no co-insurance among economic agent.

Proposition 11 predicts that a financial crisis will be more likely to *endogenously* evolve in a highly connected banking system. It confirms the empirical findings of International Monetary Fund (2009) and Bisias et al. (2012) that there existed a large distress dependence among major banks before the 2008 financial crisis, when the banking system became unprecedentedly connected. It is also consistent with the observation of Acharya (2009), who argues that banks choose correlated investments due to a pecuniary externality: a failed bank reduces counterparties' profitability through an increase in the market-clearing rate for deposits.

6 Discussion and Conclusion

This paper studies banks' incentives to choose their risk exposure in financial networks, where banks are connected through cross-holdings of unsecured debts. In contrast to previous literature that focuses on the co-insurance mechanism for exogenous shocks, I show that connected banks will ex-ante choose to expose to greater risks due to the anticipation of precisely this co-insurance. The co-insurance (or "cross-subsidy") asymmetrically distort successful banks' upside profits. It hence creates a moral hazard through a risk-taking externality. In addition, connected banks coordinate to expose to correlated risks, aggravating the systemic fragility. By studying banking networks, this paper sheds new insights on the following government policies, which are particularly important in our growingly complex financial systems.

[15] For example, $\rho_{ij} = 1$ and $\rho_{ji} = 0$ is not compatible because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not symmetric. For another example, $\rho_{ij} = 1$, $\rho_{jk} = 1$, $\rho_{ik} = 0$ is not compatible because $\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$ is not positive semi-definite.

Government Bailout

In contrast to the conventional wisdom, this paper argues that a government bailout will reduce connected banks' risk-taking incentives. With a government bailout, banks are no longer concerned about the ex-post contagion and distortion. They will ex-ante choose to expose to lower risks. With that being said, a government bailout can induce other moral hazards such that excessive network formation. The degree of the optimal bailout is hence an empirical question and deserves future research.

Deposit Insurance

the deposit insurance has been argued to reduce panics and prevent banks runs (Diamond and Dybvig, 1983). However, they may have unintended consequences. For example, deposit insurance reduces depositors' incentives to monitor banks (Demirgüç-Kunt and Detragiache, 2002). This paper argues that it also reduces deposit rates' *pricing disciplining* ability: the invisible hand. I conclude that deposit insurance scheme contributes to the systemic fragility, particularly in well-connected financial systems.

Transparency

Deposit rates' *pricing disciplining* eliminates the network risk-taking externality. One assumption is that the interbank relationship needs to be observable. I hence argue that the transparency of the banking network is important for mitigating connected banks' choices of risk exposure. The transparency policy is particularly effective in regulating the shadow banking sector. The shadow liabilities (e.g. repo agreements) are not insured, so their creditors will have greater incentives to observe the interbank relationships.

Network-Adjusted Capital Regulation

I argue that an individual bank's equity buffer not only reduces its' own risk-taking but also reduces the choices of risk exposure of all connected banks in a financial system. From this multiplier effect of equity buffers, I propose that the existing risk-based capital ratio requirement should also take into account the complexity of interbank connections. Specifically, I propose that banks in highly connected networks and banks with more counterparties need to hold more liquid tier 1 capital.

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APPENDIX: PROOFS

PROOF OF LEMMA 2: By assumption $\underline{Z} \geqslant v + \bar{d}$, a bank will not default whenever its project succeeds. Therefore, the total interbank payment of a successful bank is $d_i = \bar{d}$, independent of its choice of risk exposure Z_i. For a failed bank, its cash flow that will contribute to the interbank payment system is $e_i = 0$, also independent of its choice of risk exposure.

After reordering by $\boldsymbol{\omega}$, equation 2 becomes:

$$\begin{aligned} d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) &= \bar{d} && \forall \omega_i = s \\ d_i^*(\boldsymbol{\omega};\boldsymbol{Z}) &= \left\{ \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega};\boldsymbol{Z}) - v \right\}^+ && \forall \omega_i = f \end{aligned} \tag{11}$$

It is easy to see that the vector of risk exposure $\boldsymbol{Z}$ does not enter the system of equations. As a result, the fixed point $(d_1^*(\boldsymbol{\omega}), ...d_N^*(\boldsymbol{\omega}))$ is constant in $\boldsymbol{Z}$. □

Before proving proposition 1, it is useful to have the following auxiliary lemma.

PROPOSITION 1.A *the payment vector $\boldsymbol{d}^*$ is weakly increasing in any bank's cash flow $\tilde{e}_j$. In particular, $\boldsymbol{d}^*(\boldsymbol{\omega})$ is higher when any bank's project succeeds ($\omega_j = s$) compared with when it fails ($\omega_j = f$)..*[16]

Proof. The above lemma is identical to Eisenberg and Noe (2000) Lemma 5. The payment equilibrium (equation 2) is a fixed point of the form $\boldsymbol{d}^* = \Phi(\boldsymbol{d}^*; \tilde{e}_j)$. Since both min and max operator preserve monotonicity, Φ is increasing in $\tilde{e}_j$. By monotone selection theorem (Milgrom and Roberts, 1990 Theorem 1), the fixed point $\boldsymbol{d}^*$ is increasing in $\tilde{e}_j$.

□

PROOF OF LEMMA 1: The FOC and SOC for equation 6 over Z_i:

$$\begin{aligned} F(Z_i;\boldsymbol{Z_{-i}}) &= P'(Z_i)(Z_i - v) + P(Z_i) - P(Z_i)'\mathcal{D}(\boldsymbol{Z_{-i}}) = 0 \\ S(Z_i;\boldsymbol{Z_{-i}}) &= P''(Z_i)(Z_i - v) + 2P'(Z_i) - P(Z_i)''\mathcal{D}(\boldsymbol{Z_{-i}}) < 0 \end{aligned}$$

From assumption 1, it is easy to easy $S(Z_i;\boldsymbol{Z_{-i}}) < 0$. After taking the total derivative on FOC, we have

$$\frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}\mathcal{D}(\boldsymbol{Z_{-i}})} = -\frac{\partial F(\widehat{Z}_i;\boldsymbol{Z_{-i}})/\partial \mathcal{D}(\boldsymbol{Z_{-i}})}{\partial F(\widehat{Z}_i;\boldsymbol{Z_{-i}})/\partial Z_i} = \frac{P'(\widehat{Z}_i)}{S(\widehat{Z}_i;\boldsymbol{Z_{-i}})} > 0 \tag{12}$$

The above inequality implies that whatever increases bank i's risk-taking distortion $\mathcal{D}(\boldsymbol{Z_{-i}})$ will increase its choice of risk exposure $\widehat{Z}_i$. In particular, we want to see the effect from bank m's risk

[16]Throughout this paper, whenever the ordering of a vector is mentioned, it refers to a pointwise ordering. i.e $\boldsymbol{x} \geqslant \boldsymbol{y} \Leftrightarrow x_i \geqslant y_i$ for all i. For the following text, all orderings are weakly.

exposure Z_m. To do so, we vary bank m's risk exposure from Z_m to Z'_m with $Z'_m > Z_m$. Denote $\boldsymbol{Z'_{-i}}$ the new risk-exposure vector that differs from $\boldsymbol{Z_{-i}}$ only in Z_m. Then,

$$\begin{aligned}
&\mathcal{D}(\boldsymbol{Z'_{-i}}) - \mathcal{D}(\boldsymbol{Z_{-i}}) \\
= &\sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Pr(Z'_m) \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}}) \Big) + \Big(1 - \Pr(Z'_m) \Big) \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}}) \Big) \Big] \\
&- \sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Pr(Z_m) \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}}) \Big) + \Big(1 - \Pr(Z_m) \Big) \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}}) \Big) \Big] \\
= &\sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Big(\Pr(Z'_m) - \Pr(Z_m) \Big) \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=f}}) - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega^{m=s}}) \Big) \Big] \geqslant 0
\end{aligned}$$

where $\boldsymbol{\omega_{-i-m}}$ denotes the state of nature vector for all banks except bank i and m, where $\boldsymbol{\omega^{m=s}}$ denotes the state of nature vector that appends $\boldsymbol{\omega_{-i-m}}$ with $\omega_m = s$ and $\omega_i = s$, and where $\boldsymbol{\omega^{m=f}}$ denotes the state of nature vector that appends $\boldsymbol{\omega_{-i-m}}$ with $\omega_m = f$ and $\omega_i = s$. The last inequality is by Lemma 1.A. Therefore the network risk-taking distortion will be larger if any bank $m \neq i$ exposes to greater risks. From equation 12, we have

$$\frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}Z_{-i}} = \frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}\mathcal{D}(\boldsymbol{Z_{-i}})} \frac{\mathrm{d}\mathcal{D}(\boldsymbol{Z_{-i}})}{\mathrm{d}Z_{-i}} > 0 \quad \forall i \quad \text{and} \quad -i$$

□

PROOF OF PROPOSITION 2: By Lemma 2, $d_i^*(\boldsymbol{\omega}; \boldsymbol{Z})$ is constant in $\boldsymbol{Z}$. The payment equilibrium in any state of nature is the fixed point to a system of equations (equation 11). Denote the fixed point as $\boldsymbol{d}^* = \Phi(\boldsymbol{d}^*)$, where Φ a continuous mapping with a convex and compact domain $[0, \bar{d}]^N$. By Brouwer fixed point theorem, the payment $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ exists for all $\boldsymbol{\omega}$ and $\boldsymbol{Z}$.

By Lemma 1, $\mathrm{d}\widehat{Z}_i/Z_{-i} \geqslant 0$ for all i and $-i$. It implies the Nash equilibrium is a supermodular game. The domain for the risk-exposure vector $[\underline{Z}, \bar{Z}]^N$ is a complete lattice. By Tarski's theorem, the fixed point to the first order conditions $F(Z_i^*; \boldsymbol{Z_{-i}^*}) = 0$ exists. The equilibrium risk exposure $\boldsymbol{Z}^* = (Z_1^*, ..., Z_N^*)$ is this fixed point. □

PROOF OF CORROLARY 1: Define Z^N and Z^S as the equilibrium risk exposure of a connected and a stand-alone bank respectively. Formally, they are defined as

$$\begin{aligned}
P'(Z^N)(Z^N - v) + P(Z^N) - P(Z^N)'\mathcal{D}(\boldsymbol{Z^N}) = 0 \\
P'(Z^S)(Z^S - v) + P(Z^S) - P(Z^S)' \cdot 0 = 0
\end{aligned}$$

By equation 12, $\mathrm{d}Z^N/\mathrm{d}\mathcal{D}(\boldsymbol{Z^N}) > 0$. Also, we have $\mathcal{D}(\boldsymbol{Z^N}) > 0$ because $P(Z) > 0$ for all Z. Therefore, $Z^N > Z^S$. □

PROOF OF LEMMA 3: For any state of nature $\boldsymbol{\omega}$, conjecture that there exists two vectors, $\boldsymbol{a}(\boldsymbol{\omega})$

and $\boldsymbol{b}(\boldsymbol{\omega})$, such that $d_i^*(\boldsymbol{\omega}) = \{a_i(\boldsymbol{\omega})\bar{d} - b_i(\boldsymbol{\omega})v\}^+$. By definition, they satisfy equation 11. After plugging $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ into equation 11, we have $(a_i, b_i) = (1, 0)\ \forall \omega_i = s$, and

$$\begin{aligned} d_i^*(\boldsymbol{\omega}) &= \left\{ \sum_{\omega_j = s} \theta_{ij}\bar{d} + \sum_{j \in \mathcal{F}_{\omega}^+} \theta_{ij}\Big(a_j(\boldsymbol{\omega})\bar{d} - b_j(\boldsymbol{\omega})v\Big) - v \right\}^+ \\ &= \left\{ \Big(\sum_{j \in \mathcal{F}_{\omega}^+} \theta_{ij} a_j(\boldsymbol{\omega}) + \sum_{\omega_j = s} \theta_{ij} \Big)\bar{d} - \Big(\sum_{j \in \mathcal{F}_{\omega}^+} \theta_{ij} b_j(\boldsymbol{\omega}) + 1 \Big) v \right\}^+ \qquad \forall \omega_i = f \end{aligned}$$

where $\mathcal{F}_{\boldsymbol{\omega}}^+ \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v \geqslant 0\}$. We call it "in-the-money" failed banks. Similarly, define $\mathcal{F}_{\boldsymbol{\omega}}^- \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v < 0\}$ as the "out-of-money" failed banks, and $\mathcal{S}_{\boldsymbol{\omega}} \equiv \{i : \omega_i = s)$ as successful banks.

Per the conjecture, we need $\forall \omega_i \in f$

$$a_i(\boldsymbol{\omega}) = \sum_{j \in \mathcal{F}_{\omega}^+} \theta_{ij} a_j(\boldsymbol{\omega}) + \sum_{\omega_j = s} \theta_{ij} \tag{13}$$

$$b_i(\boldsymbol{\omega}) = \sum_{j \in \mathcal{F}_{\omega}^+} \theta_{ij} b_j(\boldsymbol{\omega}) + 1 \tag{14}$$

Since the RHS of above equations are increasing in $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ respectively, the fixed points exist by Tarski's theorem. The conjecture is hence verified. Since equation 13 and 14 hold for all failed banks, it is true for banks in $\mathcal{F}_{\boldsymbol{\omega}}^+$. We rewrite the above equations in a matrix form for banks in $\mathcal{F}_{\boldsymbol{\omega}}^+$.

$$\boldsymbol{a}_+(\boldsymbol{\omega}) = \boldsymbol{\Theta}_{++}\boldsymbol{a}_+(\boldsymbol{\omega}) + \boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s \tag{15}$$

$$\boldsymbol{b}_+(\boldsymbol{\omega}) = \boldsymbol{\Theta}_{++}\boldsymbol{b}_+(\boldsymbol{\omega}) + \mathbb{1}_+ \tag{16}$$

where $\boldsymbol{a}_+(\boldsymbol{\omega})$ and $\boldsymbol{b}_+(\boldsymbol{\omega})$ are truncated vectors of $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ only with rows that belong to $\mathcal{F}_{\boldsymbol{\omega}}^+$. Similarly, $\boldsymbol{\Theta}_{++}$ is a truncated matrix of $\boldsymbol{\Theta}$ with rows and columns that belong to $\mathcal{F}_{\boldsymbol{\omega}}^+$, and $\boldsymbol{\Theta}_{+\boldsymbol{s}}$ is the truncated matrix of $\boldsymbol{\Theta}$ where each row belongs to $\mathcal{F}_{\boldsymbol{\omega}}^+$ and each column belongs to $\mathcal{S}$. $\mathbb{1}_+$ and $\mathbb{1}_s$ are column vectors of ones with appropriate dimension. Note that $\boldsymbol{\Theta}_{++}$, $\boldsymbol{\Theta}_{+\boldsymbol{s}}$, $\mathbb{1}_+$, and $\mathbb{1}_s$ are all state-contingent. To conserve space, I suppress their underscript $\boldsymbol{\omega}$.

By the Markovian property of $\boldsymbol{\Theta}$ (row-sum equals to one), we have $\boldsymbol{\Theta}_{++}\mathbb{1}_+ + \boldsymbol{\Theta}_{+-}\mathbb{1}_- + \boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s = \mathbb{1}_+$. By equation 15

$$\boldsymbol{a}_+(\boldsymbol{\omega}) = (\boldsymbol{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s < \mathbb{1}_+ \tag{17}$$

After plugging $(\boldsymbol{a}_+, \boldsymbol{b}_+)$ into the network risk-taking distortion, We can rewrite $\mathcal{D}(\boldsymbol{Z}_{-\boldsymbol{i}})$ in a matrix form as

$$\mathcal{D}(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\bar{d} - \Big(\Theta_{is} \mathbb{1}_s \bar{d} + \Theta_{i+} (\boldsymbol{a}_+ \bar{d} - \boldsymbol{b}_+ v) + \Theta_{i-} \cdot 0 \Big) \Big]$$
$$= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{i+} \Big((\mathbb{1}_+ - \boldsymbol{a}_+) \bar{d} + \boldsymbol{b}_+ v \Big) + \boldsymbol{\Theta}_{i-} \mathbb{1}_- \bar{d} + \boldsymbol{\Theta}_{is} \mathbb{1}_s \cdot 0 \Big]$$

Each part of the above definition has a clean interpretation: $\boldsymbol{\Theta}_{i+}[(\mathbb{1}_+ - \boldsymbol{a}_+)\bar{d} + \boldsymbol{b}_+ v]$ is bank i's subsidy to "in-the-money" failed banks , $\boldsymbol{\Theta}_{i-}\mathbb{1}_-\bar{d}$ is bank i's subsidy to "out-of-money" failed banks, and $\boldsymbol{\Theta}_{is}\mathbb{1}_s \cdot 0$ is bank i's subsidy to successful banks.

To prove lemma 3, compare three financial networks with same $\boldsymbol{\Theta}$ and N but different interbank liabilities $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$, with $\bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 = \varepsilon$. To prove the monotonicity and concavity, it suffice to prove $\mathcal{D}^3(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^2(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^1(\mathbf{Z}_{-i})$ and $\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i})$ with inequality happens somewhere.

Observe that $\mathcal{F}^+_{\omega} \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v \geqslant 0\}$ is a function of $\bar{d}$. We hence denote $\mathcal{F}^+_1(\boldsymbol{\omega})$, $\mathcal{F}^+_2(\boldsymbol{\omega})$, and $\mathcal{F}^+_3(\boldsymbol{\omega})$ the set of "in-the-money" failed bank in state $\boldsymbol{\omega}$ for network $(\bar{d}_1, \boldsymbol{\Theta}, N)$, $(\bar{d}_2, \boldsymbol{\Theta}, N)$, and $(\bar{d}_3, \boldsymbol{\Theta}, N)$ respectively. By monotone selection theorem (similar to lemma 1.A), $d^{3*}_i(\boldsymbol{\omega}) \geqslant d^{2*}_i(\boldsymbol{\omega}) \geqslant d^{1*}_i(\boldsymbol{\omega}), \forall i \in \mathcal{N}$ and $\boldsymbol{\omega} \in \Omega$. That implies $\mathcal{F}^+_1(\boldsymbol{\omega}) \subseteq \mathcal{F}^+_2(\boldsymbol{\omega}) \subseteq \mathcal{F}^+_3(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega$.

Let's consider the following four cases: (1) $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (2) $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (3) $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (4) $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

Case I: $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$

From equation 15 and 16, it's easy to see that $\boldsymbol{a}^{\mathbf{1}}_+ = \boldsymbol{a}^{\mathbf{2}}_+ = \boldsymbol{a}^{\mathbf{3}}_+$ and $\boldsymbol{b}^{\mathbf{1}}_+ = \boldsymbol{b}^{\mathbf{2}}_+ = \boldsymbol{b}^{\mathbf{3}}_+$. We also have $\boldsymbol{\Theta}_{\boldsymbol{i}+}$, $\mathbb{1}_+$, $\boldsymbol{\Theta}_{\boldsymbol{i}-}$, and $\mathbb{1}_-$ unchanged for the three networks. Therefore,

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{\boldsymbol{i}+} (\mathbb{1}_+ - \boldsymbol{a}_+)(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}_{\boldsymbol{i}-} \mathbb{1}_- (\bar{d}_3 - \bar{d}_2) \Big] > 0$$
$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{\boldsymbol{i}+} (\mathbb{1}_+ - \boldsymbol{a}_+)(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}_{\boldsymbol{i}-} \mathbb{1}_- (\bar{d}_2 - \bar{d}_1) \Big] > 0$$

The last inequality is due to equation 17. With $\bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 = \varepsilon$, we have $\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) = \mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) > 0$

Case II: $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

We first compare the interbank liabilities $\bar{d}_2$ with $\bar{d}_1$. In some state of nature $\boldsymbol{\omega}$, some otherwise "out-of-money" failed banks for $(\bar{d}_1, \boldsymbol{\Theta}, N)$ become "in-the-money" for $(\bar{d}_2, \boldsymbol{\Theta}, N)$. Denote those banks $t_1, t_2, ..., t_T$, where $T \geqslant 1$. Due to continuity of the payment equilibrium in terms of $\bar{d}$

(equation 2). There exists $\bar{d}_1 < \tilde{d}_1 < \tilde{d}_2 ... < ... < \tilde{d}_S < \bar{d}_2$ (where $1 \leqslant S \leqslant T$), such that when the interbank liabilities $\bar{d} = \tilde{d}_s$, some bank t_t is exactly "in-the-money", or $\tilde{a}_t(\boldsymbol{\omega})\tilde{d}_s - \tilde{b}_t(\boldsymbol{\omega})v = 0$. In other words, this margin bank t is "in-the-money" when $\bar{d} \in [\tilde{d}_s, \tilde{d}_{s+1})$ and "out-of-money" when $\bar{d} \in (\tilde{d}_{s-1}, \tilde{d}_s]$ respectively. Denote $\widetilde{\mathcal{D}}^s(\mathbf{Z}_{-i})$ the network risk-taking distortion at those cut-offs $\tilde{d}_s$. We have

$$
\begin{aligned}
\mathcal{D}^2(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}^S(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{2}}_+)(\bar{d}_2 - \tilde{d}_S) + \boldsymbol{\Theta}^{\mathbf{2}}_{i-}\mathbb{1}_-(\bar{d}_2 - \tilde{d}_S)\Big] > 0 \\
\widetilde{\mathcal{D}}^{s+1}(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}^s(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}^{\boldsymbol{s}}_+)(\tilde{d}_{s+1} - \tilde{d}_s) + \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i-}\mathbb{1}_-(\tilde{d}_{s+1} - \tilde{d}_s)\Big] > 0 \\
\widetilde{\mathcal{D}}^1(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{1}}_{i+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{1}}_+)(\tilde{d}_1 - \bar{d}_1) + \boldsymbol{\Theta}^{\mathbf{1}}_{i-}\mathbb{1}_-(\tilde{d}_1 - \bar{d}_1)\Big] > 0
\end{aligned}
\tag{18}
$$

where each column of $\boldsymbol{\Theta}^{\mathbf{1}}_{i+}$ corresponds to an "in-the-money" failed bank at the state $\boldsymbol{\omega}$ in a network with $\bar{d} \in [\bar{d}_1, \tilde{d}_1]$. Each column of $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}$ corresponds to an "in-the-money" failed bank at the state $\boldsymbol{\omega}$ in a network with $\bar{d} \in [\tilde{d}_s, \tilde{d}_{s+1}]$. Each column of $\boldsymbol{\Theta}^{\mathbf{2}}_{i+}$ corresponds to an "in-the-money" failed bank at the state $\boldsymbol{\omega}$ in a network with $\bar{d} \in [\tilde{d}_S, \bar{d}_2]$. The same notation applies to $\boldsymbol{a}_+$ and $\widetilde{\boldsymbol{\Theta}}_{i-}$ as well. They are state-contingent, and to conserve space we suppress the underscript.

The above inequalities show that $\mathcal{D}^2(\mathbf{Z}_{-i}) \geqslant \widetilde{\mathcal{D}}^S(\mathbf{Z}_{-i}) \geqslant ... \geqslant \widetilde{\mathcal{D}}^2(\mathbf{Z}_{-i}) \geqslant \widetilde{\mathcal{D}}^1(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^1(\mathbf{Z}_{-i})$ and hence the monotonicity result follows. To prove the concavity, we observe that $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}\mathbb{1}_+ + \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i-}\mathbb{1}_- = \boldsymbol{\Theta}_{if}\mathbb{1}_f$ for all s and $\boldsymbol{\omega}$. By definition, $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}$ and $\tilde{\boldsymbol{a}}^{\boldsymbol{s}}_+$ are sub-matrix of $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s+1}}_{i+}$ and $\tilde{\boldsymbol{a}}^{\boldsymbol{s+1}}_+$ respectively. Hence we have

$$
\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}^{\boldsymbol{s}}_+) + \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i-}\mathbb{1}_- > \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s+1}}_{i+}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}^{\boldsymbol{s+1}}_+) + \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s+1}}_{i-}\mathbb{1}_- \qquad \forall s = 1, ..., S-1
$$

After summing every difference in equation 18 and replacing all of RHS with the first line, i.e. the smallest, we have

$$
\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) > \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{2}}_+)(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}^{\mathbf{2}}_{i-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]
$$

Since $\mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$, we have the following identity as in case I,

$$
\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{2}}_+)(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}^{\mathbf{2}}_{i-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]
$$

Hence $\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i})$ and the concavity follows.

Case III: $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is identical to case II with a slight twist. Instead of replacing all RHS of equation

18 with the first line, we replace it with the last line. Hence, we obtain,

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}^{\mathbf{2}}_{i+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{2}}_+)(\bar{d}_3 - \bar{d}_2) + \mathbf{\Theta}^{\mathbf{2}}_{i-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$
$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) = \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}^{\mathbf{2}}_{i+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{2}}_+)(\bar{d}_2 - \bar{d}_1) + \mathbf{\Theta}^{\mathbf{2}}_{i-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

The monotonicity and concavity result follows.

Case IV: $\mathcal{F}^+_1(\omega) \subset \mathcal{F}^+_2(\omega) \subset \mathcal{F}^+_3(\omega)$ for some ω.

The proof is a combination of case 2 and case 3:

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}^{\mathbf{2}}_{i+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{2}}_+)(\bar{d}_3 - \bar{d}_2) + \mathbf{\Theta}^{\mathbf{2}}_{i-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$
$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) > \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}^{\mathbf{2}}_{i+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{2}}_+)(\bar{d}_2 - \bar{d}_1) + \mathbf{\Theta}^{\mathbf{2}}_{i-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

The monotonicity and concavity result follows.

Because $\mathcal{F}^+_1(\omega) \subseteq \mathcal{F}^+_2(\omega) \subseteq \mathcal{F}^+_3(\omega)$ for all $\omega \in \Omega$, Case I-IV (or some combination of them) exhaust all the possibilities. □

PROOF OF PROPOSITION 3: By lemma 1, the Nash Equilibrium for risk exposure $\boldsymbol{Z}^*$ is a supermodular game. By lemma 3, bank's expected profit exhibits an increasing difference in Z_i and $\bar{d}$. Then the Pareto-dominant equilibrium risk exposure is increasing in $\bar{d}$ (Milgrom and Roberts 1990, theorem 6). □

PROOF OF COROLLARY 2: In network $(\bar{d}_1, \mathbf{\Theta}, N)$ and in some state of nature $\tilde{\omega}$, all failed banks are "in-the-money"; that means $\mathbf{\Theta}_{++}\mathbb{1}_+ + \mathbf{\Theta}_{+s}\mathbb{1}_s = \mathbb{1}_+$. As a result, equation 17 becomes $\boldsymbol{a}_+(\tilde{\omega}) = (\boldsymbol{I}_+ - \mathbf{\Theta}_{++})^{-1}\mathbf{\Theta}_{+s}\mathbb{1}_s = \mathbb{1}_+$. That implies $\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) = 0$ for all $\bar{d}_2 > \bar{d}_1$. In other words, $\mathcal{D}(\mathbf{Z}_{-i}; \bar{d})$ reaches the maximum when every failed banks are 'in-the-money" in all possible states of nature. Suppose this is the case, then we can rewrite failed banks' equilibrium payment (equation 11) as

$$\boldsymbol{d}^*_f(\omega) = \mathbf{\Theta}_{ff}\boldsymbol{d}^*_f(\omega) + \mathbf{\Theta}_{fs}\mathbb{1}_s\bar{d} - \mathbb{1}_f v \qquad \forall\omega$$

It implies

$$\boldsymbol{d}^*_f(\omega) = (\mathbf{I}_f - \mathbf{\Theta}_{ff})^{-1}(\mathbf{\Theta}_{fs}\mathbb{1}_s\bar{d} - \mathbb{1}_f v) = \mathbb{1}_f\bar{d} - (\mathbf{I}_f - \mathbf{\Theta}_{ff})^{-1}\mathbb{1}_f v \qquad \forall\omega$$

The interbank payments received by the successful banks are

$$\boldsymbol{\Theta}_{\boldsymbol{sf}}\boldsymbol{d}_{\boldsymbol{f}}^{*}(\boldsymbol{\omega})+\boldsymbol{\Theta}_{\boldsymbol{ss}}\mathbb{1}_{s}\bar{d}=\mathbb{1}_{s}\bar{d}-\boldsymbol{\Theta}_{\boldsymbol{sf}}(\mathbf{I}_{f}-\boldsymbol{\Theta}_{ff})^{-1}\mathbb{1}_{f}v \qquad \forall\boldsymbol{\omega}$$

That means successful banks' network distortion vector in that particular state is $\vec{\mathcal{D}}(\boldsymbol{\omega})=\boldsymbol{\Theta}_{\boldsymbol{sf}}(\mathbf{I}_{f}-\boldsymbol{\Theta}_{ff})^{-1}\mathbb{1}_{f}v$. By the network symmetry, the expected distortion conditional on the set $\boldsymbol{f}$ fails will be the ratio of column sum of $\vec{\mathcal{D}}(\boldsymbol{\omega})$ and the number of columns. That is

$$\mathbb{E}[\mathcal{D}^{max}(\boldsymbol{\omega})|\text{set } f \text{ fails}]=\frac{\mathbb{1}_{s}'\boldsymbol{\Theta}_{sf}(\mathbf{I}_{f}-\boldsymbol{\Theta}_{ff})^{-1}\mathbb{1}_{f}v}{\mathbb{1}_{s}'\mathbb{1}_{s}}=\frac{\mathbb{1}_{f}'\mathbb{1}_{f}}{\mathbb{1}_{s}'\mathbb{1}_{s}}v$$

Then a bank's unconditional expected network distortion is $\sum_{\boldsymbol{f}}\frac{\mathbb{1}_{f}'\mathbb{1}_{f}}{\mathbb{1}_{s}'\mathbb{1}_{s}}v\cdot\Pr(\mathcal{F}=\boldsymbol{f})$. Again due to the symmetry, the permutation among the failed banks is irrelevant. Therefore, the maximum network risk-taking distortion is

$$\mathcal{D}^{max}(\boldsymbol{Z}_{-\boldsymbol{i}})=\sum_{f=1}^{N-1}\frac{f}{N-f}\cdot v\cdot\binom{N-1}{f}\Big[P(Z_{-i})\Big]^{N-1-f}\Big[1-P(Z_{-i})\Big]^{f}$$

It is worth noting that $\mathcal{D}^{max}(\boldsymbol{Z}_{-\boldsymbol{i}})$ is independent of the network topology, $\boldsymbol{\Theta}$. □

PROPOSITION 4: Let's separately analyze the two types of networks.

I, Complete Network

The complete network is fully symmetric: for failed banks, they are either altogether "in-the-money" or "out-of-money". That means we have either $\mathcal{F}^{+}(\boldsymbol{\omega})=\mathcal{F}(\boldsymbol{\omega})$ or $\mathcal{F}^{+}(\boldsymbol{\omega})=\varnothing$. Let's solve the payment equilibrium (equation 13 and 14) in those two types of states of nature.

1. For $\boldsymbol{\omega}$ where $\mathcal{F}^{+}(\boldsymbol{\omega})=\mathcal{F}(\boldsymbol{\omega})$, $a_i(\boldsymbol{\omega})=1$ and $b_i(\boldsymbol{\omega})=1/(1-\sum_{j\in\mathcal{F}_{\omega}}\theta_{ij})$ for all $\omega_i=f$.
2. For $\boldsymbol{\omega}$ where $\mathcal{F}^{+}(\boldsymbol{\omega})=\varnothing$, $a_i(\boldsymbol{\omega})=\sum_{\omega_j=s}\theta_{ij}$ and $b_i(\boldsymbol{\omega})=1$ for all $\omega_i=f$.

From the definition of $\mathcal{F}(\boldsymbol{\omega})^{+}\equiv\{i:\omega_j=f,a_i\bar{d}-b_iv\geqslant 0\}$, we know $\mathcal{F}(\boldsymbol{\omega})^{+}=\mathcal{F}(\boldsymbol{\omega})$ if and only if $\sum_{\omega_j=s}\theta_{ij}\bar{d}\geqslant v$. In summary

$$d_i^{C}(\boldsymbol{\omega})=\begin{cases}\bar{d} & \forall \quad \omega_i=s\\ \left(\bar{d}-\frac{1}{\sum_{\omega_j\in s}\theta_{ij}}v\right)^{+} & \forall \quad \omega_i=f\end{cases}$$

where $1/\sum_{\omega_j=s}\theta_{ij}=(N-1)$ / # of successful banks. That implies conditioning on m numbers of banks fail, $d_i^C(\boldsymbol{\omega})$ is independent of $\boldsymbol{\omega}$. Hence, we can rewrite

$$
\begin{aligned}
\mathcal{D}^C(\mathbf{Z}_{-i}) &= \sum_{m=1}^{N-1} \Big(\bar{d} - \underbrace{\Big(\bar{d} - \frac{N-1}{N-m}v\Big)^+ \cdot \frac{m}{N-1}}_{\text{payment from failed banks}} - \underbrace{\bar{d} \cdot \frac{N-1-m}{N-1}}_{\substack{\text{payment from}\\ \text{successful banks}}}\Big) \cdot \Pr(m \text{ banks failed}) \\
&= \sum_{m=1}^{N-1} \min\Big(\frac{m \cdot v}{N-m}, \frac{m \cdot \bar{d}}{N-1}\Big) \cdot \Pr(m \text{ banks failed}) \qquad (19)
\end{aligned}
$$

where

$$\Pr(m \text{ banks failed}) = \binom{N-1}{m}\Big(1 - P(Z_{-i})\Big)^m \Big(P(Z_{-i})\Big)^{N-1-m}$$

II, Ring Network

For a failed bank, there are three scenarios: a) its debtor succeeded. b) its debtor failed but "in-the-money". c) its debtor failed and "out-the-money". Let's solve the payment equilibrium (equation 13 and 14) in those three types of states of nature.

1. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{S}(\boldsymbol{\omega})$, $a_i(\boldsymbol{\omega}) = 1$ and $b_i(\boldsymbol{\omega}) = 1$.
2. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{F}^+(\boldsymbol{\omega})$, $a_i(\boldsymbol{\omega}) = a_{i-1}(\boldsymbol{\omega})$ and $b_i(\boldsymbol{\omega}) = b_{i-1}(\boldsymbol{\omega}) + 1$.
3. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{F}^-(\boldsymbol{\omega})$, $a_i(\boldsymbol{\omega}) = 0$ and $b_i(\boldsymbol{\omega}) = 1$.

By induction, we have

$$d_i^R(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \Big(\bar{d} - K_i(\boldsymbol{\omega})v\Big)^+ & \forall \quad \omega_i = f \end{cases}$$

where $K_i(\boldsymbol{\omega}) \equiv \min\{o : \omega_{i-o} = s\}$ is the number of failed debtors in the chain before reaching the first successful bank. Conditioning on m number(s) of banks failed, the total interbank payments received by a successful bank i is

$$\sum_j \theta'_{ij} d_j^R(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \text{w.p.} \quad \binom{N-2}{N-2-m}\Big/\binom{N-1}{m} \\ \Big(\bar{d} - v\Big)^+ & \text{w.p.} \quad \binom{N-3}{N-2-m}\Big/\binom{N-1}{m} \\ \ldots\ldots & \\ \big(\bar{d} - mv\big)^+ & \text{w.p.} \quad \binom{N-2-m}{N-2-m}\Big/\binom{N-1}{m} \end{cases} \qquad (20)$$

The first line corresponds to the scenario where i's direct debtor succeeded. In this case, bank i will receive an interbank payment of $\bar{d}$. Conditioning on m number of bank failed, the probability of this scenario is $\binom{N-2}{N-2-m}\Big/\binom{N-1}{m}$. Similarly, the second line corresponds to the scenario where i's direct debtor failed but its debtor's debtor succeeded. In this case, bank i will receive an interbank payment of $(\bar{d}-v)^{+}$. The probability of this scenario is $\binom{N-3}{N-2-m}\Big/\binom{N-1}{m}$. The same logic applies till all m banks failed. It is easy to confirm by Hockey-stick identity (emma I.A) that the total probability in equation 20 is one. Finally, we have

$$
\begin{aligned}
&\sum_{m=1}^{N-1}\left[\bar{d}-\sum_{l=0}^{m}\left(\bar{d}-lv\right)^{+}\binom{N-2-l}{N-2-m}\Big/\binom{N-1}{m}\right]\cdot \Pr(m \text{ banks failed})\\
\leqslant &\sum_{m=1}^{N-1}\left[\bar{d}-\left(\sum_{l=0}^{m}\left(\bar{d}-lv\right)\binom{N-2-l}{N-2-m}\Big/\binom{N-1}{m}-\bar{d}\cdot\frac{N-1-m}{N-1}\right)^{+}-\bar{d}\cdot\frac{N-1-m}{N-1}\right]\\
&\quad\cdot \Pr(m \text{ banks failed}) \qquad\qquad\qquad\qquad\qquad\qquad (\textbf{\textit{By lemma I.B}})\\
=&\sum_{m=1}^{N-1}\left(\bar{d}-\left(\bar{d}-\frac{N-1}{N-m}v\right)^{+}\cdot\frac{m}{N-1}-\bar{d}\cdot\frac{N-1-m}{N-1}\right)\cdot \Pr(m \text{ banks failed}) \qquad (\textbf{\textit{By lemma I.A}})\\
=&\mathcal{D}^{C}(\boldsymbol{Z_{-i}})
\end{aligned}
$$

It's worth noting that $\mathcal{D}^{R}(\boldsymbol{Z_{-i}})=\mathcal{D}^{C}(\boldsymbol{Z_{-i}})=\mathcal{D}^{max}(\boldsymbol{Z_{-i}})$ if $\bar{d}-mv\geqslant 0$ for all m. A necessary and sufficient condition is $\bar{d}\geqslant(N-1)v$. It confirms Corollary 2. Finally, by monotone selection theorem, the equilibrium risk exposure of banks in a complete network is larger than that of banks in a ring network. □

PROPOSITION 5: By binomial theorem, we can rewrite equation 2 as

$$\mathcal{D}^{max}(\boldsymbol{Z_{-i}})=\frac{1-P(Z_{-i})-[1-P(Z_{-i})]^{N}}{P(Z_{-i})}\cdot v$$

It is immediate that $\mathrm{d}\mathcal{D}^{max}(\boldsymbol{Z_{-i}})/\,\mathrm{d}N>0$. By monotone selection theorem, each bank's maximum risk exposure Z_i^* is increasing in the number of banks N in the network. □

PROPOSITION 6: Denote the central clearing counterparty (CCP) as bank 0. Because the CCP has no outside liability, it's always "in-the-money". Hence, the payment equilibrium when m banks fail can be solved by

$$
\begin{aligned}
d_s^* &= \bar{d}\\
d_f^* &= (d_0^*/N-v)^{+}\\
d_0^* &= (N-m)\cdot d_s^*+m\cdot d_f^*
\end{aligned}
$$

The above fixed point system is solved as

$$d_i^{CCP}(\boldsymbol{\omega})=\begin{cases}\bar{d} & \forall \quad \omega_i=s\\ \left(\bar{d}-\frac{N}{N-m}v\right)^{+} & \forall \quad \omega_i=f\end{cases}$$

As a result, the risk-taking distortion to a successful bank is

$$\mathcal{D}^{CCP}(\mathbf{Z}_{-i}) = \sum_{m=1}^{N-1} \Big(\bar{d} - \underbrace{\Big(\bar{d} - \frac{N}{N-m} v \Big)^{+} \cdot \frac{m}{N}}_{\text{payment from failed banks}} - \underbrace{\bar{d} \cdot \frac{N-m}{N}}_{\substack{\text{payment from} \\ \text{successful banks}}} \Big) \cdot \Pr(m \text{ banks failed})$$

$$= \sum_{m=1}^{N-1} \min \Big(\frac{m \cdot v}{N-m}, \frac{m \cdot \bar{d}}{N} \Big) \cdot \Pr(m \text{ banks failed}) \tag{21}$$

Compare equation 21 with 19, it's easy to see that $\mathcal{D}^{CCP}(\mathbf{Z}_{-i}; \bar{d}) = \mathcal{D}^{C}(\mathbf{Z}_{-i}; \frac{N-1}{N} \bar{d})$. □

PROPOSITION 7 Let's separately analyze the two cases:

I, No Deposit Insurance

Bank i's depositors' the expected return is

$$\mathbb{E}\Big[\Pi_i^D(\boldsymbol{\omega}; v_\Theta^*, \mathbf{Z}_\Theta^*)\Big] = \mathbb{E}\Big\{ \min\Big[v_\Theta^*, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\mathbf{Z}_\Theta^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) \Big] \Big\}$$

$$= \underbrace{\mathbb{E}\Big\{ e_i(\mathbf{Z}_\Theta^*, \boldsymbol{\omega}) \Big\}}_{=P(Z_\Theta^*) \cdot Z_\Theta^*} + \underbrace{\mathbb{E}\Big\{ \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) \Big\}}_{=0} - \underbrace{\mathbb{E}\Big\{ \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\mathbf{Z}_\Theta^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) - v_\Theta^* \Big)^{+} \Big\}}_{=P(Z_\Theta^*)(Z_\Theta^* - v_\Theta^*) - P(Z_\Theta^*) \mathcal{D}_\Theta(Z_\Theta^*)}$$

$$= P(Z_\Theta^*) \cdot \Big(v_\Theta^* + \mathcal{D}_\Theta(Z_\Theta^*) \Big)$$

With a slight abuse of notation, the subscript Θ represents the full network structure $(\bar{d}, \boldsymbol{\Theta}, N)$. The second line follows the first line because for all $x, y \in \mathbb{R}$, $\min(x, y) = y - (y - x)^{+}$. From the symmetry assumption, $\mathbb{E}[d_j^*(\boldsymbol{\omega})] = \mathbb{E}[d_i^*(\boldsymbol{\omega})]$, $\forall i \neq j$. Hence $\mathbb{E}[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega})] = 0$. Plugging $\mathbb{E}[\Pi_i^D(\boldsymbol{\omega}; v_\Theta^*, \mathbf{Z}_\Theta^*)]$ to the equilibrium condition

$$\beta \cdot P(Z_\Theta^*) \cdot \Big(v_\Theta^* + \mathcal{D}_\Theta(Z_\Theta^*) \Big) - M = 0 \tag{22}$$

where bank's risk exposure Z_Θ^* is the result of a Nash equilibrium defined by equation 4. Explicitly,

$$P'(Z_\Theta^*) \cdot \Big(Z_\Theta^* - v_\Theta^* - \mathcal{D}_\Theta(Z_\Theta^*) \Big) + P(Z_\Theta^*) = 0 \tag{23}$$

Equation 22 and 23 jointly determine banks' equilibrium risk exposure as

$$P'(Z_\Theta^*) \cdot \Big(Z_\Theta^* - \frac{M}{\beta \cdot P(Z_\Theta^*)} \Big) + P(Z_\Theta^*) = 0 \tag{24}$$

It is easy to see that the equilibrium risk exposure $Z^*_\Theta = Z^*$ is independent of the network structure $(\bar{d}, \mathbf{\Theta}, N)$. From equation 22, the equilibrium deposit rates are determined by

$$v^*_\Theta = \frac{M}{\beta \cdot P(Z^*)} - \mathcal{D}_\Theta(Z^*)$$

From proposition 4, $\mathcal{D}_S < \mathcal{D}_R(Z) \leqslant \mathcal{D}_C(Z)$ for all Z. Finally, we have $v^{C*} \leqslant v^{R*} < v^{S*}$.

II, With Deposit Insurance

With deposit insurance, $\Pi^D_i(\boldsymbol{\omega}) = v^*_i$ for all $\boldsymbol{\omega}$. The equilibrium condition becomes:

$$\beta \cdot v^*_\Theta - M = 0$$

Or $v^*_\Theta = v^* = M/\beta$, independent of the network structure. Plugging into equation 23, we have

$$P'(Z^*_\Theta)\Big(Z^*_\Theta - \frac{M}{\beta} - \mathcal{D}_\Theta(Z^*_\Theta)\Big) + P(Z^*_\Theta) = 0$$

It's identical to the benchmark case with fixed deposit rates. Corollary 1 and proposition 4 implies $Z^{C*} \geqslant Z^{R*} > Z^{S*}$. □

PROPOSITION 8: Denote (v^*_o, Z^*_o) and (v^*_n, Z^*_n) as the equilibrium deposit rates and risk exposure when the interbank network is observable or unobservable to depositors, respectively.

Without network observability, deposit rates' equilibrium condition is

$$\beta \cdot P(\widetilde{Z}_n) \cdot v^*_n - M = 0$$

where $\widetilde{Z}_n$ is the anticipated risk exposure under depositors' belief where there is no network distortion. i.e

$$P'(\widetilde{Z}_n) \cdot (\widetilde{Z}_n - v^*_n) + P(\widetilde{Z}_n) = 0$$

Banks, however, do observe the network structure when determining their optimal risk exposure. Hence, the equilibrium (v^*_n, Z^*_n) is characterized by

$$\beta \cdot P(\widetilde{Z}_n) \cdot v^*_n - M = 0 \tag{25}$$
$$P'(\widetilde{Z}_n) \cdot (\widetilde{Z}_n - v^*_n) + P(\widetilde{Z}_n) = 0 \tag{26}$$
$$P'(Z^*) \cdot \Big(Z^*_n - v^*_n - \mathcal{D}_\Theta(Z^*_n)\Big) + P(Z^*_n) = 0 \tag{27}$$

On the other hand, when the interbank network is fully observable to depositors, the depositors' belief about banks' risk exposure is correct. The equilibrium (v_o^*, Z_o^*) is charactered by

$$\beta \cdot P(Z_o^*) \cdot \Big(v_o^* + \mathcal{D}_\Theta(Z_o^*)\Big) - M = 0 \tag{28}$$

$$P'(Z_o^*) \cdot \Big(Z_o^* - v_o^* - \mathcal{D}_\Theta(Z_o^*)\Big) + P(Z_o^*) = 0 \tag{29}$$

Equation 25-26 and 28-29 imply that $Z_o^* = \widetilde{Z}_n$. Equation 26-27 imply $Z_n^* > \widetilde{Z}_n$. Therefore $Z_n^* > Z_o^*$ $\square$

LEMMA 4 The proof is similar to that of lemma 3. In any state of nature $\boldsymbol{\omega}$, the payment vector for "in-the-money" failed banks is $\boldsymbol{d}_+^* = \boldsymbol{\Theta}_{++}\boldsymbol{d}_+^* + \boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(r - v)$, or

$$\boldsymbol{d}_+^* = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(r - v))$$

To conserve space I suppress the state $\boldsymbol{\omega}$ in $\boldsymbol{d}_+^*(\boldsymbol{\omega})$, $\boldsymbol{\Theta}_{++}(\boldsymbol{\omega})$, $\boldsymbol{\Theta}_{+\boldsymbol{s}}(\boldsymbol{\omega})$, $\mathbb{1}_s(\boldsymbol{\omega})$ and $\mathbb{1}_f(\boldsymbol{\omega})$. We can again write the risk-taking distortion in a matrix form

$$\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbb{1}_+\bar{d} - \boldsymbol{d}_+^*) + \boldsymbol{\Theta}_{\boldsymbol{i}-}\mathbb{1}_-\bar{d}\Big] \tag{30}$$

To prove the lemma 4, compare three financial systems with different sizes of equity buffers r_1, r_2, r_3 , with $\bar{r}_3 - \bar{r}_2 = \bar{r}_2 - \bar{r}_1 = \varepsilon$. Similar to the proof of lemma 3, we need to consider the following four cases.

Case I: $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$

For all $\boldsymbol{\omega}$, $\boldsymbol{d}_+^*$ is linearly increasing in r: $\boldsymbol{d}_+^{\mathbf{3}*} - \boldsymbol{d}_+^{\mathbf{2}*} = \boldsymbol{d}_+^{\mathbf{2}*} - \boldsymbol{d}_+^{\mathbf{1}*} = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\mathbb{1}_+\varepsilon > 0$. Then it is easy to see that the network risk-taking distortion is linearly decreasing in r.

$$\mathcal{D}^3(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}^2(\mathbf{Z}_{-\boldsymbol{i}}) = \mathcal{D}^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}^1(\mathbf{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\boldsymbol{d}_+^{\mathbf{1}*} - \boldsymbol{d}_+^{\mathbf{2}*})\Big] < 0$$

Case II: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

We first compare the equity r_2 with r_1. In some state of nature $\boldsymbol{\omega}$, some otherwise "out-of-money" failed banks for $(\bar{d}, \boldsymbol{\Theta}, N; r_1)$ become "in-the-money" for $(\bar{d}_2, \boldsymbol{\Theta}, N; r_2)$. Denote those banks $t_1, t_2, ..., t_T$, where $T \geqslant 1$. Due to the continuity of the payment equilibrium in terms of r (equation 8), there exists $r_1 < \tilde{r}_1 < \tilde{r}_2 ... < ... < \tilde{r}_S < r_2$ (where $1 \leqslant S \leqslant T$), such that when the equity buffer $r = \tilde{r}_s$, some banks t_t are exactly "in-the-money". As a result, those margin banks t_t are "in-the-money" when $r \in (\tilde{r}_s, \tilde{r}_{s+1})$ and "out-of-money" when $r \in (\tilde{r}_{s-1}, \tilde{r}_s)$ respectively.

Denote $\widetilde{\mathcal{D}}^s(\boldsymbol{Z}_{-i})$ the network risk-taking distortion when $r = \tilde{r}_s$. We have

$$\begin{aligned}
\mathcal{D}^2(\boldsymbol{Z}_{-i}) - \widetilde{\mathcal{D}}^S(\boldsymbol{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\tilde{\boldsymbol{d}}^{\boldsymbol{S}*}_{+} - \boldsymbol{d}^{\mathbf{2}*}_{+})\Big] \leqslant 0 \\
\widetilde{\mathcal{D}}^{s+1}(\boldsymbol{Z}_{-i}) - \widetilde{\mathcal{D}}^s(\boldsymbol{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}(\tilde{\boldsymbol{d}}^{\boldsymbol{s}*}_{+} - \tilde{\boldsymbol{d}}^{\boldsymbol{s+1}*}_{+})\Big] \leqslant 0 \qquad \forall s = 1, ..., S-1 \qquad (31) \\
\widetilde{\mathcal{D}}^1(\boldsymbol{Z}_{-i}) - \mathcal{D}^1(\boldsymbol{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{1}}_{i+}(\boldsymbol{d}^{\mathbf{1}*}_{+} - \tilde{\boldsymbol{d}}^{\mathbf{1}*}_{+})\Big] \leqslant 0
\end{aligned}$$

Summing above equations, it is easy to see that $\mathcal{D}^2(\boldsymbol{Z}_{-i}) - \widetilde{\mathcal{D}}^1(\boldsymbol{Z}_{-i}) \leqslant 0$. It then remains to prove the concavity. By construction, $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}$ is a submatrix of $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s+1}}_{i+}$. We also know $\tilde{\boldsymbol{b}}^s_+ = (\mathbf{I}^s_+ - \boldsymbol{\Theta}^{\boldsymbol{s}}_{++})^{-1}\mathbb{1}^s_+$ is a submatrix of $\tilde{\boldsymbol{b}}^{s+1}_+$. This is due to the construction that at the cutoff $r = \tilde{r}_s$, bank t can be treated either as ITM or OTM. With those two facts, we have

$$\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}(\mathbf{I}^s_+ - \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{++})^{-1}\mathbb{1}^s_+ < \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s+1}}_{i+}(\mathbf{I}^{s+1}_+ - \boldsymbol{\Theta}^{\boldsymbol{s+1}}_{++})^{-1}\mathbb{1}^{s+1}_+$$

After summing every difference in equation 31 and replacing all of RHS with the $\boldsymbol{\Theta}^2_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^2_{++})^{-1}\mathbb{1}^2_+$ (the largest), we have

$$\mathcal{D}^2(\boldsymbol{Z}_{-i}) - \mathcal{D}^1(\boldsymbol{Z}_{-i}) > \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big]$$

Since $\mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$, we have the following identity as in case I,

$$\mathcal{D}^3(\boldsymbol{Z}_{-i}) - \mathcal{D}^2(\boldsymbol{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big]$$

Hence $\mathcal{D}^3(\boldsymbol{Z}_{-i}) - \mathcal{D}^2(\boldsymbol{Z}_{-i}) < \mathcal{D}^2(\boldsymbol{Z}_{-i}) - \mathcal{D}^1(\boldsymbol{Z}_{-i})$ and the concavity follows.

Case III: $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is identical to case II with a slight twist. When comparing r_3 with r_2. Again replacing all RHS of equation 31 with $\boldsymbol{\Theta}^2_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^2_{++})^{-1}\mathbb{1}^2_+$, now the smallest. Hence, we obtain,

$$\mathcal{D}^3(\boldsymbol{Z}_{-i}) - \mathcal{D}^2(\boldsymbol{Z}_{-i}) < \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big] = \mathcal{D}^2(\boldsymbol{Z}_{-i}) - \mathcal{D}^1(\boldsymbol{Z}_{-i})$$

Case IV: $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is a combination of case 2 and case 3:

$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) > \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \mathbf{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big]$$
$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \sum_{\omega_{-i}} \Pr(\omega_{-i})\Big[\mathbf{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \mathbf{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big]$$

The monotonicity and concavity result follows.

Because $\mathcal{F}^+_1(\omega) \subseteq \mathcal{F}^+_2(\omega) \subseteq \mathcal{F}^+_3(\omega)$ for all $\omega \in \Omega$, Case I-IV (or some combination of them) exhaust all the possibilities. □

PROPOSITION 9: The first and second order conditions of maximizing bank's expected profit (equation 9) over its choice of risk exposure Z_i:

$$F(Z_i; \mathbf{Z}_{-i}, r) = P'(Z_i)(Z_i + r - v) + P(Z_i) - P(Z_i)'\mathcal{D}(\mathbf{Z}_{-i}; r)$$
$$S(Z_i; \mathbf{Z}_{-i}, r) = P''(Z_i)(Z_i + r - v) + 2P'(Z_i) - P(Z_i)''\mathcal{D}(\mathbf{Z}_{-i}; r)$$

Taking the total derivative of FOC, we have

$$\frac{\mathrm{d}Z^*_i}{\mathrm{d}r} = -\frac{\frac{\partial F}{\partial \mathcal{D}}\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}r} + \frac{\partial F}{\partial r}}{S(Z_i; \mathbf{Z}_{-i}, r)} = \frac{1}{-S}\Big[-P'(Z_i)\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}r} + P'(Z_i)\Big] < 0 \quad \forall \mathbf{Z}_{-i}$$

where $P'(Z_i) < 0$ is the direct effect of an equity buffer and $\mathrm{d}\mathcal{D}/\mathrm{d}r < 0$ is the network effect. □

PROPOSITION 10: The proof is similar to the proof of lemma 4. The payment vector for "in-the-money" failed banks is

$$\boldsymbol{d}^*_+(\omega) = \begin{cases} (\mathbf{I}_+ - \mathbf{\Theta}_{++})^{-1}(\mathbf{\Theta}_{+s}\mathbb{1}_s\bar{d} + \mathbb{1}_+(t - v)) & \text{if} \quad \#\{l|\omega_l = f\} \geqslant n \\ (\mathbf{I}_+ - \mathbf{\Theta}_{++})^{-1}(\mathbf{\Theta}_{+s}\mathbb{1}_s\bar{d} + \mathbb{1}_+(0 - v)) & \text{if} \quad \#\{l|\omega_l = f\} < n \end{cases}$$

Let $t_2 - t_1 = \varepsilon$. We again have two cases: (1) $\mathcal{F}^+_2(\omega) = \mathcal{F}^+_1(\omega)$ for all ω. (2) $\mathcal{F}^+_1(\omega) \subset \mathcal{F}^+_2(\omega)$ for some ω.

Denote the bailout event indicator $\mathbb{1}[\#\{l|\omega_l = f\} > n]$ as $\mathcal{B}(\omega)$. Since $n < N$, $\mathcal{B}(\omega) = 1$ for some ω. For case 1,

$$\boldsymbol{d}^{\mathbf{2}*}_+(\omega) - \boldsymbol{d}^{\mathbf{1}*}_+(\omega) = \mathcal{B}(\omega)(\mathbf{I}_+ - \mathbf{\Theta}_{++})^{-1}\mathbb{1}_+\varepsilon \quad \forall \omega \in \mathbf{\Omega}$$

From equation 30,

$$\begin{aligned}&\mathcal{D}^2(\boldsymbol{Z_{-i}})-\mathcal{D}^1(\boldsymbol{Z_{-i}})\\&\qquad=\sum_{\boldsymbol{\omega}_{-i}}\Pr(\boldsymbol{\omega_{-i}})\Big[\boldsymbol{\Theta_{i+}}(\boldsymbol{d^{1*}_{+}}-\boldsymbol{d^{2*}_{+}})\Big]=\sum_{\boldsymbol{\omega}_{-i}}-\mathcal{B}(\boldsymbol{\omega}^{i=s})\Pr(\boldsymbol{\omega_{-i}})\Big[\boldsymbol{\Theta_{i+}}(\mathbf{I}_{+}-\boldsymbol{\Theta}_{++})^{-1}\mathbb{1}_{+}\varepsilon\Big]<0\end{aligned}$$

The proof of case 2 is identical to case 2 of lemma 3 and 4. I omit here to avoid repetition. □

PROPOSITION 11: Note that conditional on the state of nature $\boldsymbol{\omega}$, the payment vector $\boldsymbol{d}^*$ is still independent of the risk vector $\boldsymbol{Z}$ or correlation matrix $\boldsymbol{\lambda}$. Now, Compare bank i's expected profit when it chooses between λ_{ij} and $\widetilde{\lambda}_{ij}$ with $\widetilde{\lambda}_{ij}>\lambda_{ij}$

$$\begin{aligned}&\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};Z_i,\boldsymbol{\widetilde{\lambda}_{ij}})\Big]-\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};Z_i,\boldsymbol{\lambda_i})\Big]=\\&\qquad-\sum_{\boldsymbol{\omega}_{-i-j}}\Big(\bar{d}-\sum_l\theta_{il}d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,j=s}})\Big)\cdot\Pr(\boldsymbol{\omega}_{-i-j}|\omega_i=s,\omega_j=s)\cdot P(Z_j)\cdot(\widetilde{\lambda}_{ij}-\lambda_{ij})\\&\qquad+\sum_{\boldsymbol{\omega}_{-i-j}}\Big(\bar{d}-\sum_l\theta_{il}d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,j=f}})\Big)\cdot\Pr(\boldsymbol{\omega}_{-i-j}|\omega_i=s,\omega_j=f)\cdot P(Z_j)\cdot(\widetilde{\lambda}_{ij}-\lambda_{ij})\end{aligned}$$

Suppose $\lambda^*_{j,k}=1$ for all $k\neq i$. That implies $\Pr(\boldsymbol{\omega}_{-i-j}|\omega_i=s,\omega_j=s)=1$ if and only if every element of $\boldsymbol{\omega}_{-i-j}$ is s. Similarly, $\Pr(\boldsymbol{\omega}_{-i-j}|\omega_i=s,\omega_j=f)=1$ if and only if every element of $\boldsymbol{\omega}_{-i-j}$ is f.

By Lemma 1.A in the appendix above, $\sum_l\theta_{il}d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,-i=s}})\geqslant\sum_l\theta_{il}d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,-i=f}})$. That implies bank i's expected profit is increasing in its project's dependence λ_{ij} with other banks. Therefore, for all $\boldsymbol{Z}$, bank i's choices of conditional dependence with bank j won't deviate from $\lambda^*_{i,j}=1$. With perfect correlation, the network risk-taking distortion disappears: $\mathcal{D}(\boldsymbol{Z^*_{-i}},\mathbb{1})=0$ for all $\boldsymbol{Z^*_{-i}}$. Hence, the equilibrium is characterized by

$$\begin{aligned}&\lambda^*_{ij}=1\quad\forall i,j\in\mathcal{N}\\&P'(Z^*_i)(Z^*_i-v)+P(Z^*_i)=0\quad\forall i\in\mathcal{N}\end{aligned}$$

And $\rho^*_{ij}=1$ for all i,j. □

INTERNET APPENDIX: OMITTED PROOFS

LEMMA I.A [Hockey-stick Identity]
For all $n > r$, we have

$$\text{(i)} \quad \sum_{l=r}^{n} \binom{l}{r} = \binom{n+1}{r+1} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{l=r}^{n} \binom{l}{r}(n-l) = \binom{n+1}{r+1}\frac{n-r}{r+2}$$

PROOF
We proceed by induction. For an initial $n = r + 1$

$$\text{(i)} \qquad \binom{r}{r} + \binom{r+1}{r} = \binom{r+2}{r+1}$$

$$\text{(ii)} \qquad \binom{r}{r} * 1 + \binom{r+1}{r} * 0 = \binom{r+2}{r+1} * \frac{1}{r+2} = 1$$

The above equations are to confirm the initial conditions hold. Now suppose that for $n = k$, the two equality holds. For $n = k + 1$, we have

$$\text{(i)} \qquad \sum_{l=r}^{k+1} \binom{l}{r} = \sum_{l=r}^{k} \binom{l}{r} + \binom{k+1}{r} = \binom{k+1}{r+1} + \binom{k+1}{r} = \binom{k+2}{r+1}$$

$$\text{(ii)} \qquad \sum_{l=r}^{k+1} \binom{l}{r}(k+1-l) = \sum_{l=r}^{k} \binom{l}{r}(k+1-l) = \binom{k+1}{r+1}\frac{k-r}{r+2} + \binom{k+1}{r+1} = \binom{k+2}{r+1}\frac{k+1-r}{r+2}$$

Q.E.D by induction. □

LEMMA I.B [Triangle Inequality]
For any sequence $\{A_i\}$ and $B \in \mathbb{R}$ with $B < \max_i(A_i)$, we have

$$\sum_i \Big(A_i\Big)^+ \geqslant \Big(\sum_i A_i - B\Big)^+ + B$$

PROOF Without loss of generality, let $A_0 = \max_i(A_i)$

$$\sum_i \Big(A_i\Big)^+ - B = \sum_{i \neq 0} \Big(A_i\Big)^+ + (A_0 - B)^+ \geqslant \Big(\sum_i A_i - B\Big)^+$$

□