Unbiased Capital Allocation in an Asymptotic Single Risk Factor (ASRF) Model of Credit Risk

by

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Abstract

This paper derives unbiased capital allocation rules for portfolios in which credit risk is driven by a single common factor and idiosyncratic risk is fully diversified. The methodology for setting unbiased capital allocations is developed in the context of the Black-Scholes-Merton (BSM) equilibrium model. The methodology is extended to develop an unbiased capital allocation rule for the Gaussian ASRF structural model of credit risk. Unbiased capital allocations are shown to depend on yield to maturity as well as probability of default, loss given default, and asset correlations. Unbiased capital allocations are compared to capital allocations that are set equal to unexpected loss in a Gaussian credit loss model—an approach that is widely applied in the banking industry and used to set minimum bank regulatory capital standards under the Basel II Internal Ratings Based (IRB) approach. The analysis demonstrates that the Gaussian unexpected loss approach substantially undercapitalizes portfolio credit risk relative to an unbiased capital allocation rule. The results include a suggested correction for the IRB capital assignment function. The corrected capital rule calls for a substantial increase in minimum capital requirements over the existing Basel II IRB regulatory capital function.

Key words: economic capital, credit risk internal models, Basel II, Internal Ratings Approach

JEL Classification: G12, G20, G21, G28
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1. Introduction

The market value of equity in a bank's capital structure functions as a buffer that protects all bank creditors from potential loss.[1] Other things equal, an increase in bank equity value raises the probability that a bank will fully perform on its contractual obligations. In the risk management literature, bank equity is often called economic capital, and the process of selecting the amount of equity in the bank's capital structure is called capital allocation. In practice, many banks use value-at-risk (VaR) techniques to set economic capital allocations.[2] VaR methods attempt to maximize bank leverage while ensuring that the potential default rate on a bank's outstanding debt is below a maximum target rate selected by management.[3]

This paper revisits the capital allocation problem for a portfolio of credit risks. We develop a methodology for constructing unbiased portfolio capital allocations in the context of the Black and Scholes (1973) and Merton (1974) (BSM) equilibrium asset pricing model. We apply this methodology to derive a closed form expression for unbiased economic capital allocations under Gaussian asymptotic single risk factor (G-ASRF) credit risk assumptions. The unbiased G-ASRF capital allocation estimator we develop is based on a credit portfolio's full return distribution and capital allocations are the solution to a well-defined capital structure optimization problem.

The G-ASRF assumptions are consistent with a class of models that includes the Gaussian credit loss model (GCLM) pioneered by Vasicek (1991), and extended by Finger

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[1] The capital allocation issues discussed herein apply to non-bank firms as well, but the discussion will be written to address the capital allocation problem faced by a bank.

[2] See, for example, the Basel Committee on Banking Supervision (1999).

[3] The constraint can also be described a minimum bank solvency margin (1 minus the bank's expected default rate).

(1999), Schönbucher (2000), Gordy (2003) and others. The GCLM is widely utilized throughout the industry in many risk measurement and capital allocation applications. The GCLM methodology focuses on estimating the distribution of portfolio credit losses and sets capital allocations equal to unexpected credit loss (UL). UL is defined as the difference between the target- rate-of-default quantile of the portfolio's estimated loss distribution the portfolio's expected loss measured from the same distribution. This unexpected loss methodology (UL-GCLM) is used to set regulatory capital requirements for banks under the Basel II Internal Ratings Based (IRB) approach.

We compare unbiased capital allocations set using the BSM equilibrium model to capital allocations that are set using the unbiased capital allocation estimator that is derived for the G-ASRF framework, and to capital allocations that are set using the UL-GCLM methodology. The results show that the unbiased G-ASRF capital allocation estimator provides a close approximation to the capital allocations that are set using the BSM methodology. The results also show that when capital requirements are set using the UL-GCLM methodology, capital is less than ¼ the magnitude needed to generate the target solvency margin. This finding has important ramifications for bank regulatory policy because the recently adopted Basel II Advanced Internal Ratings Based (A-IRB) approach sets regulatory capital requirements using the UL-GCLM method.

The bias in the UL-GCLM capital allocation methodology can be attributed to four separate sources. A primary source of bias owes to the GCLM's failure to recognize interest income generated by fully performing portfolio credits. In a portfolio context, full-performance credits create profits that offset losses that are generated by defaulting credits. Because the CGLM ignores profits, portfolio diversification benefits are improperly measured. As a consequence, the CGLM produces biased measures of the potential portfolio losses. In particular, the critical values in the loss tail of a credit portfolio's future value distribution—values that are key inputs into the capital allocation process–are downward biased (i.e., potential losses are over-estimated).

In terms of magnitude, the most important source of bias in the UL-GCLM capital allocation methodology is the use of UL to estimate capital requirements. This source of bias is opposite in sign (i.e., required capital estimates are understated) compared to the bias that owes to diversification mismeasurement.

A third source of bias in the UL-GCLM capital allocation methodology is the failure to rigorously account for the passage of time. This source of bias leads to an underestimate of required capital. Time is not an independent input in the GCLM and the functional form of the model's loss distribution and the associated UL measure are invariant to the capital allocation time horizon. Time is recognized only to the extent that it changes the probability of default (PD) and loss given default (LGD) values that are input into the model. In contrast, subsequent analysis will show that an unbiased capital allocation rule must explicitly account for the passage of time to ensure that the required compensations for time and credit risks are fully recognized in the construction of a capital allocation estimate.

A fourth source of bias in the UL-GCLM capital allocation methodology is the model's inability to accurately reproduce the negative return tail of the credit portfolio's return distribution. The sign of this bias depends on the solvency margin target that is used to set capital allocations. The model's inability to accurately reproduce future portfolio value distributions owes in part to the Gaussian factors driving uncertainty but also to the simplifying assumption that loss given default (LGD) is a fixed value. Equilibrium capital allocation models (e.g., the BSM model) are based on lognormal or other processes and importantly also include fully endogenous LGDs. As a consequence, equilibrium models recognize the random nature of LGDs or the statistical co-dependence among LGDs and between LGDs and PDs. These features generate return distribution characteristics that cannot be accurately reproduced in a G-ASRF framework.

Kupiec (2004c) compares the Basel II A-IRB capital requirements to the capital allocations set using a full BSM equilibrium model of credit risk and shows that the A-IRB approach sets minimum regulatory capital requirements at levels that are only about 20 percent of the capital that is required to achieve the regulatory target solvency margin of 99.9 percent. The results of this study explain the source of the bias in A-IRB capital requirements. Because the A-IRB approach is based on the UL-GCLM methodology for setting capital, the A-IRB incorporates the UL-GCLM biases. An alternative A-IRB capital function can be formulated to produce minimum capital requirements that are consistent with policy goals. In addition to PD, LGD, and correlation assumptions, this alternative A-IRB capital function requires yield to maturity (YTM) as an input. It also uses a multiplier to remove the bias generated by the Gaussian uncertainty and fixed LGD assumptions associated with the simple G-ASRF framework. In this study, the multiplier is calibrated using the BSM equilibrium model of credit risk.

The methodology for setting unbiased capital allocations is established in the context of the BSM equilibrium model of credit risk. The use of an equilibrium model ensures that the resulting capital allocation rule is consistent with equilibrium relationships that exist between PDs, LGDs, YTMs, and asset correlations. These relationships determine the benefits that can be achieved by diversifying idiosyncratic risk (i.e., trading off profit against credit losses) and implicitly determine the level of leverage that can be safely used to fund a fixed-income portfolio. The G-ASRF model, in contrast, does not incorporate the equilibrium restrictions that must be maintained among PDs, LGDs, YTMs, and correlations, and so it is not an ideal medium for exploring first principles.

An outline of this paper follows. Section 2 summarizes the general methodology for constructing unbiased economic capital allocations and demonstrates that unexpected loss measures of capital are downward biased. Section 3 revisits the credit risk capital allocation problem in the context of the Black-Scholes-Merton (BSM) model. Section 4 derives semi-closed form unbiased credit risk capital allocation rules for a single common factor version of the BSM model in which idiosyncratic risk is fully diversified. Section 5 reviews the GCLM. Section 6 introduces a new G-ASRF portfolio return model. Section 7 reviews the UL-GCLM capital allocation methodology. Section 8 derives a methodology for estimating unbiased capital allocations in the G-ASRF return model. Section 9 reports the results of a calibration exercise in which UL-GCLM capital allocation rules are compared to the unbiased capital allocation estimates from the G-ASRF return model and unbiased capital allocations estimated using the full BSM model. This exercise is used to estimate a multiplier for alternative solvency margin targets and recommend a reformulation for the Basel II A-IRB minimum capital rule. Section 10 concludes the paper.

2. UNBIASED BUFFER STOCK CAPITAL FOR CREDIT RISKS

The construction of an unbiased economic capital allocation is simplified when portfolios are composed of investments with market values that cannot become negative. For instruments on which losses have the potential to exceed their initial market value, as they can for example on short positions, futures, derivatives, or other structured products, economic capital calculations must be modified from the techniques described subsequently. In these circumstances, ensuring a minimum solvency margin may require changing portfolio investment shares and may not be achieved by altering the composition of the capital structure alone.[4] For purposes of the analysis that follows, portfolio composition is restricted to include only long positions in fixed income claims that may generate losses that are bounded above by the initial market value of the credit.

Defining an Appropriate Value-at-Risk (VaR) Measure

Let T represent the capital allocation horizon of interest. The purchased asset A, has an initial market value A_0 and a time T random value of $\tilde{A}_T$ with a cumulative density function $\Psi(\tilde{A}_T, A_T)$, and a probability density function $\psi(\tilde{A}_T, A_T)$. Let $\Psi^{-1}\left(\tilde{A}_T, 1-\alpha\right)$ represent the inverse of the cumulative density function of $\tilde{A}_T$ evaluated at $1-\alpha$, $\alpha \in [0,1]$. Define an α coverage VaR measure, $VaR(\alpha)$, as,

$$VaR(\alpha) = A_0 - \Psi^{-1}\left(\tilde{A}_T, 1-\alpha\right) \tag{1}$$

$VaR(\alpha)$ is a measure of loss that could be exceeded by at most $(1-\alpha)$ of all potential future value realizations of $\tilde{A}_T$. Expression (1) measures value-at-risk loss relative to the initial market value of the asset. When credit risk losses are bounded above by the initial invested amount, A_0, $\Psi^{-1}\left(\tilde{A}_T, 1-\alpha\right)$ is bounded below by 0.

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[4] See Kupiec (2004a) for a discussion of the capital allocation problem in these instances.

Unbiased Capital Allocation for Credit Risk

Assume, hypothetically, that a bank follows a capital allocation rule that sets equity capital equal to $VaR(\alpha)$. By definition, there is at most a $100(1-\alpha)$ percent probability that the investment's value will ever post a loss larger than $VaR(\alpha)$. The amount that must be borrowed to finance this investment under this capital allocation rule is $A_0 - VaR(\alpha)$. If the bank borrows $A_0 - VaR(\alpha)$, it must promise to pay back more than $A_0 - VaR(\alpha)$ if equilibrium interest rates and credit risk compensation are positive. Because the $VaR(\alpha)$ capital allocation rule ignores the equilibrium returns that are required by bank creditors, the probability that the bank will default on its funding debt under a $VaR(\alpha)$ capital allocation rule is greater than $(1-\alpha)$ if the bank's debts can only be satisfied by raising funds through the sale of $\tilde{A}_T$ at time T.[5]

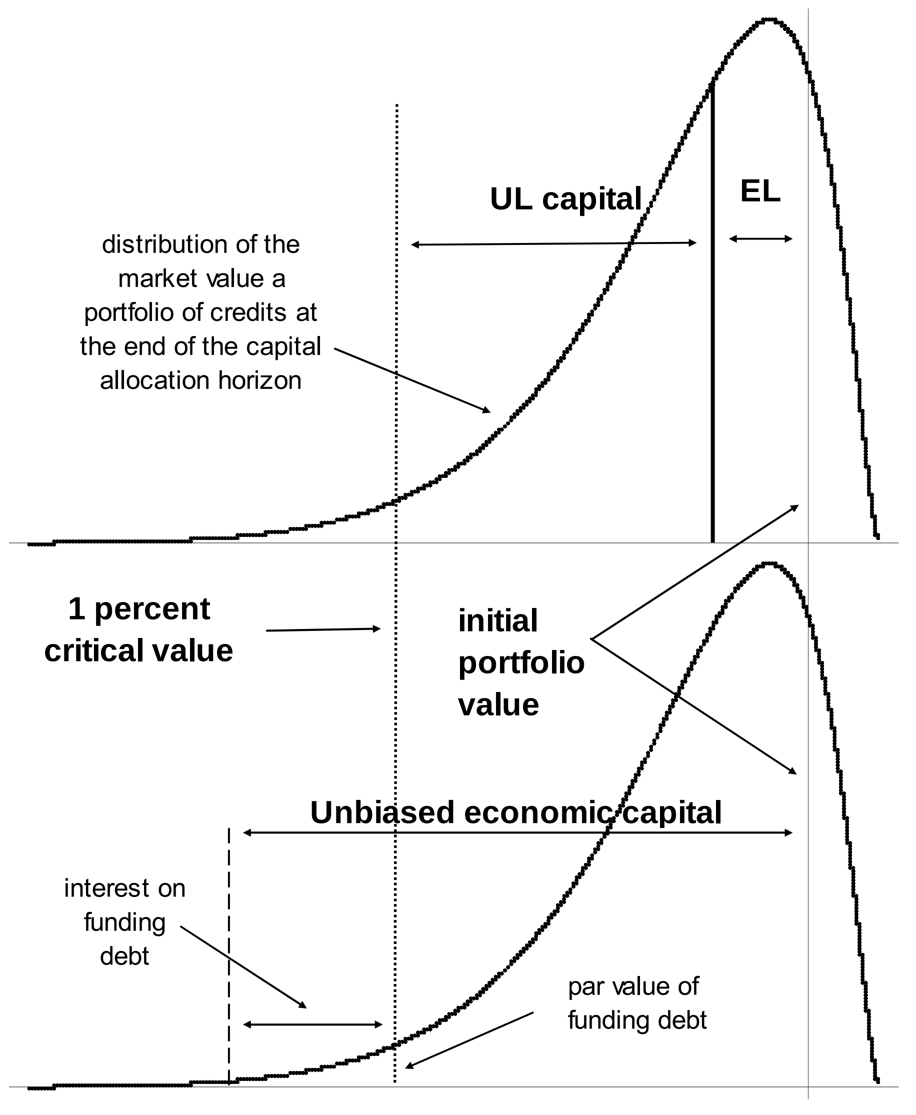
The literature on capital allocation often recommends setting economic capital equal to unexpected loss, commonly defined as, $UL(\alpha)=VaR(\alpha)-EL$, where, $EL = A_0 - \left(\int_{-\infty}^{A_0} \psi(\tilde{A}_T, A_T)\, dA_T \right)^{-1} \int_{-\infty}^{A_0} A_T\, \psi(\tilde{A}_T, A_T)\, dA_T$. The second term in the expression for EL is the expected end-of-period asset value conditional on the asset posting a loss. Because $0 \le EL \le A_0$, $VaR(\alpha) - EL \le VaR(\alpha)$. Since the default rate associated with a $VaR(\alpha)$ capital allocation rule exceeds $(1-\alpha)$ when interest rates and credit risk compensation are positive, it follows that the true default rate associated with a $UL(\alpha)$ capital allocation rule will exceed $(1-\alpha)$ if $EL>0$.

The unbiased economic capital allocation rule for a $(1-\alpha)$ target default rate (solvency rate α) is: set equity capital equal to $VaR(\alpha)$ plus the interest that will accrue on the bank's borrowings. Alternatively, set the par (maturity) value of the funding debt equal to $VaR(\alpha)$ and estimate the proceeds that will be generated by the funding debt issue. The

[5] A formal proof is given in Proposition 1 in the Appendix.

difference between the market value of the purchased asset and the proceeds from the funding debt issue is the economic capital needed to fund the investment and satisfy the solvency rate target. This capital allocation rule generalizes to the portfolio context. The unexpected loss and unbiased capital allocation methodologies are illustrated in Figure 1 for a 99 percent target solvency rate. Figure 1 illustrates the downward bias associated with the UL capital allocation methodology.

Figure 1: Alternative Capital Allocation Methodolgies



In order to estimate the equilibrium interest cost on funding debt, one must go beyond the tools of value-at-risk and employ formal asset pricing models or use empirical approximations to value a bank's funding debt. The following section modifies the BSM model to price the bank's funding debt issue and provide a rigorous derivation of the unbiased economic capital allocation methodology.

3. Unbiased Capital Allocation in a Black-Scholes-Merton (BSM) Model

If the risk-free term structure is flat and a firm issues only pure discount bond, and asset values follow geometric Brownian motion, under certain simplifying assumptions,[6] BSM established that the market value of a firm's debt issue is equal to the discounted value of the bond's par value (at the risk free rate), less the market value of a Black-Scholes put option written on the value of the firm's assets. The put option has a maturity identical to the bond's maturity, and a strike price equal to the par value of the bond. More formally, if B_0 represents the bond's initial equilibrium market value, and Par the bond's promised payment at maturity date *M*, BSM establish,

$$B_0 = Par\ e^{-r_f M} - Put(A_0, Par, M, \sigma), \tag{2}$$

where r_f represents the risk free rate and $Put(A_0, Par, M, \sigma)$ represents the value of a Black-Scholes put option on an asset with an initial value of A_0, a strike price of Par, maturity M, and an instantaneous return volatility of σ.

The default (put) option is a measure of the credit risk of the bond. Merton (1974), Black and Cox (1976), and others show that the model will generalize as to term structure assumptions, coupon payments, default barrier assumptions, and generalized volatility

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[6] There are no taxes, transactions are costless, short sales are possible, trading takes place continuously, if borrowers and savers have access to the debt market on identical risk-adjusted terms, and investors in asset markets act as perfect competitors.

structures. The capital allocation discussion that follows uses the simplest formulation of the BSM model.[7]

Modifying the BSM Model for Credit Risk Capital Allocation

In the original BSM model, the underlying firm assets exhibit market risk. To examine portfolio credit risk issues, it is necessary to modify the BSM model so that the underlying assets in an investment portfolio are themselves risky fixed income claims. Consider the case in which a bank's only asset is a risky BSM discount bond issued by an unrelated counterparty. Assume that the bank will fund this bond with its own discount debt and equity issues. In this setting, the bank's funding debt issue can be valued as a compound option.

Let $\tilde{A}_T$ and Par_P represent, respectively, the time T value of the assets that support the discount debt investment and the par value of the bond. Let Par_F represent the par value of the discount bond that is issued by the bank to fund the investment. To simplify the discussion, we restrict attention to the case where the maturity of the bank's funding debt matches the maturity of the BSM asset (both equal to M).[8] The end-of-period cash flows that accrue to the funding debt holders are,

$$Min\left[Min\left(\tilde{A}_M, Par_P\right), Par_F\right] . \tag{3}$$

In BSM model, the firm's underlying assets evolve in value according to geometric Brownian motion.

$$dA = \mu A\, dt + \sigma A\, dW \tag{4}$$

where dW is a standard Weiner process. Equation (4) implies that the physical probability distribution for the value of the underlying assets at time T is,

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[7] That is, it assumed that the term structure is nonstochastic and flat, asset volatility is constant, the underlying asset pays no dividend or convenience yield, and all debt securities are pure discount issues.

[8] Kupiec (2004a) derives the pricing expression for the funding debt in the so-called mark-to-market setting when the bank's funding debt matures before the investment.

$$\tilde{A}_T \sim A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\tilde{z}} \tag{5}$$

where $\tilde{z}$ is a standard normal random variable.

Equilibrium absence of arbitrage conditions impose restrictions on these asset's drift rate, $\mu = r_f + \lambda\sigma,$ where λ is the market price of risk. If $dA^\eta = (\mu - \lambda\sigma)A^\eta dt + A^\eta \sigma\, dz$ is defined as the "risk neutralized" process under the equivalent martingale measure, the underlying end-of-period asset value distribution under the equivalent martingale measure, $\tilde{A}_T^\eta$, is,

$$\tilde{A}_T^\eta \sim A_0\, e^{\left(r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\tilde{z}} \tag{6}$$

The initial equilibrium market value of the bank's discount bond issue is the discounted (at the risk free rate) expected value of the end-of-period funding debt cash flows taken with respect to the probability distribution for $\tilde{A}_M^\eta$. In the held-to-maturity case, when the maturity of the investment and the bank's funding debt match (both equal to M), the initial market value of the bank's funding debt is,

$$E^\eta\left[Min\left[Min\left(\tilde{A}_M, Par_P\right), Par_F\right]\right]e^{-r_f M} \tag{7}$$

The notation $E^\eta[\ ]$ denotes the expected value operator with respect to the probability density for $\tilde{A}_M^\eta$.

Unbiased Capital Allocation

Assume that the bank is investing in a BSM risky discount bond of maturity M. At maturity, the payoff of the bank's purchased bond is given by $Min\left[Par_P, \tilde{A}_M\right]$. Let $\Phi(x)$ represent the cumulative standard normal distribution function evaluated at x, and let $\Phi^{-1}(\alpha)$ represent the inverse of this function for $\alpha \in [0,1]$. The upper bound on the par (maturity) value of the funding debt that can be issued under the target solvency constraint is, $Par_F(\alpha) = \Psi^{-1}\left(\tilde{A}_T, 1-\alpha\right) = A_0\, e^{\left[\mu - \frac{\sigma^2}{2}\right]M + \sigma\sqrt{M}\,\Phi^{-1}(1-\alpha)}$. The initial market value of this funding debt issue is given by, $B_{F0}(\alpha)$,

$$B_{F0}(\alpha) = E^{\eta}\left[Min\left[Min\left(\tilde{A}_M, Par_P\right), Par_F(\alpha)\right]\right] e^{-r_f M}. \tag{8}$$

The initial equity allocation consistent with the target solvency rate α, $E(\alpha)$, is,

$$E(\alpha) = B_0 - B_{F0}(\alpha). \tag{9}$$

Portfolio Capital

The approach for formulating unbiased capital allocations can be generalized to the portfolio setting. Let $\tilde{A}_M^P$ represent the random end-of-period value of the investment portfolio where the notation includes the superscript *P* to designate portfolio values. The portfolio value's cumulative distribution function is given by, $\Psi\left(\tilde{A}_M^P, A_M^P\right)$. The maximum par value of the funding debt consistent with the target solvency margin of α is, $Par_F^P(\alpha) = \Psi^{-1}\left(\tilde{A}_M^P, 1-\alpha\right)$, and the equity capital, $E^P(\alpha)$, necessary to satisfy the minimum solvency requirement is,

$$E^P(\alpha) = \sum_{\forall i} B_{i0} - e^{-r_f M}\, E^{\overline{\eta}}\left[Min\left[\tilde{A}_M^P, \Psi^{-1}\left(\tilde{A}_M^P, 1-\alpha\right)\right]\right] \tag{10}$$

where $\sum_{\forall i} B_{i0}$ is the initial market value of the investment portfolio and $E^{\overline{\eta}}[\ ]$ represents an expectation taken with respect to the risk neutralized multivariate distribution of asset prices which support the bonds in the investment portfolio. In most cases, expression (10) requires the evaluation of high order integral that does not have closed-form solution. The next section considers portfolio capital allocation under ASRF assumptions which reduce the complexity of the capital calculations.

4. Unbiased Capital Allocation in an Asymptotic Single Factor BSM Model

The BSM framework can accommodate any number of factors in the underlying specification for asset price dynamics. Capital allocation calculations can be simplified if a portfolio is well-diversified and asset values are driven by a single common factor in addition to individual idiosyncratic factors.

Let dW_M represents a standard Wiener process common in all asset price dynamics, and dW_i represents an independent standard Weiner process idiosyncratic to the price dynamics of asset i. Assume that the asset price dynamics for firm i are given by,

$$dA_i = \mu A_i\, dt + \sigma_M A_i\, dW_M + \sigma_i A_i\, dW_i, \tag{11}$$

$$dW_i\, dW_j = \rho_{ij} = 0, \quad \forall i, j.$$
$$dW_i\, dW_M = \rho_{im} = 0, \forall i.$$

Under these dynamics, asset prices are log normally distributed,

$$\tilde{A}_{iT} = A_{i0}\, e^{\left[r_f + \lambda\sigma_M - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right]T + \left(\sigma_M \tilde{z}_M + \sigma_i \tilde{z}_i\right)\sqrt{T}}, \tag{12}$$

where $\tilde{z}_M$ and $\tilde{z}_i$ are independent standard normal random variables. Under the equivalent martingale change of measure, asset values at time T are distributed,

$$\tilde{A}_{iT}^{\eta} = A_{i0}\, e^{\left[r_f - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right]T + \left(\sigma_M \tilde{z}_M + \sigma_i \tilde{z}_i\right)\sqrt{T}}. \tag{13}$$

The correlation between geometric asset returns is,

$$Corr\left[\frac{1}{T}\ln\left(\frac{\tilde{A}_{it}}{A_{i0}}\right), \frac{1}{T}\ln\left(\frac{\tilde{A}_{jt}}{A_{j0}}\right)\right] = \frac{\sigma_M^2}{\left(\sigma_M^2 + \sigma_i^{\,2}\right)^{\frac{1}{2}}\left(\sigma_M^2 + \sigma_j^{\,2}\right)^{\frac{1}{2}}}, \forall i, j. \tag{14}$$

If the model is further specialized so that the volatilities of assets' idiosyncratic factors are assumed identical, $\sigma_i = \sigma_j = \overline{\sigma}, \forall i, j,$ the pair-wise asset return correlations are,

$$\rho = Corr\left[\frac{1}{T}\ln\left(\frac{\tilde{A}_{it}}{A_{i0}}\right), \frac{1}{T}\ln\left(\frac{\tilde{A}_{jt}}{A_{j0}}\right)\right] = \frac{\sigma_M^2}{\sigma_M^2 + \overline{\sigma}^2} \quad \forall i, j. \tag{15}$$

BSM Bond Return Distributions

The *T*-period rate of return on BSM risky bond i that is held to maturity is,

$$\tilde{M}_{iT} = \frac{1}{B_{i0}}\left(Min\left(\tilde{A}_{iT}, Par_i\right)\right) - 1. \tag{16}$$

For bonds or loans with conventional levels of credit risk, $\tilde{M}_{iT}$ is bounded in the interval $[-1, a]$, where a is a finite constant. When return realizations are in the range, $-1 < M_{iT} < 0$, M_{iT} represents the loss rate on the bond held to maturity. For realizations in the range, $0 < M_{iT} < \frac{Par_i}{B_{i0}} - 1$, the bond has defaulted on its promised payment terms, but the bond has still generates a positive return. A fully performing bond posts a return equal to $\frac{Par_i}{B_{i0}} - 1 < a$ which is finite.

A bond's physical rate of return distribution (16) has an associated equivalent martingale rate distribution,

$$\tilde{M}_{iT}^{\eta} = \frac{1}{B_{i0}}\left(Min\left(\tilde{A}^{\eta}{}_{iT}, Par_i\right)\right) - 1. \tag{17}$$

By construction, expressions (16) and (17) have identical support.

Asymptotic Portfolio Return Distribution

The *T*-period return on a portfolio of *n* risky individual credits, $\tilde{M}_T^P$, is

$$\tilde{M}_T^P \equiv \frac{\sum_{i=1}^{n} \tilde{M}_{iT} B_{i0}}{\sum_{i=1}^{n} B_{i0}} \tag{18}$$

Let $\left(\tilde{M}_T^P \middle| \tilde{z}_M = z_M\right) = \tilde{M}_T^P \middle| z_M$ represent the portfolio return conditional on a realization of the common market factor, $\tilde{z}_m = z_M$,

$$\tilde{M}_T^P \middle| z_M = \frac{\sum_{i=1}^{n} \tilde{M}_{iT} \middle| z_M \;\; B_{i0}}{\sum_{i=1}^{n} B_{i0}} \tag{19}$$

If $\psi\left(\tilde{M}_{iT} \middle| z_M, M_{iT}\right)$ represents the conditional return density function, under the single common factor assumption for asset price dynamics, $\psi\left(\tilde{M}_{iT} \middle| z_M, M_{iT}\right)$ and $\psi\left(\tilde{M}_{jT} \middle| z_M, M_{jT}\right)$ are independent for $\forall\, i \neq j$.[9]

Consider a portfolio composed of equal investments in individual bonds that share identical *ex ante* credit risk profiles. That is, assume that the bonds in the portfolio are identical regarding par value $\{Par_i = Par_j, \forall\, i, j\}$, maturity $\{T\}$, and volatility characteristics, $\{\sigma_i = \sigma_j = \bar{\sigma}, \forall\, i, j\}$. The ASRF assumptions imply that the bonds will have

[9] Independence in this non-Gaussian setting requires that an observation of the return to bond j be uninformative regarding the conditional distribution function for bond i, $\Pr\left(\tilde{M}_{it} \mid z_M\right) < a = \Pr\left(\left(\tilde{M}_{it} \mid z_M\right) < a \text{ given that } \tilde{M}_{jt} = M_{jt}\right), \forall a, i \neq j.$ This condition is satisfied under the single common factor model assumption.

conditional returns that are independent and identically distributed with a finite mean. As the number of bonds in portfolio, N, grows without bound, because $\tilde{M}_{iT} \big| z_M$ are independent and identically distributed, the Strong Law of Large Numbers requires,

$$\underset{n\to\infty}{Lim}\left[\tilde{M}_T^P \mid z_M\right] = \underset{n\to\infty}{Lim}\left[\frac{\sum_{i=1}^{n}\tilde{M}_{iT}\big|z_M}{n}\right] \xrightarrow[a.s]{} E\left[\psi\left(\tilde{M}_{iT}\big|z_M\right)\right] \; \forall\, z_M \tag{20}$$

The notation $a.s.$ indicates "almost sure" convergence (convergence with probability one).

The conditional expected value in expression (20) can be expressed as,

$$\begin{aligned}\underset{n\to\infty}{Lim}\left[\tilde{M}_T^P\big|z_M\right] &= \frac{Par_i}{B_{i0}}\left[1-\Phi\left(z_{iT}^*(z_M)\right)\right] \\ &+ \frac{Q(z_M)}{B_{i0}}\left[1-\Phi\left(-z_{iT}^*(z_M)+\gamma_{iT}\right)\right]-1\end{aligned} \tag{21}$$

where,

$$\mu_{iT}(z_M) = \ln[A_{i0}] + \left[r_f + \lambda\sigma_M - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right]T + z_M\sigma_M\sqrt{T}$$

$$\gamma_{iT} = \sigma_i\sqrt{T}$$

$$z_{iT}^*(z_M) = \frac{\ln[Par_i] - \mu_{iT}(z_M)}{\gamma_{iT}}$$

$$Q(z_M) = e^{\mu_{iT}(z_M)+\frac{\gamma_{iT}^2}{2}}$$

The unconditional portfolio return distribution is as an implicit function of z_M,

$$\Pr\left[\tilde{M}_T^P < \left(\begin{array}{l}\frac{Par_i}{B_{i0}}\left[1-\Phi\left(z_{iT}^*(z_M)\right)\right] \\ \qquad +\frac{Q(z_M)}{B_{i0}}\left[1-\Phi\left(-z_{iT}^*(z_M)+\gamma_{iT}\right)\right]\end{array}\right)-1\right]=\Phi(z_M),\ z_M\in(-\infty,\infty) \tag{22}$$

The unconditional distribution for the investment portfolio's end-of-period T value is,

$$\Pr\left[\sum_{i=1}^n B_{i0}\left(\tilde{M}_T^P+1\right)<\left(\frac{\sum_{i=1}^n B_{i0}}{B_{i0}}\right)\left(\begin{array}{l}Par_i\left[1-\Phi\left(z_{iT}^*(z_M)\right)\right] \\ \qquad +Q(z_M)\left[1-\Phi\left(-z_{iT}^*(z_M)+\gamma_{iT}\right)\right]\end{array}\right)\right] \tag{23}$$

$$=\Phi(z_M),\ z_M\in(-\infty,\infty)$$

The T-period equivalent martingale unconditional return distribution, $\tilde{M}_T^{P\eta}$, can be written,

$$\Pr\left[\tilde{M}_T^{P\eta} < \left(\begin{array}{l}\frac{Par_i}{B_{i0}}\left[1-\Phi\left(z_{iT}^{\eta*}(z_M)\right)\right] \\ \qquad +\frac{Q^\eta(z_M)}{B_{i0}}\left[1-\Phi\left(-z_{iT}^{\eta*}(z_M)+\gamma_{iT}\right)\right]\end{array}\right)-1\right]=\Phi(z_M), \tag{24}$$

$$z_M\in(-\infty,\infty)$$

where,

$$\mu_{iT}^\eta(z_M)=\ln[A_{i0}]+\left[r_f-\frac{1}{2}\left(\sigma_M^2+\sigma_i^2\right)\right]T+z_M\sigma_M\sqrt{T}$$

$$z_{iT}^{\eta*}(z_M)=\frac{\ln[Par_i]-\mu^\eta{}_{iT}(z_M)}{\gamma_{iT}}$$

$$Q^\eta(z_M)=e^{\mu_{iT}^\eta(z_M)+\frac{\gamma_{iT}^2}{2}}$$

The unconditional equivalent martingale distribution for the portfolio's time-T value is,

$$\Pr\left[\sum_{i=1}^{n} B_{i0}\left(\tilde{M}_T^{P\eta}+1\right) < \left(\frac{\sum_{i=1}^{n} B_{i0}}{B_{i0}}\begin{pmatrix} Par_i\left[1-\Phi\left(z_{iT}^{\eta *}(z_M)\right)\right] \\ +Q^{\eta}(z_M)\left[1-\Phi\left(-z_{iT}^{\eta *}(z_M)+\gamma_{iT}\right)\right]\end{pmatrix}\right)\right] \tag{25}$$

$$= \Phi(z_M),\ z_M \in (-\infty,\infty)$$

Unbiased Portfolio Capital Allocation

The maximum par value of a funding debt issue consistent with a target solvency rate of α is $Par_F^P(\alpha)=\Psi^{-1}\left(\sum_{i=1}^{n} B_{i0}\left(\tilde{M}_T^{P}+1\right), 1-\alpha\right)$. If this par value is expressed as a proportion of the investment portfolio's initial market value, the optimal par value of funding debt is $par_F^P(\alpha)=\Psi^{-1}\left(\tilde{M}_T^{P}+1, 1-\alpha\right)$. In the ASRF BSM case, the par value of funding debt can be determined by setting $z_M=\Phi^{-1}(1-\alpha)$ and using expression (21) to solve for the end-of-horizon portfolio critical value,

$$par_F^P(\alpha)=\begin{pmatrix} \frac{Par_i}{B_{i0}}\left[1-\Phi\left(z_{iT}^{*}\left(\Phi^{-1}(1-\alpha)\right)\right)\right] \\ +\frac{Q(z_M)}{B_{i0}}\left[1-\Phi\left(-z_{iT}^{*}\left(\Phi^{-1}(1-\alpha)\right)+\gamma_{iT}\right)\right]\end{pmatrix} \tag{26}$$

To solve for the market value of the funding debt, it is necessary to solve for $\left(1-\hat{\alpha}^{\eta}\right)$, the probability that the funding debt will default under the equivalent martingale distribution,

$$\left(1-\hat{\alpha}^{\eta}\right) \ni par_F^P(\alpha)=\begin{pmatrix} \frac{Par_i}{B_{i0}}\left[1-\Phi\left(z_{iT}^{\eta *}\left(\Phi^{-1}\left(1-\hat{\alpha}^{\eta}\right)\right)\right)\right] \\ +\frac{Q^{\eta}(z_M)}{B_{i0}}\left[1-\Phi\left(-z_{iT}^{\eta *}\left(\Phi^{-1}\left(1-\hat{\alpha}^{\eta}\right)\right)+\gamma_{iT}\right)\right]\end{pmatrix} \tag{27}$$

The critical value of the equivalent martingale market factor that satisfies expression (27) is,

$$\hat{z}_M = \Phi^{-1}(1-\alpha) + \frac{\lambda}{\sqrt{T}} \tag{28}$$

and the risk neutral probability that the funding debt will default is,

$$1 - \hat{\alpha}^{\eta} = \Phi(\hat{z}_M) \tag{29}$$

Expressed as a proportion of the investment portfolio's initial market value, the initial market value of the funding issue, $b_{F0}^{P}(\alpha)$, is equal to the expected value of the bond's discounted terminal gross return, where the expectation is taken with respect to the equivalent martingale measure and discounting occurs at r_f,

$$b_{F0}^{P}(\alpha) = e^{-r_f T}\begin{pmatrix} \int_{-\infty}^{\hat{z}_M}\left[\frac{Par_i}{B_{i0}}\left[1-\Phi\left(z_{iT}^{\eta *}(z_M)\right)\right]\right]\phi(z_M)\,dz_M \\ +\int_{-\infty}^{\hat{z}_M}\left[\frac{Q^{\eta}(z_M)}{B_{i0}}\left[1-\Phi\left(-z_{iT}^{\eta *}(z_M)+\gamma_{iT}\right)\right]\right]\phi(z_M)\,dz_M \\ +\hat{\alpha}^{\eta}\, par_F^{P}(\alpha) \end{pmatrix} \tag{30}$$

The economic capital allocation for the portfolio, expressed as a proportion of the portfolio's initial market value, $K^{BSM}(\alpha)$ is,

$$K^{BSM}(\alpha) = 1 - e^{-r_f T}\begin{pmatrix} \int_{-\infty}^{\hat{z}_M}\left[\frac{Par_i}{B_{i0}}\left[1-\Phi\left(z_{iT}^{\eta *}(z_M)\right)\right]\right]\phi(z_M)\,dz_M \\ +\int_{-\infty}^{\hat{z}_M}\left[\frac{Q^{\eta}(z_M)}{B_{i0}}\left[1-\Phi\left(-z_{iT}^{\eta *}(z_M)+\gamma_{iT}\right)\right]\right]\phi(z_M)\,dz_M \\ +\hat{\alpha}^{\eta}\, par_F^{P}(\alpha) \end{pmatrix} \tag{31}$$

The dollar value capital requirement is $\sum_{i=1}^{n} B_{i0}\, K^{BSM}(\alpha)$. Because idiosyncratic risk is fully diversified, when an additional credit is added to the portfolio, the marginal capital required to maintain the target solvency margin is equal to the average capital requirement for the portfolio provided that capital is measured as a proportion of the investment's initial market value (expression (31)).

5. The Asymptotic Single Risk Factor (ASRF) Loss Model

The Gaussian asymptotic single risk factor model of credit losses (GCLM), pioneered by Vasicek (1991), has become the industry standard model used for credit risk capital allocation. The GCLM specifies capital allocation as a function of individual credit's PD, LGD, maturity, and EAD. Studies by Finger (1999), Belkin, et. al. (1998), Lucas (2001) and Gordy (2003) have extended the GCLM and developed computational methods that facilitate the adoption of GCLM–based capital allocation methods. The GCLM is the model that is used to set minimum regulatory capital requirements under the Basel II IRB approach.

The GCLM is based on a set of assumptions that mimic the way credit risks arise in the BSM model. In contrast to the equilibrium absence of arbitrage foundations of the BSM approach, the GCLM does not recognize the equilibrium relations that must hold between asset valuation processes, LGD assumptions, and default thresholds. The BSM model, for example, is an arbitrage-free model of credit risk with PDs, LGDs, credit yields, and asset valuation dynamics that are internally consistent with capital market equilibrium conditions. The GCLM does not have this consistency property and will allow any PDs, LGDs and correlations to be entered as inputs. The potential shortcomings associated with ignoring these equilibrium relationships are offset with substantial simplifications to the mathematics that are required to evaluate credit risk and allocate capital.

The standard GCLM specification follows. It is assumed that uncertainty associated with the time T value of firm i's assets can be modeled as a standard normal random variable with the following properties,

$$\begin{aligned} &\tilde{V}_{iT} = \sqrt{\rho}\,\tilde{e}_{MT} + \sqrt{1-\rho}\,\tilde{e}_{iT} \\ &\tilde{e}_{MT} \sim \phi(e_{MT}) \\ &e_{iT} \sim \phi(e_{iT}), \\ &E(\tilde{e}_{iT}\,\tilde{e}_{jT}) = E(\tilde{e}_{MT}\,\tilde{e}_{jT}) = 0\ \forall\, i,\, j. \end{aligned} \tag{32}$$

$\tilde{V}_{iT}$ is assumed to be normally distributed with $E\left(\tilde{V}_{iT}\right)=0,$ and $E\left(\tilde{V}_{iT}^{2}\right)=1.$ $\tilde{e}_{MT}$ is the market factor common to all firm asset values. The correlation between asset values is ρ.

Firm i is assumed to default on its debt at time T when $\tilde{V}_{iT} < D_i$. The loss incurred should the firm default, LGD, is specified exogenously. In most applications, LGD is measured as a proportion of the initial loan amount and typically is set equal to a constant value calibrated from historical loss data. Given the GCLM assumptions, the unconditional probability that firm i will default on its debt at time T is, $PD = \Phi\left(D_i\right).$ The time-horizon T does not play an independent role in the GCLM model and so the subscript T will be suppressed in the remainder of the discussion. In practice, time is implicitly recognized through the selection of input values for PD and LGD.

Consider a portfolio composed of n credits that have identical initial market values, correlations, ρ, and default thresholds, $D_i = D.$ In order to derive the loss distribution for a portfolio of these credits, it is useful to define an indicator function,

$$\tilde{I}_i = \begin{cases} 1 & if \quad \tilde{V}_i < D \\ 0 & otherwise \end{cases} \tag{33}$$

Under the GCLM assumptions, $\tilde{I}_i$ has a binomial distribution with an expected value of $\Phi(D)$. Define $\tilde{I}_i \mid e_M$ to be the value of the indicator function conditional on a realized value for e_M. Conditional default indicators are independent and identically distributed binomial random variables,

$$\begin{aligned}
&E\left(\tilde{I}_i \mid e_M\right) = \Phi\left(\frac{D-\sqrt{\rho}\, e_M}{\sqrt{1-\rho}}\right), \quad \forall\, i \\
&E\left(\tilde{I}_i \mid e_M - E\left(\tilde{I}_i \mid e_M\right)\right)^2 = \Phi\left(\frac{D-\sqrt{\rho}\, e_M}{\sqrt{1-\rho}}\right)\left(1-\Phi\left(\frac{D-\sqrt{\rho}\, e_M}{\sqrt{1-\rho}}\right)\right), \quad \forall\, i \\
&E\left(\left(\tilde{I}_i \mid e_M\right)\left(\tilde{I}_j \mid e_M\right)\right) = 0, \quad \forall\, i \neq j.
\end{aligned} \tag{34}$$

Define $\tilde{X} \mid e_M$ as the proportion of credits in the portfolio that default conditional on a realization of e_M, $\tilde{X} \mid e_M = \dfrac{\sum_{i=1}^{n}\left(\tilde{I}_i \mid e_M\right)}{n}$. Because $\tilde{I}_i \mid e_M$ are independent and identically distributed, the Strong Law of Large Numbers requires, for all e_M,

$$\lim_{n\to\infty}\left(\tilde{X} \mid e_M\right) = \lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}\left(\tilde{I}_i \mid e_M\right)}{n}\right) \xrightarrow[a.s.]{} E\left(\tilde{I}_i \mid e_M\right) = \Phi\left(\frac{D-\sqrt{\rho}\, e_M}{\sqrt{1-\rho}}\right) \tag{35}$$

The unconditional distribution function of $\tilde{X}$ can be derived by the change of variable technique using expression (35) and information on the density of $\tilde{e}_M$. Because realized values of *X* are monotonically decreasing in e_M,

$$\begin{aligned}
\Pr\left[\tilde{X} \leq x\right] &= \Pr\left[\tilde{e}_M \geq \frac{D-\Phi^{-1}(x)\sqrt{1-\rho}}{\sqrt{\rho}}\right] \\
&= \Pr\left[-\tilde{e}_M \leq \frac{\sqrt{1-\rho}\;\Phi^{-1}(x)-D}{\sqrt{\rho}}\right] = \Phi\left(\frac{\sqrt{1-\rho}\;\Phi^{-1}(x)-D}{\sqrt{\rho}}\right)
\end{aligned} \tag{36}$$

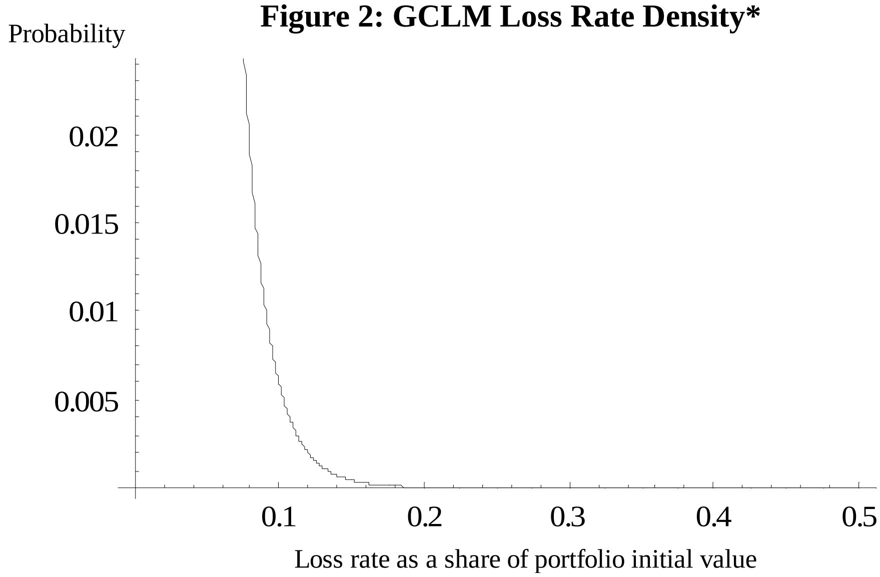
Substituting for the default barrier, $D = \Phi^{-1}(PD)$, the unconditional cumulative distribution function for $\tilde{X}$, the proportion of portfolio defaults, is given by,

$$\Pr\left[\tilde{X} \le x\right] = \Phi\left(\frac{\sqrt{1-\rho}\ \Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right), \quad x \in [0,1] \tag{37}$$

The unconditional probability density function for $\tilde{X}$ is given by,

$$\frac{\delta \Phi\left(\frac{\Phi^{-1}(x)\sqrt{1-\rho} - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)}{\delta x} = \sqrt{\frac{1-\rho}{\rho}}\ Exp\left[\frac{1}{2}\left(\Phi^{-1}(x)\right)^2 - \frac{1}{2\rho}\left(\Phi^{-1}(PD) - \sqrt{1-\rho}\ \Phi^{-1}(x)\right)^2\right] \tag{38}$$

Figure 2: GCLM Loss Rate Density*



* Asymptotic portfolio of credits with PD=1 percent, LGD=50 percent, and correlation= 20 percent.

If LGD represents the proportion of a credit's initial value that is lost if a credit defaults, portfolio credit losses measured as a proportion of initial portfolio value have the following distribution,

$$\Pr\left[LGD\cdot\tilde{X}\le LGD\cdot x\right]=\Phi\left(\frac{\Phi^{-1}(x)\sqrt{1-\rho}-\Phi^{-1}(PD)}{\sqrt{\rho}}\right),\quad x\in[0,1] \tag{39}$$

Figure 2 plots the probability density function for the GCLM loss rate on a portfolio comprised of credits with $PD=.01$, $\rho=.20$, and $LGD=.50$.

6. PORTFOLIO RETURNS UNDER GAUSSIAN ASRF ASSUMPTIONS

Expression (39) describes the cumulative probability distribution for the credit losses that may be realized on a portfolio credits under Gaussian ASRF (G-ASRF) assumptions. To estimate an unbiased capital allocation, it is necessary to derive the end-of-horizon return distribution for a portfolio of credits.[10] Let *YTM* represent the return on the initial market value of an individual credit that will be earned should the credit mature without defaulting. *YTM* is the conventional yield-to-maturity measure for simple loans or bonds. Conditional on a realized value of $\tilde{X}=X$, the end-of-horizon return on a portfolio composed of credits that have identical initial market values, correlations, ρ, and default thresholds, $D_i=D$, is given by,

$$\left(R_p \mid X\right)=YTM-(YTM+LGD)\,X \tag{40}$$

The portfolio's conditional end-of-period value is monotonically decreasing in X.

The unconditional cumulative return distribution for the asymptotic portfolio ($n\to\infty$), $R_p\left(\tilde{X}\right)$ is,

[10] In the remainder of the discussion, consistent with the Basle A-IRB capital rules, the horizon is assumed to be 1 year.

$$\Pr\left[R_p\left(\tilde{X}\right) \le R_P\left(X\right)\right] = 1 - \Pr\left[\tilde{X} < \left(\frac{YTM - R_p}{YTM + LGD}\right)\right]$$

$$= \Phi\left(\frac{\Phi^{-1}\left(PD\right) - \sqrt{1-\rho}\;\Phi^{-1}\left(\frac{YTM - R_p}{YTM + LGD}\right)}{\sqrt{\rho}}\right), \quad R_p \in \left[-(1-LGD), YTM\right] \tag{41}$$

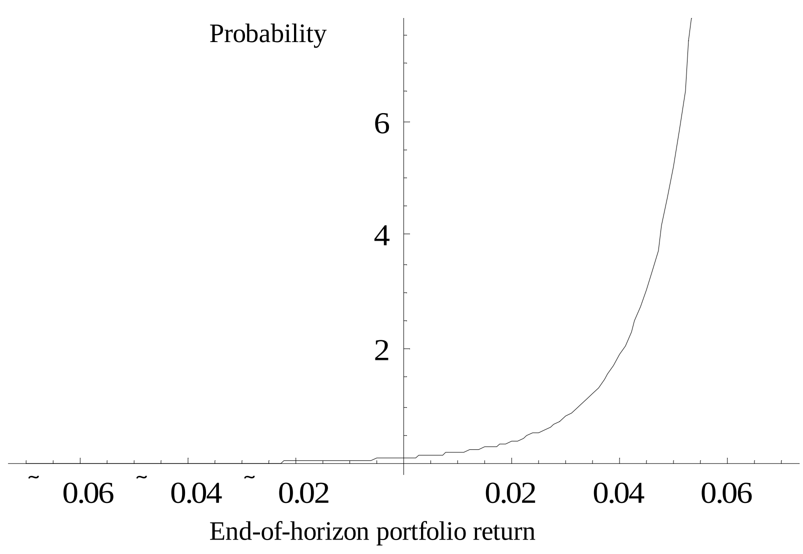
The unconditional probability density function for $\tilde{R}_P$ is given by,

$$\frac{\delta\left[\Phi\left(\frac{\Phi^{-1}\left(PD\right) - \sqrt{1-\rho}\;\Phi^{-1}\left(\frac{YTM - R_p}{YTM + LGD}\right)}{\sqrt{\rho}}\right)\right]}{\delta R_P}$$

$$= \frac{1}{YTM + LGD}\sqrt{\frac{1-\rho}{\rho}}\; Exp\left[\frac{1}{2}\left(\Phi^{-1}\left(\frac{YTM - R_p}{YTM + LGD}\right)\right)^2 - \frac{1}{2\rho}\left(\Phi^{-1}(PD) - \sqrt{1-\rho}\;\Phi^{-1}\left(\frac{YTM - R_p}{YTM + LGD}\right)\right)^2\right] \tag{42}$$

Figure 3 plots the end-of-horizon return distribution for a G-ASRF asymptotic portfolio comprised of credits with $PD = .01$, $\rho = .20$, $YTM = .07$, and $LGD = .50$.

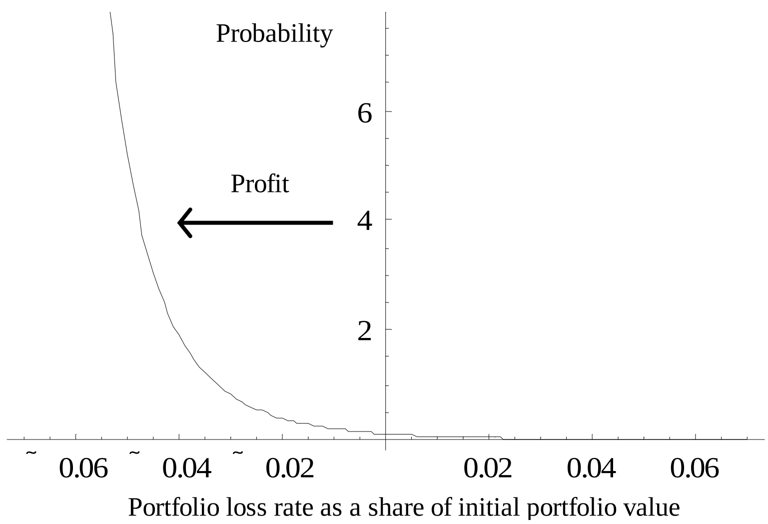
Figure 3: G-ASRF Portfolio Return Density*



*Asymptotic portfolio of credits with PD=1 percent, LGD=50 percent, GND=7 percent, and correlation=20 percent.

The negative of the portfolio return distribution defines the portfolio loss distribution. Figure 4 plots the end-of-horizon G-ASRF portfolio loss distribution for a portfolio comprised of credits with $PD = .01$, $\rho = .20$, $YTM = .07$, and $LGD = .50$.

Figure 4: G-ASRF Portfolio Loss Rate Density Calculated from Total Portfolio Returns



*Asymptotic portfolio of credits with PD=1 percent, LGD=50 percent, GND=7 percent, and correlation=20 percent.

The distributions plotted in Figures 2 and 4 represent different measures of the loss rate on an identical portfolio of credits. The potential losses plotted in Figure 2 exceed those in Figure 4 because the GCLM does not recognize the diversification benefits that are provided by fully performing credits. Table 1, which reports the 99 percent critical values of both loss rate distributions, further clarifies the difference between these loss distribution models and indicates that the GCLM model substantially overstates potential portfolio losses.[11]

The GCLM truncates a fully-performing credit's loss to zero. As a consequence, the positive returns earned by credits are not recognized. The positive returns earned on fully performing credits are the most important source of diversification in a fixed income portfolio subject to credit risk, and these diversification benefits are not recognized by the GCLM.

Table 1: Alternative Measures of the Critical Value of an G-ASRF Portfolio Loss Distribution

Probability of default in percent	99 percent coverage critical value from GCLM distribution (in percent of portfolio initial value)	99 percent coverage critical value from G-ASRF loss distribution calculated from total portfolio returns (in percent of portfolio initial value)
1	3.763	-2.711
2	6.431	0.331
3	8.685	2.981
4	10.672	5.173
5	12.479	7.226

Estimates for portfolios with correlation=20 percent, LGD=50 percent and GND=7 percent.

7. Capital Allocation using GCLM Unexpected Loss

Credit value-at-risk (VaR) techniques recommend setting economic capital equal to *UL*. For a solvency margin target of α, $UL(\alpha)$ is defined as the difference between the $(1-\alpha)$ critical value of the GCLM loss rate distribution, less the portfolio's expected loss

[11] This does not imply that the GCLM overestimates capital requirements. In the GCLM framework, capital requirements are set equal to unexpected loss, and the unexpected loss measure imparts a large bias that is opposite in sign.

rate, $LGD \cdot PD.$ The $(1-\alpha)$ critical value of the GCLM loss rate distribution that sets a coverage rate of α is given by,

$$LGD \cdot \Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}} \right) \tag{43}$$

The capital allocation measured as a percentage of the investment portfolio's initial value is,

$$K^{GCLM}(\alpha) = LGD \cdot \Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}} \right) - LGD \cdot PD \tag{44}$$

Expression (44) is the basic formula that is used to set minimum regulatory capital requirements under the Basel II A-IRB approach.[12]

8. Unbiased Capital Allocation under G-ASRF Assumptions

Estimation of an unbiased capital allocation is a two step process requiring: (a) an estimate the maximum maturity value of funding debt that can be supported by an investment portfolio while maintaining the target solvency margin; and, (b) an estimate of the current market value of this optimally-sized debt issue. The capital allocation that satisfies the solvency margin target is the difference between the market value of the investment portfolio and the issuance proceeds from the funding debt issue. This section applies this methodology to the G-ASRF return model. Throughout this discussion, consistent with

[12] See Basel Committee on Banking Supervision (2004), p. 49, paragraph 272. In addition to this equation, the Basel II IRB approach includes an equation that defines the regulatory correlation factor, ρ, and an additional maturity adjustment multiplier for expression (44).

Basel II IRB assumptions, the capital allocation time horizon and the maturities of the credits in the investment portfolio are assumed to be 1 year.

Under a target solvency margin of α, the par value of the funding debt is determined by the $(1-\alpha)$ critical value of the G-AFRF return distribution,

$$\Psi^{-1}\left(\tilde{R}_P, 1-\alpha\right) = YTM - (YTM + LGD)\Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right) \tag{45}$$

The par value of the funding debt measured as a proportion of the investment portfolio's initial value is,

$$par_F^{G-ASRF}(\alpha) = \left(1 + YTM - (YTM + LGD)\Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)\right) \tag{46}$$

Following the procedures that were used to set an unbiased capital allocation under the BSM model, measured as a share of the investment portfolio's initial market value, an unbiased capital allocation in the G-ASRF model is,

$$K^{G-ASRF}(\alpha) = 1 - e^{-r_f} E^{\bar{\eta}}\left[1 + YTM - (YTM + LGD)\Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)\right] \tag{47}$$

Because the G-ASRF model is not specified in a way that ensures consistency with absence of arbitrage equilibrium conditions, it does not contain the information necessary to construct the equivalent martingale measure to price the funding debt. In some cases it may be possible to closely approximate the funding debt's initial market value using observed yield data.

If the portfolio is a 100 percent debt financed, in the absence of taxes or government safety net subsidies to the bank, the Modigliani-Miller (1958) theorem ensures that the equilibrium interest rate required on the bank's funding debt at the time of issuance is equal to the *YTM* on the credits in the banks investment portfolio.[13] When the share of equity funding is increased above zero, the equilibrium *YTM* at issuance will decline. Provided that bank leverage does not exceed 100 percent, the *YTM* on the credits in the bank's investment portfolio is an upper bound on the equilibrium issuance *YTM* for the bank's funding debt. Using *YTM* as a conservative estimate of the required market rate of return on the bank's funding debt at issuance, the initial market value of the funding debt measured as a proportion of the investment portfolio's initial market value is approximately,

$$b_{F0}^{G-ASRF} \approx \frac{1}{1+YTM}\left(1+YTM-(YTM+LGD)\Phi\left(\frac{\sqrt{\rho}\,\Phi^{-1}(\alpha)+\Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)\right) \tag{48}$$

Using this approximation, measured as a proportion of the portfolio's initial market value, the capital allocation required for the investment portfolio is,

$$\hat{K}^{G-ASRF}(\alpha) \approx \frac{YTM+LGD}{1+YTM}\Phi\left(\frac{\sqrt{\rho}\,\Phi^{-1}(\alpha)+\Phi^{-1}(PD)}{\sqrt{1-\rho}}\right) \tag{49}$$

The calibration analysis of the subsequent section will show that $\hat{K}^{G-ASRF}(\alpha)$ will produce mildly biased estimates of the true capital needed to achieve a target solvency margin. The G-ASRF capital estimate (49) is not expected to produce unbiased estimates because of the YTM funding cost approximation, the Gaussian uncertainty assumption, and in part because the model does not recognize the random nature of LGD. We introduce a

[13] Merton (1974) provides a more modern proof of the Modigliani-Miller theorem. Kupiec (2004b) discusses the implications of non-priced implicit or explicit safety net guarantees on a bank's capital allocation process.

multiplier, M, to correct for the bias.[14] Under this modification, individual credit and portfolio capital requirements, measured as a percentage of initial market value, are given by,

$$\hat{\hat{K}}^{G-ASRF}(\alpha) = M \cdot \left[\frac{YTM + LGD}{1+YTM} \Phi\left(\frac{\sqrt{\rho}\ \Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}} \right) \right] \tag{50}$$

The subsequent section will calibrate *M* for different solvency margin targets using the BSM model as the equilibrium model of credit risk. The use of a different equilibrium model benchmark will produce a different estimated value for the multiplier in expression (50).

9. Capital Allocations under Alternative Methodologies

In the analysis that follows, portfolio capital requirements are calculated using the unbiased capital allocation rule for an ASRF BSM model (expression (31)), the UL-GCLM methodology (expression (44)), and the G-ASRF return model unbiased capital allocation estimate (expression (49)) for asymptotic portfolios with a wide range of risk characteristics. Capital allocations are estimated for the 99.9 percent solvency margin and the BSM capital allocation is taken as the benchmark or "true" capital needed to achieve the solvency margin target. All bias is measured relative to the BSM benchmark.

The assumptions regarding the asset price dynamics that are maintained throughout the analysis appear in Table 2. All individual credits are assumed to have identical firm specific risk factor volatilities of 20 percent. The common factor has a volatility of 10 percent

[14] It may be possible to alter the bias in capital allocations by modeling LGD as a correlated random variable as suggested for example in Fyre (2000), or Pykhtin (2003), but such modifications will not attenuate the other sources of bias, and it is possible that these alternative sources of bias may be offsetting.

and the market price of risk is set at 10 percent. The risk free rate is 5 percent. The market and firm specific factor volatilities imply an underlying geometric asset return correlation of 20 percent.

All credits in an asymptotic portfolio are assumed to have the same initial value and all share an identical *ex ante* credit risk profile that is determined by the par value and maturity of the credit. For a given maturity, the par values of individual credits are altered to change the credit risk characteristics of a portfolio. The calibration analysis focuses on a one-year capital allocation horizon.

Table 2: Calibration Assumptions

risk free rate	$r_f = .05$
market price of risk	$\lambda = .10$
market factor volatility	$\sigma_M = .10$
Firm specific volatility	$\bar{\sigma}_i = .20$
Initial market value of assets	$A_0 = 100$
correlation between asset returns	$\rho = .20$

Under the single common factor BSM model assumptions, the physical probability that a BSM bond defaults is,

$$PD = \Phi\left(z_i^{df}\right)$$

$$z_i^{df} = \frac{Log\left(Par_i\right) - Log\left(A_{i0}\right) - \left(r_f + \lambda\sigma_M - \frac{\sigma_M^2 + \bar{\sigma}_i^2}{2}\right)T}{\sqrt{T}\sqrt{\sigma_M^2 + \bar{\sigma}_i^2}} \tag{51}$$

The expected value of the bond's payoff given that it defaults is,

$$E\left[\tilde{A}_{iT} \mid A_{iT} < Par_i\right] = \frac{1}{\Phi\left(z_i^{df}\right)} \int_{-\infty}^{z_i^{df}} A_{i0} e^{\left(r_f + \lambda\sigma_M - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right)T + \sqrt{T\left(\sigma_M^2 + \sigma_i^2\right)}z} \phi(z) dz . \qquad (52)$$

A bond's LGD measured from initial market value is,

$$LGD = 1 - \frac{1}{B_{i0}\Phi\left(z_i^{df}\right)} \int_{-\infty}^{z_i^{df}} A_{i0} e^{\left(r_f + \lambda\sigma_M - \frac{1}{2}\left(\sigma_M^2 + \sigma_i^2\right)\right)T + \sqrt{T\left(\sigma_M^2 + \sigma_i^2\right)}z} \phi(z) dz . \qquad (52)$$

This LGD measure does not discount the expected value of the terminal payoff in default.

Table 3: Credit Risk Characteristics of 1-Year Credits

				in percent		
par value	initial market value	probability of default in percent	expected value given default	loss given default from initial value	loss given default from par value	yield to maturity
55	52.31	0.23	51.58	1.40	6.22	5.142
56	53.26	0.30	52.45	1.53	6.35	5.145
57	54.2	0.38	53.31	1.64	6.47	5.166
58	55.15	0.48	54.17	1.78	6.60	5.168
59	56.1	0.59	55.03	1.91	6.73	5.169
60	57.04	0.73	55.88	2.03	6.87	5.189
61	57.98	0.90	56.73	2.16	7.00	5.209
62	58.92	1.09	57.57	2.29	7.14	5.227
63	59.86	1.31	58.41	2.42	7.28	5.246
64	60.8	1.57	59.25	2.55	7.43	5.263
65	61.73	1.86	60.08	2.68	7.57	5.297
66	62.66	2.20	60.90	2.80	7.72	5.330
67	63.59	2.57	61.73	2.93	7.87	5.362
68	64.51	3.00	62.54	3.05	8.03	5.410
69	65.43	3.47	63.35	3.17	8.18	5.456
70	66.34	3.99	64.16	3.28	8.34	5.517

The calibration analysis includes 16 portfolios. The risk characteristics of these portfolios' credits are reported in Table 3. Individual credit PDs range from 23 basis points—for a bond with par values of 55, to 3.99 percent for a bond with a par value of 70. The LGD characteristics (measured from initial market value) range from 1.40 percent to 3.28 percent. Low LGDs are a signature characteristic of the BSM model in a short horizon setting and the LGDs of the bonds examined in this analysis are modest relative to the observed default loss history on corporate bonds.[15] The UL-GCLM and G-ASRF return model capital allocation rules explicitly account for loss given default, so *a priori*, there is no reason to expect that any specific set of loss given default values will compromise the performance of these capital allocation methodologies.

The alternative recommendations for capital are reported in Table 4. The capital requirements generated under the UL-GCLM rule (expression (44)) are far smaller than the capital needed to achieve the regulatory target default rate of 0.1 percent. The UL-GCLM capital shortfall depends on the characteristics of the credit portfolio. If one constructs a multiplier to correct for the UL-GCLM bias, the multiplier would range from 3.8 to 5.7 for the credits analyzed in this calibration exercise. These results are consistent with the magnitude of the bias in Basel II A-IRB minimum capital requirements that has been identified in Kupiec (2004c).

The G-ASRF return model capital allocation estimator (expression (49)), while downward biased relative to the BSM capital allocation (expression (31)), produces capital

[15] Some industry credit risk models include a stochastic default barrier such as in the Black and Cox (1976) model to increase the LGD relative to a basic BSM model and thereby improve correspondence with observed market data.

estimates that are close to achieving the target solvency objective. If one constructs a multiplier to correct the estimator's average bias (as in expression (50)), the multiplier will vary according to the target solvency margin selected. Calibration results for a target 99.9 percent solvency margin reported in Table 4, suggest that a capital allocation rule based on expression (50) with a multiplier $M \approx 1.26$ would produce equity capital allocations that are approximately consistent with the targeted solvency margin of 99.9 percent. Table 5 reports the alternative model capital estimates for a 98 percent target solvency margin. At the 98 percent solvency level, the G-ASRF capital allocation estimator (expression (49)) overstates the true capital required, and so the bias correction factor in expression (50) is slightly less than 1.

Table 4: Capital Allocation for a 99.9 Percent Solvency Margin Under Alternative Models

par value	probability of default in percent	unbiased BSM portfolio capital	unbiased G-ASRF capital estimate	UL-GCLM capital estimate	implied G-ASRF multiplier
55	0.233	0.396	0.325	0.070	1.217
56	0.298	0.487	0.402	0.092	1.210
57	0.379	0.593	0.486	0.117	1.221
58	0.476	0.715	0.584	0.149	1.224
59	0.593	0.854	0.734	0.184	1.164
60	0.732	1.011	0.809	0.225	1.249
61	0.896	1.187	0.951	0.274	1.249
62	1.088	1.384	1.100	0.328	1.258
63	1.311	1.601	1.264	0.388	1.267
64	1.568	1.839	1.445	0.456	1.273
65	1.862	2.098	1.639	0.530	1.280
66	2.196	2.379	1.852	0.610	1.285
67	2.574	2.681	2.073	0.696	1.293
68	2.997	3.005	2.316	0.789	1.298
69	3.469	3.348	2.567	0.885	1.304
70	3.992	3.712	2.831	0.983	1.311
				multiplier average	1.256

10. CONCLUSIONS

Capital allocations that are set using the UL-GCLM method—the method used to set minimum regulatory capital requirements for banks under the Basel II A-IRB approach–are downward biased, and the bias is substantial in magnitude. UL-GCLM capital allocations are only about 20 percent as large as the true capital needed to achieve targeted solvency rates.

It is possible to derive an alternative method for setting capital allocations under G-ASRF assumptions that produce capital allocations that are approximately unbiased. This new methodology requires individual credits' PDs, LGDs, asset correlations, and YTMs as inputs into a capital allocation assignment function. A multiplier can be incorporated to improve the accuracy of the G-ASRF capital allocation estimator relative to the capital allocation that would be set using an equilibrium model of credit risk. We have calibrated the multiplier to be consistent with capital allocations that are set using the BSM methodology.

This new methodology for setting capital allocations provides a correction for the Basel II A-IRB capital assignment function. The new approach is computationally practical and could be easily implemented. The corrected capital rule calls for a substantial increase in minimum capital requirements over the existing Basel II A-IRB regulatory capital function.

Table 5: Capital Allocation for a 98 Percent Solvency Margin Under Alternative Models

par value	probability of default in percent	unbiased BSM portfolio capital	unbiased G-ASRF capital estimate	GCLM UL capital estimate	implied unbiased G-ASRF multiplier
55	0.233	0.095	0.100	0.019	0.950
56	0.298	0.121	0.129	0.027	0.938
57	0.379	0.152	0.163	0.035	0.933
58	0.476	0.19	0.204	0.046	0.931
59	0.593	0.235	0.248	0.059	0.948
60	0.732	0.287	0.304	0.075	0.944
61	0.896	0.348	0.370	0.095	0.941
62	1.088	0.418	0.443	0.117	0.944
63	1.311	0.498	0.527	0.143	0.945
64	1.568	0.588	0.623	0.174	0.944
65	1.862	0.69	0.730	0.208	0.945
66	2.196	0.804	0.851	0.247	0.945
67	2.574	0.93	0.982	0.290	0.947
68	2.997	1.069	1.132	0.338	0.944
69	3.469	1.221	1.291	0.390	0.946
70	3.992	1.387	1.465	0.446	0.947
				multiplier average	0.943

Appendix

Proposition 1

Should a bank set its equity capital equal to a $VaR(\alpha)$ *measure of its investment portfolio's future potential value, when interest rates are positive and investors require positive compensation for credit risk, other things held constant, the probability that the bank will be insolvent at the end of the capital allocation horizon is greater than* $(1-\alpha)$.

Proof: Let the amount that the bank must repay on its funding debt be represented by $A_0 - VaR^{A_0}(\alpha) + \zeta = \Psi^{-1}\left(\tilde{A}_T, 1-\alpha\right) + \zeta$, for some $\zeta > 0$. The inverse of the cumulative density function, $\Psi^{-1}(\tilde{A}_T, 1-\alpha)$, is continuous and monotonically decreasing in $\alpha \in [0,1]$. There is a unique value of $\hat{\alpha}$, $0 < \hat{\alpha} < \alpha$, such that $\Psi^{-1}\left(\tilde{A}_T, 1-\hat{\alpha}\right) = \Psi^{-1}\left(\tilde{A}_T, 1-\alpha\right) + \zeta$. Thus, $1-\hat{\alpha} > 1-\alpha$, or the probability that the time T value of investment asset is insufficient to discharge the principal and interest on the funding debt is in excess of $(1-\alpha)$.

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