A Simple Model for Pricing Securities with Equity, Interest-Rate, and Default Risk*

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Abstract

We develop a model for pricing derivative and hybrid securities whose value may depend on different sources of risk, namely, equity, interest-rate, and default risks. In addition to valuing such securities the framework is also useful for extracting probabilities of default (PD) functions from market data. Our model is not based on the stochastic process for the value of the firm, which is unobservable, but on the stochastic processes for interest rates and the equity price, which are observable. The model comprises a risk-neutral setting in which the joint process of interest rates and equity are modeled together with the default conditions for security payoffs. The model is embedded on a recombining lattice which makes implementation feasible with polynomial complexity. We demonstrate the simplicity of calibration of the model to market observable data. The framework is shown to nest many familiar models as special cases. The model is able to uncover not just default probabilities, but also default functions. The framework is extensible to handling correlated default risk and may be used to value distressed convertible bonds, debt-equity swaps, and credit portfolio products such as CDOs. We present numerical and calibration examples to demonstrate the applicability and implementation of our approach.

1 Introduction

Several financial securities depend on more than just one source of risk. Prominent among these are corporate bonds (which depend on interest rate risk and on credit risk of the issuing firm) and convertible bonds (which depend, in addition, on equity risk). In this paper, we offer and implement a simple, tractable model for the pricing of securities whose values may depend on one or more of three sources of risk: equity risk, credit risk, and interest-rate risk.

Our framework stitches together three standard building blocks. We begin with a term-structure model for describing riskless interest-rate movements. For specificity, we use the Heath-Jarrow-Morton [1990](henceforth, HJM) framework for this purpose, though any other standard interest-rate model could also be used in its stead. We then overlay on this structure a process for describing the evolution of equity prices. The process we choose is a generalized form of the Cox-Ross-Rubinstein [1979] (henceforth, CRR) model that allows for default of the firm issuing the equity. Finally, we tack on a hazard rate process that captures the likelihood of default at each node.

The resulting framework combines, in a single parsimonious model and accounting for correlations, the three major sources of risk. Default probabilities in our model are derived as *endogenous* functions of the information on the lattice, calibrated to default spreads. Thus, default information in the model is extracted from *both* equity- and debt-market information rather than from just debt-market information (as in reduced-form credit-risk models) or from just equity-market information (as in structural credit-risk models). In particular, default probabilities may be jointly calibrated to market prices of equity and risky debt. This allows valuation, in a single consistent framework, of hybrid debt-equity securities such as convertible bonds that are vulnerable to default, as well as of derivatives on interest rates, equity and credit. Our model can also serve as a basis for valuing credit portfolios where correlated default is an important source of risk.

Our framework has several antecedents and points of reference in the literature. It is intimately linked to the two standard approaches to credit-risk modeling: the class of reduced-form models (e.g., Duffie and Singleton [1999], Madan and Unal [2000]) and the class of structural models (Merton [1974], Black-Cox [1976], and others). As in the reduced-form approach, we represent default likelihood using a hazard-rate process. From the structural model approach, we borrow the boundary conditions on equity in the event of default, specifically, the idea that default is identified with zero equity value.

There are, however, important differences. The typical reduced-form model only considers interest-rate and default processes; thus, implementation is achieved solely using debt-market information. Our framework also incorporates the equity process, so default information may be extracted from both equity- and debt-market information. This is significant, given that equity markets are more liquid than corporate debt markets.[1]

Structural models begin with a process for the value of the firm; equity and debt are viewed as contingent claims on this value. Since the firm value process is latent, its parameters must be inferred from observed variables. Equity-market information is typically used for this purpose, but implementation in practice involves making restrictive simplifying assumptions on the debt-structure of the firm (e.g., the Moody's KMV model or the Credit Grades model). These restrictions are unnecessary in our model since we work directly with observable equity prices rather than firm values.

Our framework also extends several other models in the literature. Schönbucher [1998], [2002], and Das and Sundaram [2000] study "defaultable HJM" models which are HJM models with a default process tacked on. Our model generalizes these to also including equity processes. In particular, the Das and Sundaram [2000] results as a special case of our framework if the equity process is switched off.

Equally, our framework may also be viewed as a generalization of Amin and Bodurtha [1995] (see also Brenner, Courtadon, and Subrahmanyam [1987]). The Amin-Bodurtha model combines interest rate risk and equity risk but does not incorporate credit risk. Since there is no default, equity in their model is necessarily infinitely-lived and never gets "absorbed" in a post-default value.

Several other frameworks too are nested within our model. For example, if the equity and hazard-rate processes are switched off, we obtain the HJM model, while if the interest-rate and hazard-rate processes are switched off, we obtain the CRR model.

Some recent papers on pricing derivatives with default risk are closely related to ours. These include especially Davis and Lischka [1999] and Schönbucher [2002].[2] These models are also multivariate lattice approaches aimed at fitting market observable data. A finite-differencing approach using Fokker-Planck equations is presented in Andersen and Buffum [2002]. Other related papers in the literature include Jarrow [2001], Takahashi, Kobayashi and Nakagawa [2001], and Carayannopoulos and Kalimipalli [2001]. The latter paper, for instance, prices convertible bonds, but in a model which does not account fully for interest rate risk. All these models vary in their choice of stochastic processes and implementation of default conditions.

As in Schönbucher [2002], our model also allows for a pricing tree with absorbing default branches

[1]Incorporation of equity risk into reduced form models has also been examined in Jarrow [2001] and Mamaysky [2002]. The approach in these papers is very different from ours; it is based on deriving equity values through modeling the dividend process.

[2]There are some important operational differences between the Davis-Lischka framework and ours. One is the choice of interest-rate process: Davis-Lischka use the Hull-White model while we use a flexible HJM model. Second, the specification of the default intensity process in Davis-Lischka is restrictive; the process is perfectly correlated with the equity process. We allow the default intensity process to depend more generally on both equity returns and interest-rates as also other information.

at each node. However, our model is more parsimonious in that we do not specify an *exogenous* process for the default probability. Instead, we directly model the equity process, and make the default probability a dynamic function of both, equity and interest rates, i.e. an *endogenous* calibration of the default function. This allows us to also price derivatives indexed to equity values, which is not feasible in the exogenous form of the model. It is parsimonious, as no additional lattice dimension for default is required. Further, endogeneity imposes consistency of default risk with equity and interest rate risk.

Our lattice design allows recombination, making the implementation of the model simple and efficient; indeed, the model is fully implementable on a spreadsheet. Unlike many earlier models, we are able to *(a)* price derivatives on equity and interest rates with default risk, *(b)* extract probabilities of default endogenously in the model, *(c)* provide for the risk-neutral simulation of correlated default risk in a manner consistent with no arbitrage and consistent with equity correlations (which we believe, has not been undertaken in any model so far).

The rest of the paper proceeds as follows. In Section 2 we develop the pricing lattice in the state variables of the model in a manner that allows for additional structure to accommodate default risk. Section 3 provides the specifics of default modeling on the pricing lattice, as well as shows how default swaps are used to calibrate the model for subsequent use. Section 4 deals with implementation issues and examples. We show that the model may generate a wide range of spread curve shapes. Empirical calibration to markets is undertaken to evidence the ease of implementation. This section also explores the impact of default risk on embedded options within classic bond structures. Finally, an analysis of the model application to correlated default products is provided. Section 5 concludes by summarizing the economic and technical benefits of the model.

2 Modeling the Lattice

Representing the stochastic processes on a lattice permits valuation by dynamic programming via backward recursion. We employ a parsimonious model that can be embedded on a bivariate lattice, on which we model the joint risk-neutral evolution of defaultable equity prices and the forward interest rate curve.

2.1 A simple motivating example

Consider a simplified version of our model in which interest rates are not stochastic. Suppose equity prices evolve in continuous time according to a geometric Brownian motion but with the added twist that the equity price could suddenly jump to zero. Finally, suppose that the jump to default is governed by a constant intensity Poission process with hazard rate λ.

The prices of options on equity in this world may be determined using Merton's [1976] jump-diffusion option model. The prices of calls are analogous to a model where the firm may default, with a corresponding zero recovery rate for equity. Samuelson [1972] had provided the solution to this problem:

$$\begin{aligned} \text{Call on defaultable equity} &= \exp(-\xi T)\ BMS[S_0 e^{\xi T}, K, T, \sigma, r] \\ &= BMS[S_0, K, T, \sigma, r+\xi] \end{aligned}$$

here $BMS[.]$ is the standard Black-Merton-Scholes (BMS) option pricing model, with an initial stock price S_0, interest rate r, stock volatility σ, maturity T, and exercise price K. ξ is the default intensity, or the instantaneous rate of default. Notice that the price of the call is exactly priced by the BMS model with an adjusted risk-neutral interest rate $(r+\xi)$.

Since the defaultable call option, like the value of equity, has a zero recovery rate, it is tempting to intuit that the price of a defaultable call option should be the price of a non-defaultable call option, i.e. $BMS[S_0, K, T, \sigma, r]$, multiplied by the risk-neutral probability of survival, i.e. $\exp(-\xi T)$. However, this simple intuition would be wrong (notice this from a simple comparison with Samuelson's formula above). Not only does the discount rate need to be adjusted for the probability of default, but the drift of the risk-neutral equity process is also impacted by the jump to default compensator (ξ).[3] Therefore, special care should be taken to ensure that the correct risk-neutral processes are used for pricing defaultable securities.

This intuition may be further clarified in a discrete-time setting. Defaultable equity may be represented by the following tree, which embodies a single period of length h, wherein the stock price moves from $S(t)$ to a stochastic value $S(t+h)$. When jump to default is allowed for, the value of $S(t+h)$ is assumed to take one of three values:

$$S(t+h) = \begin{cases} uS(t) & \text{w/prob} \quad q\ \exp(-\xi h) \\ dS(t) & \text{w/prob} \quad (1-q)\ \exp(-\xi h) \\ 0 & \text{w/prob} \quad 1-\exp(-\xi h) \end{cases}$$

Here u, d are the respective "upshift" and "downshift" parameters for the changes in the stock price over interval h. Given the constant risk-neutral default intensity ξ, the probability of survival in the interval h is $\exp(-\xi h)$. Since this risk-neutral setting requires that the normalized stock price is a

[3] For an excellent exposition of default jump compensators, see Giesecke [2001].

martingale, it is easy to solve for the value of the risk-neutral probability q. Hence,

$$\exp(rh) = u\, q\, \exp(-\xi h) + d\, (1-q)\, \exp(-\xi h),$$

implying that

$$q = \frac{\exp(rh) - d\exp(-\xi h)}{u\exp(-\xi h) - d\exp(-\xi h)} = \frac{\exp[(r+\xi)h] - d}{u-d}$$

For illustrative purposes, we set $u = \exp(\sigma h)$ and $d = 1/u$, to mimic the Cox, Ross and Rubinstein (CRR) model. Suppose $r = 0.10$, $\sigma = 0.20$, $\xi = 0.01$, and $h = 0.25$. Then, the risk-neutral probability $q = 0.766203$. If there were no default risk, i.e. $\xi = 0$, then $q = 0.740548$. Hence, notice that the drift upwards tends to occur with greater probability in the presence of default, corresponding to the fact that in the risk-neutral setting, the jump to default is compensated.

In the ensuing sections, we generalize this model to apply to the case with stochastic interest rates, and stochastic default processes, so that ξ is no longer constant, nor uncorrelated with equity prices and interest rates.

2.2 Equity Model

The model for the evolution of equity prices is based on the branching process outlined in the previous section, with σ_s as the parameter governing the volatility of the equity process. Under this specification, the trinomial movement in the stock price at time t over the next period is set to be $S(t+h) = S(t)\exp[\sigma_s X_s(t)]$. $X_s(t)$ is a random variable, taking values in the set $\{+1, -1, -\infty\}$.

A probability measure is chosen such that the expected return on equity in each period, is set to $r(t)h$ and the variance of the return is $\sigma_s^2 h$. Allowing $X_s(t) = -\infty$ embeds default risk in the model, of the sort envisaged in the Duffie-Singleton [1999] model. The firm suddenly defaults in which case its stock price goes to zero, when $X_s(t) \rightarrow -\infty$. Setting the expected return to $r(t)h$ is equivalent to normalizing the equity prices by a money market account numeraire, and ensuring that the normalized prices are martingales. Since the same numeraire is also used in the case of bonds, we generate a lattice that is arbitrage-free in bond and equity markets. The choice of probability measure to satisfy the martingale requirements in the model will be taken up shortly in a subsequent section.

Representing the stochastic processes on a lattice permits valuation by dynamic programming via backward recursion. We employ a parsimonious model that can be embedded on a bivariate lattice, on which we model the joint risk-neutral evolution of equity prices and the forward interest rate curve. Our model accommodates the correlation between interest rates and equity prices and we show that our approximation converges to an exact bivariate process as the time interval, represented by h, shrinks to zero on the lattice. More importantly, we show how to embed credit risk in the model.

2.3 Term-Structure Model

Our lattice adopts the discrete-time, recombining form of the Heath-Jarrow-Morton (HJM) [1990] model, which it defaults to if there is no equity component in the derivative security being priced. We quickly review this here, before moving on to the description of the joint lattice, and readers may examine the original HJM paper for comprehensive details. Initially, we prepare the univariate HJM lattice for the evolution of the term structure, and subsequently stitch on an equity process.

The model is based on a time interval $[0, T^*]$. Periods are of fixed length $h > 0$; thus, a typical time-point t has the form kh for some integer k. At all times t, zero-coupon bonds of all maturities are available. Assuming no arbitrage, there exists an equivalent martingale measure Q for all assets. For any given pair of time-points (t, T) with $0 \leq t \leq T \leq T^* - h$, $f(t, T)$ denotes the forward rate on the default-free bonds applicable to the period $(T, T + h)$. The short rate is $f(t, t) = r(t)$. Forward rates follow the stochastic process:

$$f(t + h, T) \;=\; f(t, T) + \alpha(t, T)h + \sigma(t, T)X_f\sqrt{h}, \tag{1}$$

where α is the drift of the process and σ the volatility; and X_f is a random variable taking values in the set $\{-1, +1\}$. Both α and σ are taken to be only functions of time, and not other state variables. This is done to preserve the computational tractability of the model. Relaxing this assumption will make the model non-recombining, though technically feasible nevertheless.

We denote by $P(t, T)$ the time–t price of a default-free zero-coupon bond of maturity $T \geq t$. As usual,

$$P(t, T) \;=\; \exp\left\{-\sum_{k=t/h}^{T/h-1} f(t, kh) \cdot h\right\} \tag{2}$$

The well-known recursive representation of the drift term α of the forward-rate and spread processes, is required to complete the risk-neutral lattice. Let $B(t)$ be the time–t value of a "money-market account" that uses an initial investment of \$1, and rolls the proceeds over at the default-free short rate:

$$B(t) \;=\; \exp\left\{\sum_{k=0}^{t/h-1} r(kh) \cdot h\right\}. \tag{3}$$

The equivalent martingale measure Q is defined with respect to $B(t)$ as numeraire; thus, under Q all asset prices in the economy discounted by $B(t)$ will be martingales. Let $Z(t, T)$ denote the price of the default-free bond discounted using $B(t)$: $Z(t, T) \;=\; \frac{P(t,T)}{B(t)}$. which is a martingale under Q, for any $t < T$, i.e. $Z(t, T) = E_t[Z(t + h, T)]$. It follows that $Z(t + h, T)/Z(t, T) = (P(t + h, T)/P(t, T)) \cdot$

Table 1: Branching process and probability measure. This tableau presents the 6 branches from each node of the pricing lattice, as well as the probabilities for each branch.

X_f	X_s	Probability
1	1	$p_1 = \frac{1}{4}(1+m_1)[1-\lambda(t)]$
1	-1	$p_2 = \frac{1}{4}(1-m_1)[1-\lambda(t)]$
-1	1	$p_3 = \frac{1}{4}(1+m_2)[1-\lambda(t)]$
-1	-1	$p_4 = \frac{1}{4}(1-m_2)[1-\lambda(t)]$
1	$-\infty$	$p_5 = \frac{\lambda(t)}{2}$
-1	$-\infty$	$p_6 = \frac{\lambda(t)}{2}$

$(B(t)/B(t+h))$. Algebraically manipulating the martingale equation leads to a recursive expression relating the risk-neutral drifts α to the volatilities σ at each t:

$$\sum_{k=t/h+1}^{T/h-1} \alpha(t,kh) \;=\; \frac{1}{h^2}\ln\left(E_t\left[\exp\left\{-\sum_{k=t/h+1}^{T/h-1}\sigma(t,kh)X_f h^{3/2}\right\}\right]\right). \tag{4}$$

2.4 The Joint Process

We now connect the two processes for the term structure and the defaultable equity price together on a bivariate lattice. There are two goals here. First, we set up the probabilities of the joint process so as to achieve the correct correlation between equity returns and changes in the spot rate. Second, our lattice is set up so as to be recombining, allowing for polynomial computational complexity, providing for fast computation of derivative security prices.

Specification of the joint process requires a probability measure over the random shocks $[X_f(t), X_s(t)]$. This probability measure is chosen to (i) obtain the correct correlations, (ii) ensure that normalized equity prices and bond prices are martingales, and (iii) to make the lattice recombining. Our lattice model is hexanomial, i.e. from each node, there are 6 emanating branches or 6 states. The following table depicts the states:

Here $\lambda(t)$ is the probability of default at each node of the tree. We also associate $\lambda(t)$ with a default intensity process $\xi(t)$, such that the survival probability in time interval h is:

$$1-\lambda(t) = \exp[-\xi(t)h]. \tag{5}$$

Notice that the table contains two free parameters m_1 and m_2 in the probability measure. We solve for the correct values of m_1 and m_2 to provide a default-consistent martingale measure, with the appropriate correlation between the equity and interest rate processes, ensuring too, that the lattice recombines.

In order for the lattice to be recombining, it is essential that the drift of the process for equity prices be zero. Hence, we write the modified stochastic process for equity prices as follows:

$$\ln\left[\frac{S(t+h)}{S(t)}\right] = \sigma_s X_s(t)\sqrt{h} \tag{6}$$

and then we adjust the probability measure over $X_s(t)$ such that

$$E[\exp(\sigma_s X_s(t)\sqrt{h})] = \exp[r(t)h].$$

In addition, under the HJM model, the mean value of the random variable X_f must be zero, and its variance should be 1. These properties are verified as follows.

$$\begin{aligned}
E(X_f) &= \frac{1}{4}[1+m_1+1-m_1-1-m_2-1+m_2](1-\lambda(t)) \\
&\quad +\frac{\lambda(t)}{2}[1-1] \\
&= 0 \\
Var(X_f) &= \frac{1}{4}[1+m_2+1-m_1+1+m_2+1-m_2](1-\lambda(t)) \\
&\quad +\frac{\lambda(t)}{2}[1+1] \\
&= 1
\end{aligned}$$

Now, we compute the two conditions required to determine m_1 and m_2. We use the expectation of the equity process to determine one equation. We exploit the fact that under risk-neutrality the equity return must equal the risk free rate of interest. This leads to the following:

$$\begin{aligned}
E\left[\frac{S(t+h)}{S(t)}\right] &= E[\exp(\sigma_s X_s(t)\sqrt{h})] \\
&= \frac{1}{4}(1-\lambda(t))[e^{\sigma_s\sqrt{h}}(1+m_1)+e^{-\sigma_s\sqrt{h}}(1-m_1) \\
&\quad +e^{\sigma_s\sqrt{h}}(1+m_2)+e^{\sigma_s\sqrt{h}}(1-m_2)]+\frac{\lambda(t)}{2}[0] \qquad (7)\\
&= \exp(rh)
\end{aligned}$$

Hence the stock return is set equal to the riskfree return. This implies the following from a simplification of equation (7):

$$m_1+m_2 = \frac{\frac{4e^{r(t)h}}{1-\lambda(t)}-2(a+b)}{a-b} = A \tag{8}$$

$$a = \exp(\sigma_s \sqrt{h}) \tag{9}$$

$$b = \exp(-\sigma_s \sqrt{h}) \tag{10}$$

Our second condition comes from the correlation specification. Let the correlation (coincident with covariance for unit valued variables) between the shocks $[X_f(t), X_s(t)]$ be equal to ρ, where $-1 \leq \rho \leq 1$. A simple calculation follows (ignoring the branches of default, since the correlation in that case is undefined)xs:

$$\begin{aligned} Cov[X_f(t), X_s(t)] &= \frac{1}{4}(1-\lambda(t))[1+m_1-1+m_1-1-m_2+1-m_2] \\ &= \frac{m_1-m_2}{2}(1-\lambda(t)). \end{aligned} \tag{11}$$

Setting this equal to ρ, we get the equation

$$m_1 - m_2 = \frac{2\rho}{1-\lambda(t)} = B. \tag{12}$$

Solving the two equations (8) and (12) leads to the following solution:

$$m_1 = \frac{A+B}{2} \tag{13}$$

$$m_2 = \frac{A-B}{2} \tag{14}$$

These values may now be substituted into the probability measure in Table 1 above. Notice that since the interest rate $r(t)$ only enters the probabilities and not the random shock $X_s(t), \forall t$, the equity lattice will also be recombining, just as was the case with the HJM model for the term structure. Hence, the product space of defaultable equity and interest rates will also be recombining. As interest rates change, the probability measure will also change, but this will not impact the recombining property of the lattice.

2.5 Ensuring a valid probability measure

It is also necessary that the solutions for m_1 and m_2 be such that the resultant probabilities do not become negative or greater than 1. From Table 1, we see that the necessary condition is $-1 \leq m_i \leq +1, \quad i = 1, 2$. To see this, note that the greatest absolute value of the probabilities on the non-defaultable branches is when $\lambda = 0$. Given this, we require the following 2 conditions on m_1, so as to be valid probabilities:

$$0 \leq \frac{1}{4}[1+m_1] \leq 1, \quad 0 \leq \frac{1}{4}[1-m_1] \leq 1$$

which implies that $-1 \leq m_1 \leq +1$. The same condition is derived for m_2.

Further analysis of m_1 and m_2 gives us the properties of the measure. We may also derive the probabilities of the non-defaulting branches in Table 1 and these are as follows:

$$\begin{aligned}
p_1 &= \frac{\frac{1}{2}[e^{rh+\sigma\sqrt{h}} - e^{-\xi h}] + \frac{1}{4}\rho[e^{2\sigma\sqrt{h}} - 1]}{e^{2\sigma\sqrt{h}} - 1} \\
p_2 &= \frac{\frac{1}{2}[e^{rh+\sigma\sqrt{h}} - e^{2\sigma\sqrt{h}-\xi h}] + \frac{1}{4}\rho[e^{2\sigma\sqrt{h}} - 1]}{1 - e^{2\sigma\sqrt{h}}} \\
p_3 &= \frac{\frac{1}{2}[e^{rh+\sigma\sqrt{h}} - e^{-\xi h}] + \frac{1}{4}\rho[1 - e^{2\sigma\sqrt{h}}]}{e^{2\sigma\sqrt{h}} - 1} \\
p_4 &= \frac{\frac{1}{2}[e^{rh+\sigma\sqrt{h}} - e^{2\sigma\sqrt{h}-\xi h}] + \frac{1}{4}\rho[1 - e^{2\sigma\sqrt{h}}]}{1 - e^{2\sigma\sqrt{h}}}
\end{aligned}$$

It is easy to check that for a large range of values of the intensity ξ (from 0 to 0.15), time interval h (from 0.01 to 0.50), and values of σ and r, all probabilities lie within the acceptable range, i.e. $0 \leq p_i \leq 1$, $i = 1, 2, 3, 4$.

Of course, the preceding analysis really implies that there is a range for the value of default probability $\lambda(t)$ which is consistent with the equity and term structure processes. Hence, we can derive the corresponding values of $\lambda(t)$ that correspond to the permissible ranges for m_1, m_2 above. This results in 8 bounds, which are presented in the Table 2. After computing the value of the default probability we check that it satisfies these bounds, else we set it to be within the range values.

2.6 Time interval and the stock variance

The relationship between the time interval h and the variance of stock return is computed as follows:

$$Var[\sigma_s X_s(t)\sqrt{h} \| \text{no default}] = \frac{\sigma_s^2 h Var[X_s(t)]}{1-\lambda(t)} \tag{15}$$

$$= \frac{\sigma_s^2 h[E(X_s^2) - E(X_s)^2]}{1-\lambda(t)} \tag{16}$$

$$= \sigma_s^2 h \left[1 - (1-\lambda(t))\left(\frac{m_1+m_2}{2}\right)^2\right] \tag{17}$$

We focus in on the term $[\frac{m_1+m_2}{2}]^2$, which becomes very small as the time interval shrinks. Substituting in the appropriate variables, this term may be represented in detail as follows:

$$\left[\frac{m_1+m_2}{2}\right]^2 = \left[\frac{\frac{2e^{rh}}{1-\lambda(t)} - (e^{\sigma_s\sqrt{h}} + e^{-\sigma_s\sqrt{h}})}{e^{\sigma_s\sqrt{h}} - e^{-\sigma_s\sqrt{h}}}\right]^2 \tag{18}$$

Table 2: Bounds on default probabilities. These 8 conditions specify limit values on one period default probabilities $\lambda(t)$. Some of these conditions may not apply in the sense that they suggest negative limits for probabilities, which are superseded by the lower limit condition of zero value. The upper bound on $\lambda(t)$ will be the minimum of the positive limits in the table.

Condition on m_i	Limit Value of $\lambda(t)$
$0 \leq \frac{1}{4}[1+m_1]$	$\frac{1}{2b}[-2e^{r(t)h} - a\rho + b(\rho+2)]$
$\frac{1}{4}[1+m_1] \leq 1$	$\frac{1}{4a-2b}[-2e^{r(t)h} - a(\rho-4) + b(\rho-2)]$
$0 \leq \frac{1}{4}[1-m_1]$	$\frac{1}{2a}[-2e^{r(t)h} - a(\rho-2) + b\rho]$
$\frac{1}{4}[1-m_1] \leq 1$	$\frac{1}{2a-4b}[2e^{r(t)h} + a(\rho+2) - b(\rho+4)]$
$0 \leq \frac{1}{4}[1+m_2]$	$\frac{1}{2b}[-2e^{r(t)h} - b(\rho-2) + a\rho]$
$\frac{1}{4}[1+m_2] \leq 1$	$\frac{1}{4a-2b}[-2e^{r(t)h} - b(\rho+2) + a(\rho+4)]$
$0 \leq \frac{1}{4}[1-m_2]$	$\frac{1}{2a}[-2e^{r(t)h} - b\rho + a(\rho+2)]$
$\frac{1}{4}[1-m_2] \leq 1$	$\frac{1}{2a-4b}[2e^{r(t)h} + b(\rho-4) - a(\rho-2)]$

Notice that this term is small for the usual values of r, h, σ_s. For example, suppose the interest rate is 4%, the stock volatility is 20%, and default probability is 0%, then if $h = 0.25$, then the term is 0.0025, which is a small number. As $h \to 0$, the term $[1 - (1 - \lambda(t)) \left(\frac{m_1+m_2}{2}\right)^2] \to 1$. Therefore, as $h \to 0$, $Var[\sigma_s X_s(t)\sqrt{h}] \approx \sigma_s^2 h$, conditional on no default.

3 Credit Risk

Accounting for credit risk is achieved by adding the process for default probability $[\lambda(t)]$ to the lattice. Rather than add an extra dimension to the lattice model by embedding a separate $\lambda(t)$ process, we define one-period default probability functions at each node on the bivariate lattice, by making default a function of equity prices and interest rates at each node. There are two reasons for this. First, equity prices already reflect credit risk, and hence there is a connection between $\lambda(t)$ and equity prices. Second, default probabilities are empirically known to be connected to the term structure, and hence, may be modeled as such. Therefore, our approach entails modeling the default risk at each node as a function of the level of equity and the term structure at each node.

Our approach specifies a *conditional* $\lambda(t)$ at each node, i.e. rather than add a separate default probability process, we simply make the $\lambda(t)$s a function of the state variables of equity and interest rates. We refer to this as an *endogenous* default approach.[4] If in fact, default probabilities were added as a separate stochastic process (which we denote the *exogenous* approach, as in Davis and Lischka [1999] or Andersen and Buffum [2002]), the question of consistency conditions between $\lambda(t)$, equity and interest rates would arise, a complex situation to resolve. By positing a functional relationship of $\lambda(t)$ to the other variables, we are able to obtain a consistent lattice as well as a more parsimonious one. As noted before, $\lambda(t) = 1 - e^{-\xi(t)h}$, and we express the default intensity $\xi(t)$ as:

$$\xi[\mathbf{f}(t), S(t), t; \theta] \quad \in \quad [0, \infty) \tag{19}$$

i.e. a function of the term structure of forward rates $\mathbf{f}(t)$, the stock price $S(t)$ at each node, and time t. This function may be as general as possible. We impose the condition that is required of default intensities, i.e. $\xi(t) \geq 0$. θ is a parameter set that defines the function. This is not a new approach. A similar *endogenous* default intensity extraction has been implemented in Das and Sundaram [2000], Carayannopoulos and Kalimipalli [2001], and Acharya, Das and Sundaram [2002]. However, the settings in those papers were less general than in this one.

[4] After presentations of this model to Wall Street firms, practitioners have called our model the "two and a half" model to stand for the fact that equity, interest rate and default are modeled, leading to a 3-dimensional model. But since default is endogenously modeled, it is denoted as a half dimension, leading therefore, to a $2\frac{1}{2}$ dimension model.

Of course, in addition to the probability of default of the issuer, a recovery rate is required. In the two states in which default occurs, this recovery rate is applied. The recovery rates may be treated as constant, or as a function of the state variables in this model. It may also be pragmatic to express recovery as a function of the default intensity, supported by the empirical analysis of Altman, Brooks, Resti and Sironi [2002].

Various possible parameterizations of the default intensity function may be used. For example, the following model (subsuming the parameterization of Carayannopoulos and Kalimipalli [2001]) prescribes the relationship of the default intensity $\xi(t)$ to the stock price $S(t)$, short rate $r(t)$, and time on the lattice $(t - t_0)$.

$$\begin{aligned} \xi(t) &= h(y) \exp[a_0 + a_1 r(t) - a_2 \ln S(t) + a_3(t - t_0)] \\ &= h(y) \frac{\exp[a_0 + a_1 r(t) + a_3(t - t_0)]}{S(t)^{a_2}} \end{aligned} \tag{20}$$

For $a_2 \geq 0$, we get that as $S(t) \to 0, \xi(t) \to \infty$, and as $S(t) \to \infty, \xi(t) \to 0$. Further, we may also specify the function $h(y)$, based on a state variable y (such as the debt-equity ratio) through which other influences on the default intensity function may be imposed. This function must satisfy consistency conditions depending on its choice of state variable. For example, if y were the debt-equity ratio, then we might require that (a) $h(0) = 0$, (b) $h(\infty) = \infty$, and (c) $h'(y) > 0, \forall y$.

3.1 Example: two-period tree

Here, we present a simple illustration of a pricing tree in 3 dimensions, one each for time (t), interest rate, and stock price. This tree is in (t, i, j) space. The initial node is denoted $(1, 1, 1)$ and after one period, we have 4 nodes (interest rates go up and down, and stock price goes up and down), denoted $\{(2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$. Since the tree is recombining, after 2 periods, we will have only 9 nodes. In the illustration below, presented in Table 3, are the results of calculations for 2 periods. At each node, we show the one-period probability of default. The table presents all the details of the inputs used in the example. This table should be useful to readers who wish to implement the model, and it also details all the inputs required for building the pricing lattice. As may be seen, the approach requires a parsimonious set of inputs, all of which are observable and may be accessed from standard sources.

3.2 Calibration with credit default swaps

Default swaps may be used to calibrate the model. A default swap is a contract between two parties, whereby the buyer of the default swap pays a flat stream of insurance payments to the seller, who

Table 3: Two-Period Tree Example. In this table, we present the results of a two-period tree based on given input parameters. The example here may be useful for anyone replicating our model to check their results. The input parameters are the default function values $\{a_0, a_1, a_2, a_3\}$, the stock price S, stock volatility σ_s, correlation of term structure with stock prices ρ, and the time step on the tree h. The initial forward rate term structure and corresponding volatilities are also given. The output price lattice is recombining, and therefore, there are $(n+1)^2$ nodes at the end of the n^{th} period on the lattice. The lattice starts at node $(1, 1, 1)$ and then moves to 4 nodes in the subsequent period, and then to 9 nodes, etc. At each time step there are two axes (i, j) for interest rates and stock prices respectively. The default probability (λ) for the next period is also stated at each node, and is a function of r, S, and time. The default function is: $\lambda = 1 - \exp[-\xi h]$, where $\xi = \exp[a_0 + a_1 r + a_3 ih]/(S^{a_2})$, where i indexes nodes on the interest rate branch of the tree (see table below). Note that a_3 modulates the slope more severely when rates are low than high. Alternative specifications would be to replace i with t. It can be seen that the default probability declines as S increases, and increases in r.

Input Values

Parameter	Value	Fwd Rates	FWR Vols
a_0	0.1	0.060	0.0020
a_1	0.1	0.065	0.0019
a_2	1.0	0.070	0.0018
a_3	0.1		
S	100		
σ_s	0.40		
ρ	0.4		
h	0.5		

Output Price Lattice

[t	i	j]	r	S	λ
1	1	1	0.0600	100.0000	0.0058
2	1	1	0.0663	132.6896	0.0044
2	1	2	0.0663	75.3638	0.0077
2	2	1	0.0637	132.6896	0.0046
2	2	2	0.0637	75.3638	0.0081
3	1	1	0.0725	176.0654	0.0033
3	1	2	0.0725	100.0000	0.0058
3	1	3	0.0725	56.7971	0.0102
3	2	1	0.0700	176.0654	0.0035
3	2	2	0.0700	100.0000	0.0061
3	2	3	0.0700	56.7971	0.0108
3	3	1	0.0675	176.0654	0.0037
3	3	2	0.0675	100.0000	0.0064
3	3	3	0.0675	56.7971	0.0113

makes good any loss on default of a reference bond. The seller's payment is contingent upon default. The price of a default swap is quoted as a spread rate per annum. Therefore, if the default swap rate is 100 bps, paid quarterly, then the buyer of the insurance in the default swap would pay 25 bps of the notional each quarter to the seller of insurance in the default swap. The present value of all these payments must equal the expected loss on default anticipated over the life of the default swap. In the event of default, the buyer of protection in the default swap receives the par value of the bond less the recovery on the bond. In many cases, this is implemented by selling the bond back to the insurance seller at par value. Pricing of a default swap has been described in detail in Duffie [1999].

The increasing amount of trading in default swaps now offers a source of empirical data for calibrating the model. Other models, such as CreditGrades[5], also use default swap data. Hence, the term-structure of default swaps is now available for cross-sectional fitting of our model parameters. As an illustration of the lattice computations that may be employed for pricing, we consider the simplest form of a default swap, i.e. that written on a zero-coupon bond. A recent paper by Longstaff, Mithal and Neis [2002] undertakes an empirical comparison of default swap and bond premia in a parsimonious closed-form model.

The way the lattice is set up in our model makes it very simple to compute the default swap spread s (stated as a rate in basis points per annum). Since the probability of default is known at each node on the tree, we can compute the expected cashflow from the default swap at each node, which is just $[\lambda(t) \times \text{(Loss in Value on Default)}]$. We accumulate these values at each node and discount them back along the tree to obtain the expected present value of loss payments made by the writer of the default swap. The buyer then pays in a constant spread s each period, such that the present value of these payments equals the present value of expected loss on default.

Assume that we have "pure" default swap spreads for a range of maturities, $t = 1, 2, 3, ..., T$ years. The pure premium on a default swap is the present value of insurance payments on a defaultable zero-coupon bond. The premium is equal to the expected present value of payouts on default of the underlying zero-coupon instrument. Expectations are taken under the default-risk adjusted martingale measure described in this paper. Given any four maturities, we can calibrate the four parameters $\theta = \{a_0, a_1, a_2, a_3\}$ in the function in equations (19) and (20) by exact fitting of four default swap premia. If more than four maturities for default swap spreads are available, the parameters may be fitted using a least squares criterion.

We denote the recovery rate on default as ϕ, which may be specified in this case as constant, without loss of generality. Applying the recovery of market value (RMV) assumption on default, the

[5]This is a model developed by RiskMetrics, who use default swaps to calibrate a Merton-type model to obtain probabilities of default.

pure default swap rate is the continuous stream of payments expressed in basis points that equates the present value of these payments to the expected present value of the payoffs on the default swap. On the lattice, these values may be computed via backward recursion.

Since we are working in discrete time, careful specification of the timing of default is also required. We make the following assumptions. *(a)* In any period in which default occurs, recovery payoffs are realized at the end of the period. *(b)* Default is based on the default intensity at the beginning of the period. In order to price a credit default swap we define the following quantities.

First, we define the price of a defaultable zero-coupon bond. We denote the price of this bond at time t as $ZCCB(t)$. The pricing recursion under the RMV condition is as follows:

$$\begin{aligned} ZCCB(t) &= e^{-r(t)h} \left\{ \sum_{k=1}^{4} \hat{p}_k(t) ZCCB_k(t+h) \right\} [1-\lambda(t)(1-\phi)], \qquad (21) \\ & ZCCB(T) = 1.0 \end{aligned}$$

Here, $\hat{p}_k(t) = p_k(t)/[1-\lambda(t)], k = 1..4$ are the four probabilities for the non-default branches of the lattice, *conditional* on no default occurring, and k indexes the four states of non-default. Therefore, $\sum_{k=1}^{4} p_k(t) = 1, \forall t$. We need to price this at every node on the lattice, as we will define recovery as a fraction of the value of this bond, if and when it occurs.

Second, we compute the expected present value of all payments in the event of default of the zero-coupon bond, denoted $CDS(t)$. Again, the lattice-based recursive expression is:

$$\begin{aligned} CDS(t) &= e^{-r(t)h} \left\{ \sum_{k=1}^{4} \hat{p}_k(t) CDS_k(t+h) \right\} [1-\lambda(t)] \qquad (22) \\ &+ \lambda(t) ZCCB(t)(1-\phi), \quad CDS(T) = 0.0. \end{aligned}$$

The formula above has two components. *(i)* The first part is the present value of future possible losses on the default swap, given that default does not occur at time t. *(ii)* The second part is the present value of the loss (sustained at the end of the period). Note that the formula contains $ZCCB(t)(1-\phi)$, which is the present value of loss at the end of the period, $ZCCB(t+h)(1-\phi)$.

Third, we calculate the expected present value of a $1 payment at each point in time conditional on no default occurring. This is defined as follows:

$$G(t) = \left[e^{-r(t)h} \left\{ \sum_{k=1}^{4} \hat{p}_k(t) G_k(t+h) + 1 \right\} \right] [1-\lambda(t)], \quad G(T) = 0.0. \qquad (23)$$

This is computed since we only make default swap premium payments until the period in which the default occurs. Therefore, we wish to compute and store the value of unit $1 payments each period provided default has not occured. It is assumed that the payments are made at the end of the period conditional on no default in that period.

In order to get the annualized basis points spread (s) for the premium payments on the default swap, we equate the quantities $s \times h \times G(0) = CDS(0)$, and the premium spread is:

$$s = \frac{CDS(0)}{h \times G(0)} \times 10,000 \text{ bps.} \tag{24}$$

In the equation above, we multiply by 10,000 and divide by the time interval h in order to convert the amount into annualized basis points. We use this calculation in the illustrative examples that are provided in the following section.

4 Implementation

The ease of use of our model is demonstrated by providing some implementation examples in this section. We begin with an example showing how different parameters result in various default swap spread term structures. Next, we reverse this procedure, and show that it is easy to fit default parameters to an observed default swap term structure. Following this, we show how default risk results in differences in prices of corporate bonds, with or without convertible features. Finally, we present a section explaining how the model may be used for pricing credit correlation products.

4.1 Default Swap Spread Curves

In this section we demonstrate that the model is able to generate varied spread curve shapes. In the plots in Figure 1 we present the term structure of default swap spreads for maturities from 1 to 10 years. The figure has 2 graphs, each containing 2 plots each. The default intensity is specified as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. Keeping a_0 fixed, we varied parameters a_1 (impact of the short rate), a_2 (impact of the equity price) and a_3 (impact of time) over two values each. Four plots are the result, 2 in each graph. The other inputs to the model, such as the forward rates and volatilities, stock price and volatility, etc., are provided in the description of the figure. Comparison of the plots provides an understanding of the impact of the parameters.

When $a_3 > 0$, the term structure of default swap spreads is upward sloping, as would be expected. When $a_3 < 0$, i.e. default spreads are driven down as maturity increases. The parameter a_3 may be used to tune the model for varied slopes of credit spread curves and for different credit ratings. It is known that higher quality credits have a tendency to deteriorate in quality over time, hence $a_3 > 0$ would be plausible. On the other hand, poorer quality credits, conditional on survival, tend to upgrade, and hence $a_3 < 0$ may be appropriate. Comparison of the plots also shows the effect of parameter a_2, the coefficient of the equity price $S(t)$. As $a_2 > 0$ increases, default spreads decline as the stock price lies in the denominator of the default intensity function, as can be seen in the plots. A comparison of curves across the 2 panels in Figure 1 shows that parameter a_1, the coefficient on interest rates, has a level effect on the spread curve. In sum, our four parameter default function is flexible enough to capture a variety of economic phenomena, as well as generate a spectrum of curve shapes.

4.2 Empirical calibration

The ability to calibrate no-arbitrage models like the one presented in this paper is increasing as more instruments become available, and as advances in computing technology make numerical computation far easier than before. The market for default swaps is steadily expanding, which provides observable data on pure credit spreads for many issuers. Default swap spreads are often preferred to bond credit spreads as they are less affected by liquidity and taxes. Firms such as RiskMetrics now make available credit spread term structures, which are calibrated to default swaps. They provided us the data we use here.

4.2.1 Calibration 1

In this subsection, we present an illustrative calibration of the model to the term structure of default swap spreads of International Business Machines (ticker symbol: IBM). For comparison, we chose two dates for the calibration, 02-Jan-2002 and 28-Jun-2002. The stock price on the 2 dates was $72.00 and $121.10 respectively. Stock return volatility was roughly 40% on both dates. Recovery rates on default were assumed to be 40% and the historical correlation between short rates (i.e. 3 month Treasury bills) and the stock return of IBM was computed over the period January 2000 to June 2002; it was found to be almost zero, i.e. 0.01528. The yield curves for the chosen dates were extracted from the historical data pages provided by the Federal Reserve Board. We converted these into forward rates as required by our model. Forward rate volatilities were set to the average historical volatility over the periods January 2000 to June 2002. Our goal in this exercise is to examine how easily our model fits its four default parameters $\{a_0, a_1, a_2, a_3\}$ to default spreads of various maturities. We searched over the four parameters to best fit spreads of 1,2,3,4 year maturities. Hence, using the lattice model as a numerical equation, we have to fit four equations in four unknowns. By examining how well the calibrated model reproduces the spread curve, we get an idea of how difficult (or easy) it is to fit our model. As it turns out, the model fits the data well, as can be seen from the overlapping plots in Figure 2. There

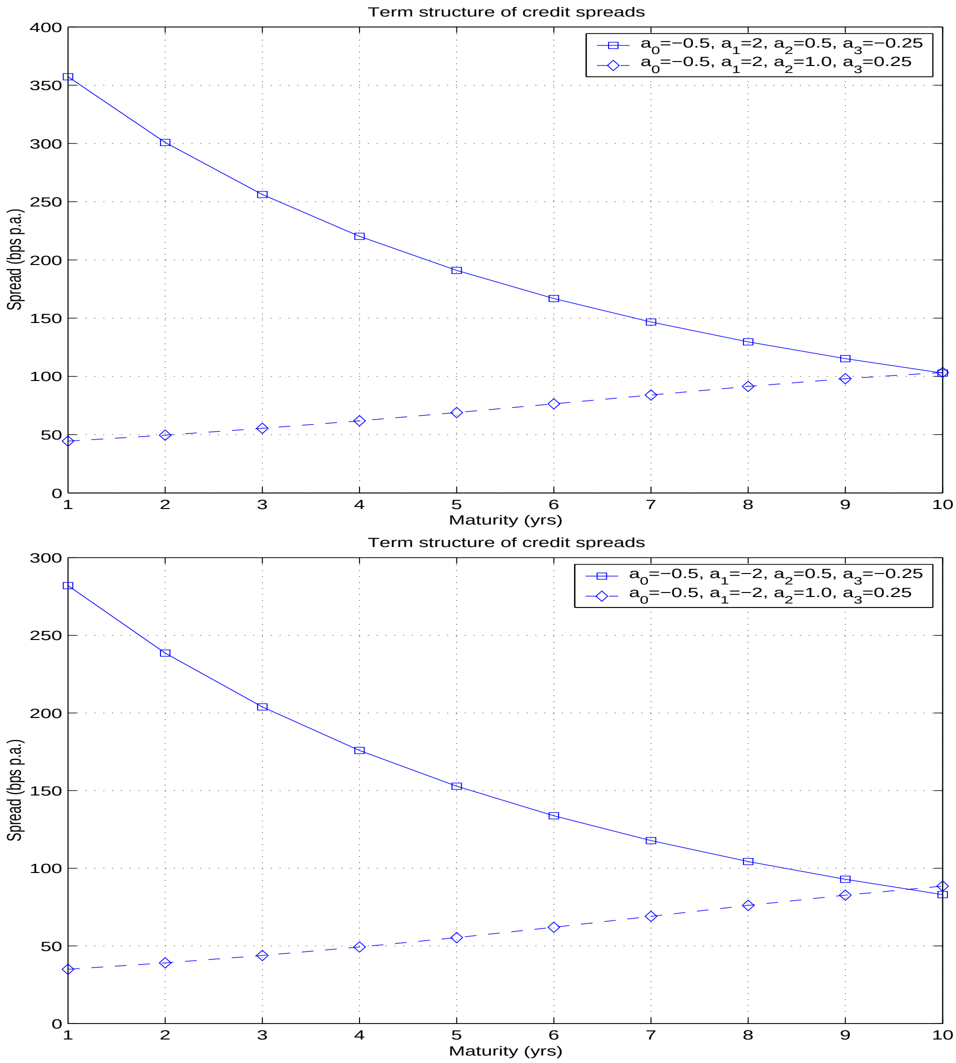

Figure 1: Term Structure of Default Swap Spreads. This figure presents the term structure of default swap spreads for maturities from 1 to 10 years. The figure has 2 graphs, each containing 2 plots each. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. Keeping all the other parameters fixed, we varied parameters a_1, a_2 and a_3. Hence, the 4 plots are the result. Periods in the model are quarterly, indexed by i. The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001\ln(i)$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005\ln(i)$. The initial stock price is 100, and the stock return volatility is 0.30. Correlation between stock returns and forward rates is 0.30, and recovery rates are a constant 40%. The default function parameters are presented on the plots.

are 4 plots – the top two relate to January 2002, and the lower two are for June 2002. Each pair of plots provides the forward curves, volatilities of forward rates, and credit spreads.

For IBM (Figure 2), credit risk increased from January 2002 to June 2002, and can be seen in the higher spreads in June, on account of worsening economic conditions in the U.S. economy. A comparison of the parameters of the default intensity function on each date provides some intuition for the impact of increasing credit risk. Notice that a_0 has becomes less negative, since the default intensity has gone up from January 2002 to June 2002. Also, a_1 has increased, making default risk more sensitive to interest rates. Since a_2 has increased, the firm's default intensity now increases at a slower rate as the stock price falls. Finally, a_3 has declined in June 2002, implying a lower slope. Hence, the model calibrates well, and also provides useful economic intuition.

4.2.2 Calibration 2

We extended the same analysis to the default swaps of a financial company, namely AMBAC Inc (ticker symbol: ABK). Default processes in the finance sector are different because firms have extreme leverage, suggesting that fitting spread curves for the financial sector is more complicated. Also, the impact of declining interest rates during this period is likely to impact bank spreads differently as compared to non-financial firms.

Our model calibrates just as easily to the default swap rates for AMBAC as it did in the case of IBM. For comparison, we calibrated the model on the same dates as we did for IBM. The results in Figure 3 portray the plots of the empirical default swap spreads and the fitted ones. It is seen that these are very close to each other, implying a good calibration.

It is interesting to note that the spreads for AMBAC have fallen from January to June 2002. This is possibly on account of declining interest rates, which usually bodes well for the finance industry. The coefficient a_3, which is negative, is more negative in June versus January, signaling that spreads have declined and the market has indicated better credit quality in the long-run. This coefficient a_2 has also increased from January to June, reducing spreads.

4.3 Impact of default risk on embedded options

The model may be easily used to price callable-convertible debt. One aspect of considerable interest is the extent to which default risk impacts the pricing of convertible debt, through its effect on the values of the call feature (related to interest rate risk) and the convertible feature (related to equity price risk). We chose an initial set of parameters to price convertible debt, and examined to what extent changing levels of default risk impacted a plain vanilla bond versus a convertible bond. The parameters and

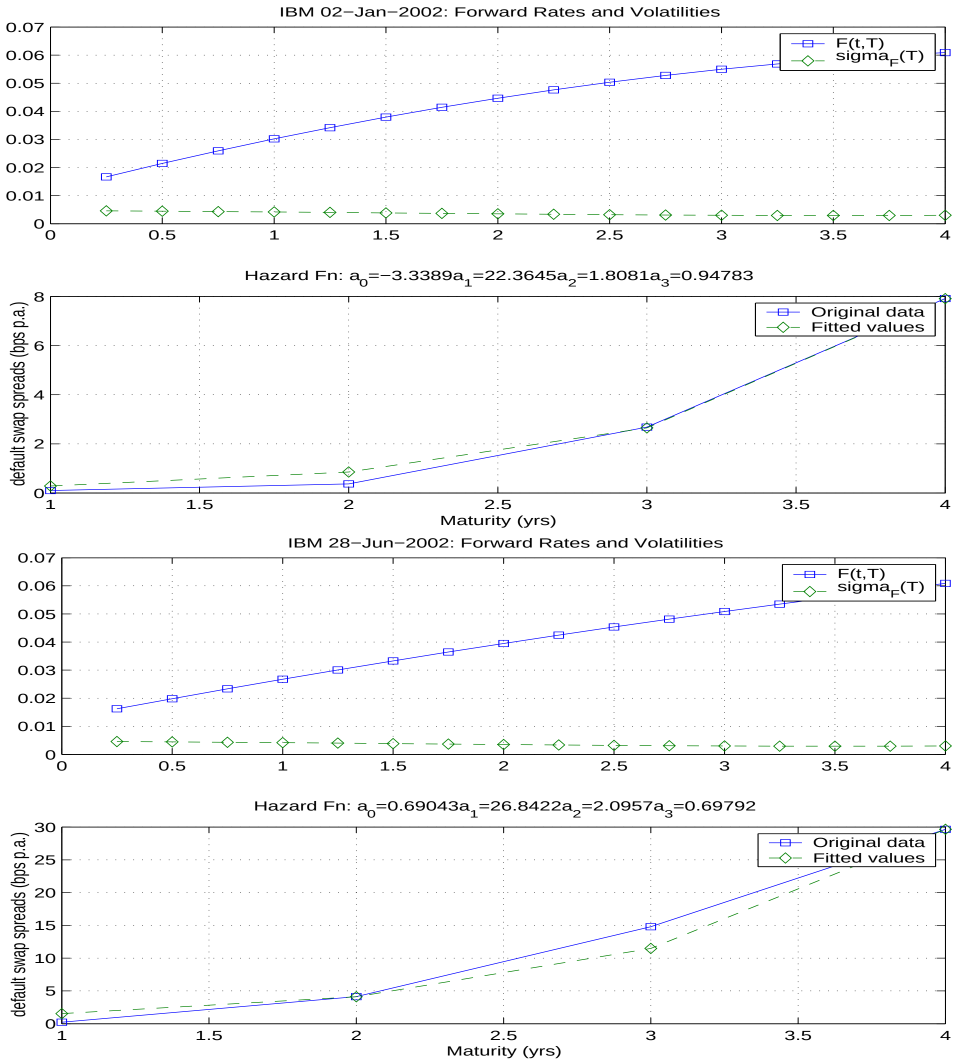

Figure 2: Fitted and Empirical Default Swap Spreads for IBM. This figure presents the fitted term structure of default swap spreads for maturities from 1 to 4 years, plotted against the original default spreads. The first and third graphs present the term structure for 2 January, 2002 and 28 June, 2002 respectively. The second and fourth graphs show the empirical and fitted credit default swap spreads. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. The fitted parameters of this function are provided in the figures. The stock price on the 2 dates was \$72.00 and \$121.10 respectively. Stock return volatility was roughly 40% on both dates. Recovery rates on default were assumed to be 40% and the correlation between short rates (i.e. 3 month tbills) and the stock return of IBM was computed over the period January 2000 to June 2002; it was found to be almost zero, i.e. 0.01528. The yield curves for the chosen dates were extracted from the historical data pages provided by the Federal Reserve Board.

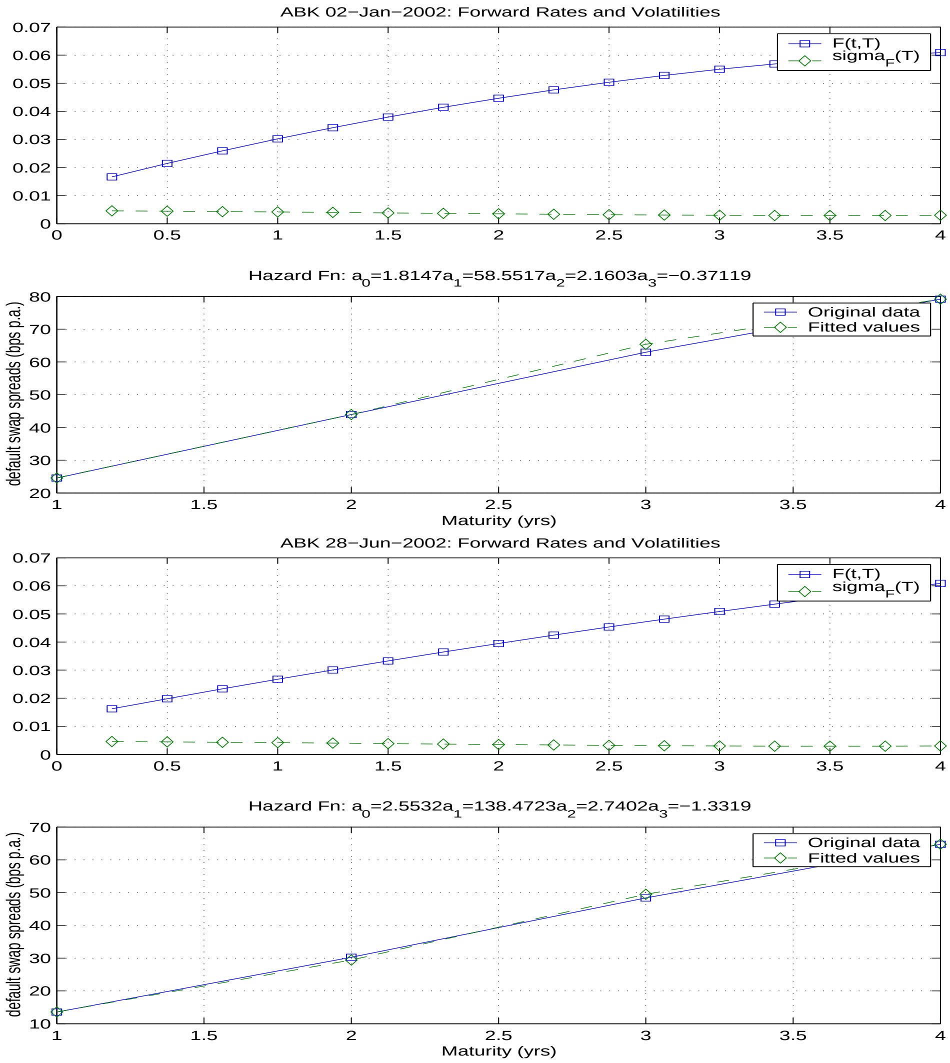

Figure 3: Fitted and Empirical Default Swap Spreads for AMBAC. This figure presents the fitted term structure of default swap spreads for maturities from 1 to 4 years, plotted against the original default spreads. The first and third graphs present the term structure for 2 January, 2002 and 28 June, 2002 respectively. The second and fourth graphs show the empirical and fitted credit default swap spreads. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a_3(t - t_0)]/S(t)^{a_2}$. The fitted parameters of this function are provided in the figures below. The stock price on the 2 dates was $58.31 and $67.20 respectively. Stock return volatility was roughly 40% on both dates. Recovery rates on default were assumed to be 40% and the correlation between short rates (i.e. 3 month tbills) and the stock return of AMBAC was computed over the period January 2000 to June 2002; it was found to be statistically zero. The yield curves for the chosen dates were extracted from the historical data pages provided by the Federal Reserve Board.

results are presented in Figure 4.

Given the base set of parameters, we varied a_0 from 0 to 4. As a_0 increases, the level of default risk increases too. For each increasing level of default risk, we plot the prices of a defaultable plain vanilla coupon bond with no call or convertible features. We also plot the prices of a callable-convertible bond. Note that this numerical experiment has been kept simple by setting $a_1 = a_3 = 0$, so that there are no interest-rate and term effects on the default probabilities.

The results comparing the plain coupon bond with a callable-convertible coupon bond are presented in Figure 4 (upper panel). The value of a_0 is varied from 0 (low default risk) to 4 (higher risk). Note that the values of bonds decline as default risk (a_0) increases. As default risk increases, the difference in price between the callable-convertible and vanilla bonds declines rapidly and eventually goes to zero. Since default risk effectively shortens the duration of the bonds, it also reduces the value of the call option. Hence, the price difference between the vanilla bond and the callable-convertible bond declines as a_0 increases. Moving from the upper to middle panel is based on one change, i.e. equity volatility was increased from 20% per year to 40% per year. The results are the same, but bond prices converge faster. Hence at high equity volatility, default risk impacts the convertible value faster, as there are more regions in the state space on our pricing tree with greater probability of default. Therefore, default risk systematically impacts the commingled values of interest rate calls and equity convertible features in debt contracts. By shortening the effective duration of the bond, both options decline in value, driving the price of the callable-convertible closer to that of the vanilla bond.

In Figure 5 we vary the dependence of default risk on equity prices. The base case is presented in the upper panel of the figure when the coefficient $a_2 = 1$. In the lower panel, we changed $a_2 = 0.75$, resulting in higher default risk. Hence, the prices are lower in the lower panel. The values of parameters for the conversion feature and for the call feature were chosen so as to make the plain bond and the callable-convertible equal in price in the upper panel. Reducing the value of a_2 to inject more default risk in fact increases the price of the callable-convertible relative to that of the plain bond, and the effect is higher for greater levels of default risk. Here, increases in default risk tend to increase the difference between equity call option values and the bond callable feature, ceteris paribus, and this drives an increasing wedge between the convertible bond and the plain bond. Further, the level of parameter a_2 also determines whether defaultable bond prices are convex or concave in default risk, and both possibilities are pictured in the two panels of Figure 5. At lower levels of default risk, the convertible bond is concave in a_0, and at higher levels it becomes convex.

Therefore, depending on market conditions, and the level of default risk, increases in default risk may increase or decrease the price differential of two bonds that have embedded options. This high-

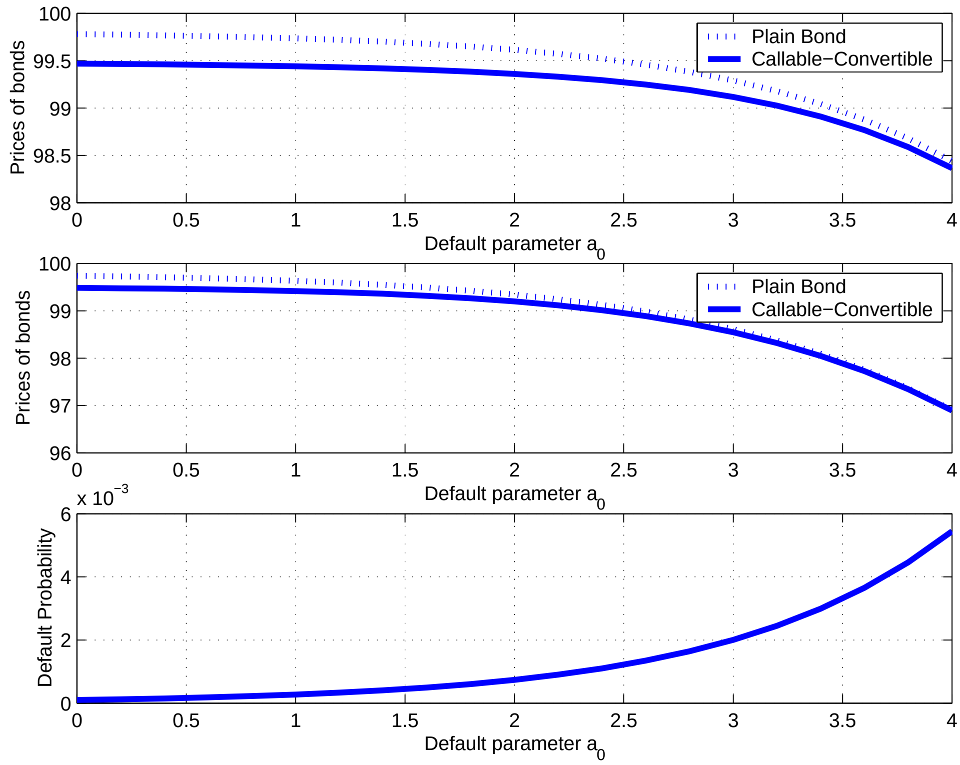

Figure 4: Comparison of callable convertible bonds and plain defaultable bonds in different volatility environments. We assumed a flat forward curve of 6%. We also assumed a flat curve for forward rate volatility of 20 basis points per period. The maturity of the bonds is taken to be 5 years, and interest is assumed paid quarterly on the bonds at an annualized rate of 6%. Default risk is based on default intensities which come from the model in equation (20). The base parameters for this function are chosen to be $a_0 = 0$, $a_1 = 0$, $a_2 = 2$, and $a_3 = 0$. Under these base parameters default risk varies only with the equity price. In our numerical experiments we will vary a_0 to examine the effect of increasing default risk. The stock price is $S(0) = 100$. The recovery rate on default is 0.4, and the correlation between the stock return and term structure is 0.25. If the bond is callable, the strike price is 100. Conversion occurs at a rate of 0.3 shares for each bond. The dilution rate on conversion is assumed to be 0.75. This figure contains three panels. The upper panel presents a comparison of bond prices when equity volatility is set to 20%, and the default probability parameter a_0 is varied on the x-axis. The middle panel shows the same comparison when the volatility is 40%. The bottom panel shows the corresponding default probability.

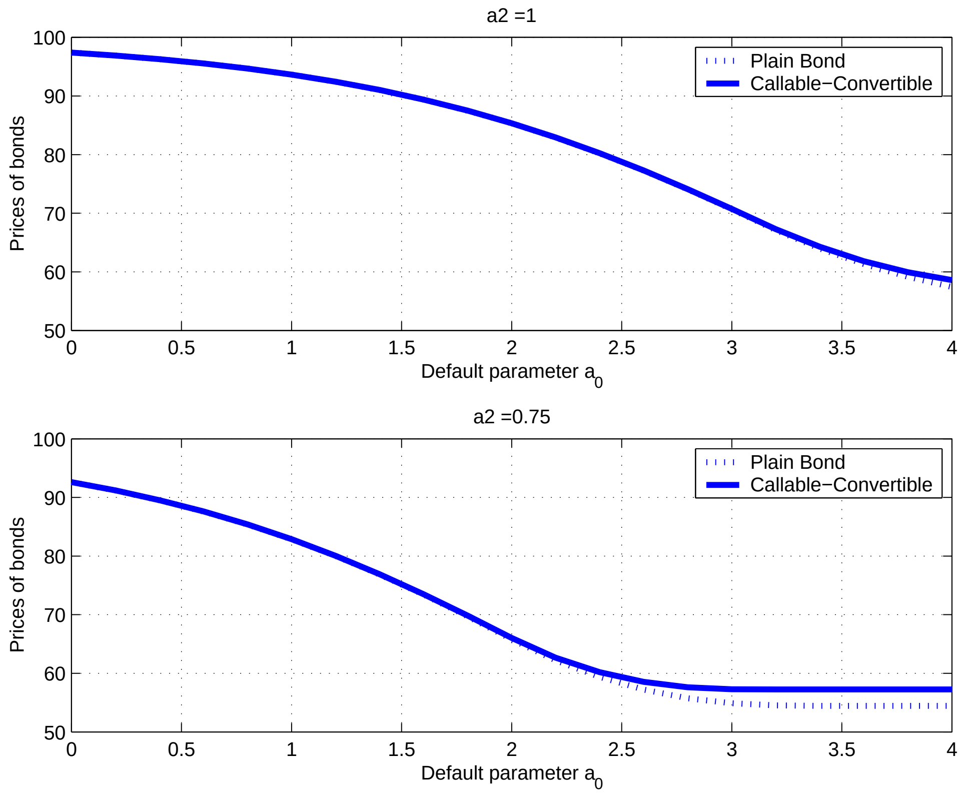

Figure 5: Comparison of default risk effects on callable convertible bonds and plain defaultable bonds for different equity dependence. We assumed a flat forward curve of 6%. We also assumed a flat curve for forward rate volatility of 20 basis points per period. The maturity of the bonds is taken to be 5 years, and interest is assumed paid quarterly on the bonds at an annualized rate of 6%. Default risk is based on default intensities which come from the model in equation (20). The base parameters for this function are chosen to be $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$. Under these base parameters default risk varies only with the equity price. In our numerical experiments we will vary a_0 to examine the effect of increasing default risk. The stock price is $S(0) = 100$ and stock volatility is 20%. The recovery rate on default is 0.4, and the correlation between the stock return and term structure is 0.25. If the bond is callable, the strike price is 105. Conversion occurs at a rate of 0.3 shares for each bond. The dilution rate on conversion is assumed to be 0.75. This figure contains two panels. The upper panel presents a comparison of bond prices when $a_2 = 1$, and the default probability parameter a_0 is varied on the x-axis. The lower panel shows the same comparison when $a_2 = 0.75$, which is higher default risk.

lights the need for careful consideration of default risk effects using an appropriate model.

4.4 Correlated default analysis

The model may be used to price credit baskets. There are many flavors of these securities, and some popular examples are n^{th} to default options, and collateralized debt obligations (CDOs). These securities may be valued using Monte Carlo simulation, under the risk-neutral measure, based on the default functions fitted using the techniques developed in this paper.

The first step in modeling default correlations is to model the correlation of default intensity amongst issuers. Since our model calibrates default functions $\xi(\mathbf{f}(t), S(t), t)$ of state variables, default correlations are determined from the correlations of the state variables, which are observable. Suppose we are given the function for default intensity of issuer $i, i = 1...n$, as $\xi_i(t) = \exp[a_0^i + a_1^i r(t) + a_3^i(t - t_0)]/S_i(t)^{a_2^i}$. Let the covariance matrix of $[r(t), S_1(t), S_2(t), ..., S_n(t)]'$ be Σ. Then, the covariance matrix of default intensities $\{\xi_i(t)\}_{i=1...n}$ is $V(t) \approx J(t)\Sigma J(t)'$, where $J(t) \in R^{n\times(n+1)}$ is the Jacobian matrix whose i^{th} row is as follows:

$$J_i(t) = \left[\frac{\partial \xi_i(t)}{\partial r(t)}, 0, ..., 0, \frac{\partial \xi_i(t)}{\partial S_i(t)}, 0, ..., 0\right] = \left[a_1^i \xi_i(t), 0, ..., 0, \frac{-a_2^i}{S_i(t)}\xi_i(t), 0, ..., 0\right]$$

We may contrast this approach with the somewhat ad-hoc practice of using equity correlations as a proxy for asset correlations, used in turn to drive default correlations in structural models. Our method is closer to the approach, also used in practice, of a factor structure that drives default correlations. However, our approach has a significant advantage over other factor models – i.e. we calibrate each default function in a manner that is based on state variables, *and* is also consistent with a no-arbitrage model over default, equity and interest-rate risks.

5 Concluding Comments

This paper presents a simple model that embeds major forms of security risk, enabling the pricing of complex, hybrid derivatives. The model achieves two distinct classes of objectives: (a) economic and (b) technical.

The following *economic* objectives are met:

- We develop a pricing model covering multiple risks, which enables security pricing for *hybrid* derivatives with default risk.
- The extraction of *easy to calibrate* default probability functions for state-dependent default.

- Using observable market inputs from equity *and* bond markets, so as to value complex securities via *relative pricing* in a no-arbitrage framework, e.g.: debt-equity swaps, distressed convertibles.
- Managing *credit portfolios* and baskets, e.g. collateralized debt obligations (CDOs) and basket default swaps.

In addition, our model delivers the following *technical* innovations:

- A hybrid defaultable model combining the ideas of both, *structural* and *reduced-form* approaches.
- A risk-neutral setting in which the joint process of interest rates and equity are modeled together with the boundary conditions for security payoffs, *after* accounting for default. The martingale measure in the paper is default consistent.
- The model is embedded on a recombining lattice, providing fast computation with *polynomial complexity* for run times.
- Cross-sectional spread data permits calibration of an *implied default probability* function which dynamically changes on the state space defined by the pricing lattice.

The model is very parsimonious, and we have been able to implement it on a spreadsheet. Further research, directed at parallelizing the algorithms in this paper, and improving computational efficiency is predicated and under way. On the economic front, the model's efficacy augurs well for extensive empirical work.

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