On the (Mis)Use of Conditional Value-at-Risk and Spectral Risk Measures for Portfolio Selection – A Comparison with Mean-Variance Analysis

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1. Introduction

Conditional Value-at-Risk and, as its natural extension, spectral risk measures have become popular risk management tools in the last decade. Originally, these risk measures have been introduced as an alternative to heavily criticized Value-at-Risk (e.g., Artzner et al. (1999), Szegö (2002)) for the assessment of solvency capital in bank regulation. In the recent literature, the *context* in which spectral risk measures are used has changed, away from the assessment of solvency capital ("risk" context) towards the use as part of an investor's objective function in portfolio selection ("decision" context). We argue that the latter is a misuse, as the regulatory concept of diversification underlying spectral risk measures is suitable for the assessment of solvency capital, but is misleading when applied to portfolio selection.

In the original context of "risk" in bank regulation, the regulatory requirements add to a bank's objective (or utility) function as a constraint:

$$\max_{X\in\mathcal{X}} \pi(X), \text{ s.t. } \rho(X) \leq \bar{\rho}. \tag{1}$$

The bank's utility function π is only allowed to be applied to those alternatives X out of its set of alternatives $\mathcal{X}$, whose solvency capital requirements $\rho(X)$ do not exceed its given solvency capital $\bar{\rho}$. In the Basel II and III frameworks, ρ corresponds to Value-at-Risk. As a theoretically more adequate alternative, spectral risk measures such as Conditional Value-at-Risk have been introduced to overcome the paradoxical results that obtain under Value-at-Risk. Especially, their definition is based on an axiomatic framework that consistently reflects these regulatory issues, and most notably the regulatory concept of subadditivity and diversification.

In the recent literature, spectral risk measures, ρ_ϕ, are also used in the context of "decision" as, e.g., in problems of portfolio selection. Here, they constitute the risk part within an investor's (μ, ρ_ϕ)-preferences, which are represented by a (μ, ρ_ϕ)-utility function, $\pi = \pi(\mu, \rho_\phi)$. In a first step, (μ, ρ_ϕ)-efficient frontiers are derived in this literature by minimizing a spectral risk measure for any given level of expected return, $\bar{\mu}$:

$$\min_{X\in\mathcal{X}} \rho_\phi(X), \text{ s.t. } E(X) = \bar{\mu} \tag{2}$$

(e.g., Adam et al. (2008), Bassett et al. (2004), Benati (2003), Bertsimas et al. (2004), De Giorgi (2002), Krokhmal et al. (2002), Rockafellar/Uryasev (2000)).

Afterwards, the same model (2) is also used to find optimal portfolios by fixing a certain level of expected return $\bar{\mu}$, and finding the corresponding portfolio on the (μ, ρ_ϕ)-efficient frontier.[1] Many applications in practice are based on such a *limited analysis*. For example, it might be that the tradeoff between reward and risk is made by, say, a higher management hierarchy by fixing $\bar{\mu}$, while the risk analyst should only assess the risk side. Conversely, the same is true when

[1]Note that approach (2) fundamentally differs from the regulatory approach given by (1). There, the regulatory constraint is not used to find optimal portfolios, but only restricts the set of alternatives $\mathcal{X}$ to the so-called acceptance set $\mathcal{A} = \{X \in \mathcal{X} | \rho(X) \leq \bar{\rho}\}$, i.e., (1) can be equivalently written as $\max\{\pi(X) | X \in \mathcal{A}\}$ (e.g., Artzner et al. (1999)). Especially, in the regulatory framework no assumption is made about the utility function π. By contrast, the portfolio selection approach (2), at least implicitly, is focused on modeling this utility function π, as the problem of finding an optimal portfolio is given by $\max\{\pi(\mu(X), \rho_\phi(X)) | X \in \mathcal{X}\}$.

imposing a fixed risk limit $\bar{\rho}$ that banks are allowed to take, and maximizing expected return. In both cases, however, the limits are exogenously determined, so that the tradeoff between reward and risk is not subject to considerations within the choice of optimal portfolios.

In order to disclose the shortcomings of spectral risk measures in portfolio selection, we need to explicitly consider this tradoff between reward and risk made by the higher hierarchy instead of restricting to a limited analysis only. We do so by applying a *tradeoff analysis*, which explicitly models an investor's (μ, ρ_ϕ)-preferences in the form of a (μ, ρ_ϕ)-utility function, $\pi = \pi(\mu, \rho_\phi)$. Finding an optimal portfolio then requires to apply the indifference curves induced by this utility function to the (μ, ρ_ϕ)-efficient frontier. To this end, we assume the *spectral utility function*

$$\pi_\phi(X) = (1 - \lambda) \cdot E(X) - \lambda \cdot \rho_\phi(X), \lambda \in [0, 1], \tag{3}$$

which naturally arises from two different perspectives. From a decision theoretic perspective, the two components $-E(X)$ and $\rho_\phi(X)$ satisfy the properties of spectral risk measures, and by forming a (negative) convex combination, the spectral utility function π_ϕ itself satisfies these properties as well. Therefore, the determination of the (μ, ρ_ϕ)-efficient frontiers and the consequent choice of optimal portfolios using a spectral utility function π_ϕ are based on one consistent axiomatic, and thus integrated, framework. For this reason, spectral utility functions are widely used to model a reward-risk tradeoff in the recent literature on insurance and production theory (see, e.g., Cai/Tan (2007) and Wagner (2010) in insurance theory, and Jammernegg/Kischka (2007) and Ahmed et al. (2007) in production theory.) From an optimization perspective, the spectral utility function (3) is the tradeoff version of the (μ, ρ_ϕ)-efficient frontier program (2), and thus also establishes an integrated framework.

As will be shown below, the limited analysis and the tradeoff analysis are equivalent approaches for finding optimal portfolios in the traditional mean-variance framework, but they are fundamentally different under spectral risk measures. As a consequence, the prevalent limited analysis is not covered by the more general tradeoff analysis, and identifies non-optimal portfolios. Some recent results on optimal portfolio selection under spectral risk measures thus have to be re-considered.

This paper is focused on those shortcomings caused by the change from "risk" to "decision", and, to our best knowledge, is the first to analyze (μ, ρ_ϕ)-efficient frontiers together with optimal portfolios within an integrated framework. Our contribution is twofold: (i) The theoretical literature on portfolio selection under spectral risk measures so far relies on the assumption of normally distributed returns. We propose a finite state space approach instead, which does not impose any particular initial distribution and allows to disclose fundamental differences between the efficient frontiers under the variance and spectral risk measures, respectively. (ii) The literature so far has been restricted to the determination of (μ, ρ_ϕ)-efficient frontiers, but does not adequately cover the choice of optimal portfolios. We address this issue by applying a spectral utility function as in (3), to build an integrated portfolio selection approach.

Within this integrated framework, spectral risk measures tend towards corner solutions. If a risk free asset exists, either the exclusive investment in the risk free asset or in the tangency portfolio instead of diversification is optimal. Similarly, without a risk free asset, diversification proves to be limited under spectral risk measures. Within the traditional mean-variance framework, however, Markowitz (1952) noted that "diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim" (p. 77). Following this prominent view, spectral risk measures appear inappropriate in the context of "decision" and portfolio selection. The reason is that they are based on a regulatory concept of diversification that differs fundamentally from diversification in the mean-variance framework. As will be shown below, diversification there is based on the tradeoff between reward and risk, whereas under spectral risk measures the dependence structure between the assets is relevant.

The paper proceeds as follows. Section 2 reviews the axiomatic framework and characterizes the regulatory concept of diversification underlying spectral risk measures. Section 3 derives the (μ, ρ_ϕ)-efficient frontiers. Section 4 analyzes the choice of optimal portfolios by using spectral utility functions. In both Sections 3 and 4, we confront these results with those from the traditional (μ, σ^2)-framework, which serves as a well-established benchmark. Section 5 discusses the economic implications of the findings. Section 6 concludes.

2. Theoretical framework

2.1. Spectral risk measures and the regulatory concept of diversification

In order to prepare for the regulatory concept of diversification underlying spectral risk measures ("risk" context), we first introduce the notion of comonotonicity (e.g., Dhaene et al. (2002)).

Definition 2.1. *Two random variables $X_1, X_2 \in \mathcal{X}$ are called comonotonic if*

$$(X_1(\omega_i) - X_1(\omega_j)) \cdot (X_2(\omega_i) - X_2(\omega_j)) \geq 0, \textit{for all } \omega_i, \omega_j \in \Omega, P(\Omega) = 1. \tag{4}$$

Two random variables are comonotonic if they increase and decrease simultaneously in their state-dependent realizations. Comonotonicity thus denotes perfect dependence between the random variables, and generalizes the concept of perfect positive correlation. We have $corr(X_1, X_2) = 1$ if and only if $X_2 = a \cdot X_1 + b, a > 0, b \in \mathbb{R}$. Perfect positive correlation implies comonotonicity, but the converse is not true. For example, while comonotonicity holds between a random variable and a constant, their correlation coefficient is zero. This will prove to be relevant for diversification between a risk free and a risky asset under the variance, whereas diversification does not pay under spectral risk measures.

We proceed with the definition of spectral risk measures (Acerbi (2002), Acerbi (2004)).[2]

Definition 2.2. *A mapping $\rho_\phi : \mathcal{X} \to \mathbb{R}$ is called spectral risk measure if it satisfies*

- *Monotonicity with respect to first order stochastic dominance: For $X_1, X_2 \in \mathcal{X}$ with $F_{X_1}(t) \geq F_{X_2}(t)$ and $t \in \mathbb{R}$, $\rho_\phi(X_1) \geq \rho_\phi(X_2)$.*

[2] The given properties differ slightly from those by Acerbi (2004) in that they do not explicitly consider law invariance and positive homogeneity. Law invariance is implied by monotonicity with respect to first order stochastic dominance (Song/Yan (2009), Section 5.1). Further, monotonicity and comonotonic additivity imply positive homogeneity, $\rho_\phi(\lambda \cdot X) = \lambda \cdot \rho_\phi(X), \lambda \geq 0$, (Schmeidler (1986), remark 1).

– *Translation invariance: For $X \in \mathcal{X}$ and $c \in \mathbb{R}$, $\rho_\phi(X+c) = \rho_\phi(X) - c$.*

– *Subadditivity: For $X_1, X_2 \in \mathcal{X}$, $\rho_\phi(X_1 + X_2) \leq \rho_\phi(X_1) + \rho_\phi(X_2)$.*

– *Comonotonic Additivity: For comonotonic $X_1, X_2 \in \mathcal{X}$, $\rho_\phi(X_1 + X_2) = \rho_\phi(X_1) + \rho_\phi(X_2)$.*

As to the above properties, spectral risk measures originally have been introduced for the assessment of solvency capital in bank regulation ("risk"). Monotonicity and translation invariance are straightforward requirements for measuring risk in monetary terms. Monotonicity states that a financial position X_1 with a larger probability of falling below a threshold t for all $t \in \mathbb{R}$ than a financial position X_2 requires more solvency capital. Since $\rho_\phi(X + \rho_\phi(X)) = \rho_\phi(X) - \rho_\phi(X) = 0$, translation invariance allows for the interpretation of $\rho_\phi(X)$ as required solvency capital.

The regulatory concept of diversification underlying spectral risk measures is captured jointly by the properties of subadditivity and comonotonic additivity, and relates exclusively to the dependence structure between financial positions. Subadditivity ensures that spectral risk measures reward diversification, as a portfolio of two financial positions does not require more solvency capital than the two single positions do. The diversification benefit results from an imperfect dependence structure between the financial positions X_1 and X_2 within a portfolio. In this case, a "high" realization in one state of the world of position X_1 (partially) compensates for a "low" realization of position X_2 in the same state of the world (and vice versa). For the special case that the two financial positions are comonotonic and "high" and "low" realizations coincide in all states of the world, such a compensational effect does not exist. Consequently, that kind of "diversification" should not be rewarded by reduced solvency capital requirements. This is captured by the additivity of spectral risk measures for comonotonic financial positions. We summarize the above argument in the following proposition (see also Cherny (2006), Theorem 5.1).

Proposition 2.3. *Let ρ_ϕ be a spectral risk measure and $X_1, X_2 \in \mathcal{X}$. Non-comonotonicity between X_1 and X_2 is a necessary condition for a positive diversification benefit, $\lambda \cdot \rho_\phi(X_1) + (1-\lambda) \cdot \rho_\phi(X_2) - \rho_\phi(\lambda \cdot X_1 + (1-\lambda) \cdot X_2) > 0, \lambda \in [0,1]$.*

The following stylized example illustrates the regulatory concept of diversification underlying spectral risk measures.

Example 2.4. Let $X_0 = x_0$ be a risk free asset and X_1, X_2 and X_3 be risky assets with state-dependent returns

$$X_1 = \begin{cases} -2 & P(\omega_1) = 1/3 \\ 0 & P(\omega_2) = 1/3 \\ 3 & P(\omega_3) = 1/3 \end{cases}, \; X_2 = \begin{cases} -3 & P(\omega_1) = 1/3 \\ 1 & P(\omega_2) = 1/3 \\ 4 & P(\omega_3) = 1/3 \end{cases}, \; X_3 = \begin{cases} 4 & P(\omega_1) = 1/3 \\ 1 & P(\omega_2) = 1/3 \\ -3 & P(\omega_3) = 1/3 \end{cases}.$$

For the non-comonotonic assets X_1 and X_3 we observe a positive diversification benefit from the dependence structure: While the single positions suffer losses in state ω_1 and ω_3, respectively, the portfolios $\gamma \cdot X_1 + (1-\gamma) \cdot X_3$ for $\gamma \in [0.5; 0.67]$ have only non-negative state-dependent returns. Consequently, subadditivity ensures that building a portfolio yields reduced solvency capital requirements in this case:

$$\rho_\phi(\gamma \cdot X_1 + (1-\gamma) \cdot X_3) < \gamma \cdot \rho_\phi(X_1) + (1-\gamma) \cdot \rho_\phi(X_3), \gamma \in (0,1).$$

For the comonotonic assets X_1 and X_2, a diversification benefit from the dependence structure does not prevail: The state-dependent returns simply add up without providing any compensational effect. Hence, comonotonic additivity implies that building a portfolio does not allow to reduce solvency capital requirements in this case:

$$\rho_\phi(\gamma \cdot X_1 + (1-\gamma) \cdot X_2) = \gamma \cdot \rho_\phi(X_1) + (1-\gamma) \cdot \rho_\phi(X_2), \gamma \in [0,1].$$

As another example, building a portfolio of a risky asset $X_i, i = 1, \ldots, 3$ and the risk free asset X_0 yields

$$\rho_\phi(\beta \cdot X_i + (1-\beta) \cdot X_0) = \beta \cdot \rho_\phi(X_i) - (1-\beta) \cdot X_0, \beta \in [0,1].$$

In this case, the solvency capital requirements decrease linearly in the proportion $(1-\beta)$. Beyond, there is no (additional) diversification benefit from the dependence structure. This result is not only driven by the comonotonicity between X_i and X_0, but is also required by the property of translation invariance. Adding a certain amount of cash $(1-\beta) \cdot X_0$ to a risky asset decreases the solvency capital requirements of the portfolio by exactly this amount. $\square$

We proceed with the representation of spectral risk measures as weighted sum of quantiles.

Proposition 2.5. *Any spectral risk measure ρ_ϕ of a random variable X is of the form*

$$\rho_\phi(X) = -\int_0^1 F_X^*(p) \cdot \phi(p) \mathrm{d}p, \tag{5}$$

where $F_X^*(p) = \sup\{x \in \mathbb{R} | F_X(x) < p\}, p \in (0,1]$ are the p-quantiles of the cumulative distribution function F_X, and the risk spectrum $\phi : [0,1] \to \mathbb{R}$ satisfies

- *positivity: $\phi(p) \geq 0$ for all $p \in [0,1]$,*
- *normalization: $\int_0^1 \phi(p)\mathrm{d}p = 1$,*
- *monotonicity: $\phi(p_1) \geq \phi(p_2)$ for all $0 \leq p_1 \leq p_2 \leq 1$.*

For the proof see Acerbi (2002), Theorem 4.1. Spectral risk measures are characterized by a risk spectrum ϕ, which assigns different weights to the p-quantiles, with smaller quantiles receiving greater weights to ensure the subadditivity property. Further properties of spectral risk measures are given by Dhaene et al. (2006).

Currently, the most widely discussed spectral risk measure is Conditional Value-at-Risk (e.g., Acerbi/Tasche (2002b), Rockafellar/Uryasev (2002)). Its risk spectrum is given by

$$\phi(p) = \begin{cases} \alpha^{-1} & \text{for } 0 < p \leq \alpha \\ 0 & \text{for } \alpha < p \leq 1 \end{cases}. \tag{6}$$

Conversely, spectral risk measures can be seen as a natural extension of Conditional Value-at-Risk, as any convex combination of Conditional Value-at-Risks yields a spectral risk measure (Acerbi (2002), Proposition 2.2).

Also, the (negative) mean $\rho_\phi(X) = -E(X)$ is a spectral risk measure with $\phi(p) = 1, p \in [0,1]$. By contrast, the variance of a financial position X, $Var(X) = \sigma^2$, is not a spectral risk measure, as it satisfies none of the required properties.

2.2. Portfolio selection problems

We now change the context from "risk" to "decision" and introduce some simple portfolio selection problems, in which spectral risk measures will be applied: An investor can split their initial wealth W_0 between different assets. The return from this investment (i.e., the final wealth) is given by a random variable $X \in \mathcal{X}$ that stems from one of the following settings:

- *Setting 1:* There are two risky assets X_1 and X_2, i.e., $\mathcal{X} = \{\gamma \cdot X_1 + (1-\gamma) \cdot X_2 | \gamma \in \mathbb{R}\}$.[3] We assume the risky assets to be (μ, ρ)-efficient[4], i.e., $E(X_1) < E(X_2) \wedge \rho(X_1) < \rho(X_2)$.
- *Setting 2:* There are two risky assets X_1 and X_2, and a risk free asset X_0, i.e., $\mathcal{X} = \{\beta \cdot (\gamma \cdot X_1 + (1-\gamma) \cdot X_2) + (1-\beta) \cdot X_0 | \beta \geq 0, \gamma \in \mathbb{R}\}$. Again, we assume the risky assets to be (μ, ρ)-efficient. Moreover, we restrict the correlation coefficient to $corr(X_1, X_2) \in (-1, 1)$ to ensure that one cannot construct an additional risk free asset from the risky assets. Further assumptions about the return of the risk free asset are made in the relevant sections.

The determination of a portfolio's risk hinges crucially on the dependence structure between the risky assets. While a portfolio's variance can be calculated directly from its basic assets' variances and the correlation coefficient, the rank dependency of spectral risk measures requires the complete dependence structure to determine a portfolio's spectral risk. The theoretical literature thus mostly relies on the assumption of normally distributed returns, as in this case the correlation coefficient captures the dependence structure completely (e.g., Alexander/Baptista (2002), Alexander/Baptista (2004), De Giorgi (2002), Deng et al. (2009)). We refrain from this assumption, and apply a *state space approach* instead, which characterizes the assets $X : \Omega \to \mathbb{R}$ via their state-dependent realizations $X = (X(\omega_1), \ldots, X(\omega_n))' = (x_1, \ldots, x_n)'$ and the corresponding vector of the probabilities of the states of the world $P = (P(\omega_1), \ldots, P(\omega_n))' = (p_1, \ldots, p_n)'$, i.e., any alternative is given by the pair (X, P). This approach captures the dependence structure completely by the vectors X, and both the variance and spectral risk measures can be calculated directly from (X, P). For the ease of demonstration, the analysis remains mostly restricted to a finite state space, as certain portfolio structures "get lost" in the case of infinitely many states. However, we also refer to the case of normally distributed returns.

To our best knowledge, this paper is the first to apply the state space approach to portfolio selection problems under spectral risk measures. Unlike the previous literature, our derivation only relies on the properties of the risk measures, and does not require any assumption on the underlying random variable X. Our approach thus is more general and, as yet, proves to be

[3] $X_1 := W_0 \cdot (1 + R_1)$ and $X_2 := W_0 \cdot (1 + R_2)$ denote the returns from investing the initial wealth W_0 in assets 1 and 2.

[4] We use ρ as a placeholder for the variance σ^2 and spectral risk measures ρ_ϕ. The term "(μ, ρ)-efficient", for example, stands for (μ, σ^2)- and (μ, ρ_ϕ)-efficient.

advantageous in that it allows to disclose restrictive portfolio structures that have not become explicit elsewhere.

For the ease of demonstration, we first restrict the analysis to $m = 2$ risky assets; we later show that the results hold for more general cases as well.

The (μ, ρ)-boundary and the (μ, ρ)-efficient frontier are defined as follows.

Definition 2.6. *A portfolio $X \in \mathcal{X}$ belongs to the (μ, ρ)-boundary if for some expected return $\bar{\mu} \in \mathbb{R}$ it has minimum risk ρ.*

Definition 2.7. *A portfolio $X \in \mathcal{X}$ belongs to the (μ, ρ)-efficient frontier if there is no portfolio $\bar{X} \in \mathcal{X}$ with $E(\bar{X}) \geq E(X)$ and $\rho(\bar{X}) \leq \rho(X)$, where at least one of the inequalities is strict.*

We use the subscript $i = 1, 2$ to indicate that the $(\mu, \rho)_i$-boundaries and the $(\mu, \rho)_i$-efficient frontiers refer to Setting 1 and 2, respectively. As is common in portfolio selection, we illustrate the $(\mu, \rho)_i$-efficient frontiers in the respective (ρ, μ)-planes.[5] Extending the previous literature, we are not only interested in the (μ, ρ)-efficient frontiers themselves, but especially in their (e.g., (piecewise) linear or (strictly) concave) shape.

3. (μ, σ^2)-efficient frontiers versus (μ, ρ_ϕ)-efficient frontiers

3.1. Comonotonic subsets of alternatives

As spectral risk measures are comonotonic additive, comonotonic subsets of alternatives become an essential part of the analysis. The state space approach allows to make the comonotonic subsets of alternatives explicit via their state-dependent realizations. Let

$$\mathcal{X} = \left\{ X_\gamma = \gamma \cdot X_1 + (1-\gamma) \cdot X_2 = \begin{pmatrix} \gamma \cdot x_{1_1} + (1-\gamma) \cdot x_{2_1} \\ \vdots \\ \gamma \cdot x_{1_n} + (1-\gamma) \cdot x_{2_n} \end{pmatrix} \middle| \gamma \in \mathbb{R} \right\} \tag{7}$$

be the set of alternatives based on the two risky assets. The boundaries of the comonotonic subsets of alternatives are given by

$$\gamma_{ij} := \frac{x_{2_i} - x_{2_j}}{(x_{2_i} - x_{2_j}) - (x_{1_i} - x_{1_j})}, i = 1, \ldots, n-1, j = 2, \ldots, n, i < j. \tag{8}$$

We obtain the proportions (8) by equalizing any two portfolio realizations and solving for γ. Therefore, any γ_{ij} denotes a portfolio where there is a switch in the ranking of the realizations. Rearranging the proportions with respect to size yields the following $k+1$ comonotonic subsets of alternatives:

$$\{X_\gamma | \gamma \in (-\infty, \gamma_{ij,1:k}]\}, \{X_\gamma | \gamma \in (\gamma_{ij,1:k}, \gamma_{ij,2:k}]\}, \ldots, \{X_\gamma | \gamma \in (\gamma_{ij,k:k}, \infty)\}. \tag{9}$$

[5] Unlike the previous literature, for the variance we use the (σ^2, μ)-plane instead of the more standard (σ, μ)-plane. The reason is that for a more straightforward comparison of the choice of optimal portfolios in Section 4 we want both the induced indifference curves of the mean-variance utility function, $\pi(X) = E(X) - \frac{\lambda}{2} \cdot Var(X)$, and spectral utility functions, $\pi_\phi(X) = (1-\lambda) \cdot E(X) - \lambda \cdot \rho_\phi(X)$, to be linear. This in turn requires having the variance instead of the standard deviation on the abscissa. None of the results on the choice of optimal portfolios would change if we were to use the (σ, μ)-plane instead.

The number of comonotonic subsets depends mainly on the number of states of the world. For $n \to \infty$, k may (but does not necessarily need to) tend to infinity.

In the case of one risk free and one risky asset, the complete set of alternatives

$$\mathcal{X} = \{ X_\beta = \beta \cdot X_{\bar{\gamma}} + (1-\beta) \cdot X_0 | \, \beta \geq 0\} \tag{10}$$

is comonotonic.

3.2. Two risky assets

We start portfolio selection with analyzing the (μ, ρ)-boundaries and the (μ, ρ)-efficient frontiers. As we restrict the analysis to two risky assets, the complete set of alternatives $X_\gamma = \gamma \cdot X_1 + (1-\gamma) \cdot X_2, \gamma \in \mathbb{R}$ belongs to the $(\mu, \rho)_1$-boundaries.

First, we briefly recall the traditional (μ, σ^2)-framework. The $(\mu, \sigma^2)_1$-boundary is obtained by solving the portfolio's expected return for the proportion γ and plugging it into its variance:

$$\begin{aligned} Var(X_\gamma) &= \left(\frac{E(X_\gamma) - E(X_2)}{E(X_1) - E(X_2)}\right)^2 \cdot a + 2 \cdot \frac{E(X_\gamma) - E(X_2)}{E(X_1) - E(X_2)} \cdot b + c, \\ a &= Var(X_1) + Var(X_2) - 2 \cdot \sqrt{Var(X_1)} \cdot \sqrt{Var(X_2)} \cdot corr(X_1, X_2), \\ b &= \sqrt{Var(X_1)} \cdot \sqrt{Var(X_2)} \cdot corr(X_1, X_2) - Var(X_2), \\ c &= Var(X_2). \end{aligned} \tag{11}$$

The $(\mu, \sigma^2)_1$-boundary is a parabola that opens to the right (see Figure 1). The $(\mu, \sigma^2)_1$-efficient frontier lies on the upper branch of the parabola starting from the minimum-variance portfolio (MVP).

Proposition 3.1. *Let $\mathcal{X}$ be as in Setting 1. Then*

1. *the minimum-variance portfolio is given by $\gamma_{MVP} = -\frac{b}{a}$;*
2. *the $(\mu, \sigma^2)_1$-efficient frontier contains all portfolios $\gamma \in (-\infty, \gamma_{MVP}]$ and is a strictly concave curve for any correlation coefficient $corr(X_1, X_2) \in [-1, 1]$.*

The proof is straightforward and therefore omitted. Essentially, the strict concavity of the $(\mu, \sigma^2)_1$-efficient frontier follows from the strict convexity of the variance on $\mathcal{X}$ for any correlation coefficient $corr(X_1, X_2) \in [-1, 1]$.

We now consider (μ, ρ_ϕ)-preferences. The $(\mu, \rho_\phi)_1$-boundary is obtained by writing the portfolio's expected return as a function of its spectral risk.

As a first step, we analyze a comonotonic subset of alternatives $X_\gamma, \gamma \in [\gamma_d, \gamma_u]$ as given in (9). For any $\delta := \frac{\gamma - \gamma_d}{\gamma_u - \gamma_d} \in [0, 1]$, the spectral risk of portfolio X_γ

$$\rho_\phi(X_\gamma) = \rho_\phi(\delta \cdot X_{\gamma_d} + (1-\delta) \cdot X_{\gamma_u}) = \delta \cdot \rho_\phi(X_{\gamma_d}) + (1-\delta) \cdot \rho_\phi(X_{\gamma_u}) \tag{12}$$

can be solved for

$$\delta = \frac{\rho_\phi(X_\gamma) - \rho_\phi(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})} \tag{13}$$

and inserted into the portfolio's expected return to give the linear risk-return schedule

$$\begin{aligned} E(X_\gamma) &= \delta \cdot E(X_{\gamma_d}) + (1-\delta) \cdot E(X_{\gamma_u}) \\ &= \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})} \cdot \rho_\phi(X_\gamma) - \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})} \cdot \rho_\phi(X_{\gamma_u}) + E(X_{\gamma_u}). \end{aligned} \tag{14}$$

If the comonotonic subset of alternatives is (μ, ρ_ϕ)-efficient, i.e., $E(X_{\gamma_d}) \leq E(X_{\gamma_u}) \wedge \rho_\phi(X_{\gamma_d}) \leq \rho_\phi(X_{\gamma_u})$, (14) is linearly increasing, and linearly decreasing otherwise.

Regarding the complete set of alternatives $X_\gamma, \gamma \in \mathbb{R}$, the portfolio's spectral risk is convex on $\mathcal{X}$ due to subadditivity and positive homogeneity. As the portfolio's expected return $E(X_\gamma)$ and the proportion γ are linearly related, the portfolio's spectral risk is also a convex function of its expected return that, according to (14), is piecewise linear for comonotonic subsets of alternatives. The $(\mu, \rho_\phi)_1$-boundary thus is a piecewise linear and overall convex curve that opens to the right (see Figure 1). The $(\mu, \rho_\phi)_1$-efficient frontier lies on the upper branch of the $(\mu, \rho_\phi)_1$-boundary starting from the minimum-spectral risk portfolio (MSP); its existence and the non-emptiness of the $(\mu, \rho_\phi)_1$-efficient frontier is guaranteed by the assumption of (μ, ρ_ϕ)-efficient basic assets. We summarize the results in the following proposition.

Proposition 3.2. *Let $\mathcal{X}$ be as in Setting 1. Then*

1. *the minimum-spectral risk portfolio γ_{MSP} lies in the set $\{\gamma_{ij,1:k}, \dots, \gamma_{ij,k:k}\}$;*
2. *the $(\mu, \rho_\phi)_1$-efficient frontier contains all portfolios $\gamma \in (-\infty, \gamma_{MSP}]$ and is a concave curve that is piecewise linear for comonotonic subsets of alternatives as given in (9).*

We give the following stylized example for numerical illustration.

Example 3.3. An investor can split their initial wealth between two risky assets with state-dependent returns

$$X_1 = \begin{cases} 1 & P(\omega_1) = 1/3 \\ 2 & P(\omega_2) = 1/3 \\ 3 & P(\omega_3) = 1/3 \end{cases} \text{ and } X_2 = \begin{cases} 4 & P(\omega_1) = 1/3 \\ 0 & P(\omega_2) = 1/3 \\ 3 & P(\omega_3) = 1/3 \end{cases}.$$

As risk measures, the investor applies the variance and Conditional Value-at-Risk at the confidence level $\alpha = 0.5$. The risky assets are (μ, σ^2)-efficient, as $E(X_1) = 2 < 2.34 = E(X_2), Var(X_1) = 0.67 < 2.89 = Var(X_2)$, and they are $(\mu, CVaR_\alpha)$-efficient, as $CVaR_\alpha(X_1) = -1.34 < -1 = CVaR_\alpha(X_2)$ holds.

Figure 1 shows the $(\mu, \sigma^2)_1$-boundary. The minimum-variance portfolio is given by $X_{0.76} = (1.53; 1.71; 3)'$, and the $(\mu, \sigma^2)_1$-efficient frontier contains all portfolios $X_\gamma, \gamma \in (-\infty; 0.76]$.

Further, Figure 1 shows the $(\mu, CVaR_\alpha)_1$-boundary. The $(\mu, CVaR_\alpha)_1$-efficient frontier contains all portfolios $X_\gamma, \gamma \in (-\infty; 0.8]$, with $X_{0.8} = (1.6; 1.6; 3)$ as the minimum-Conditional Value-at-Risk portfolio. The linear segments correspond to the portfolios $\gamma \in (-\infty; 0.34]$ $(x_2 \leq x_3 \leq x_1)$,

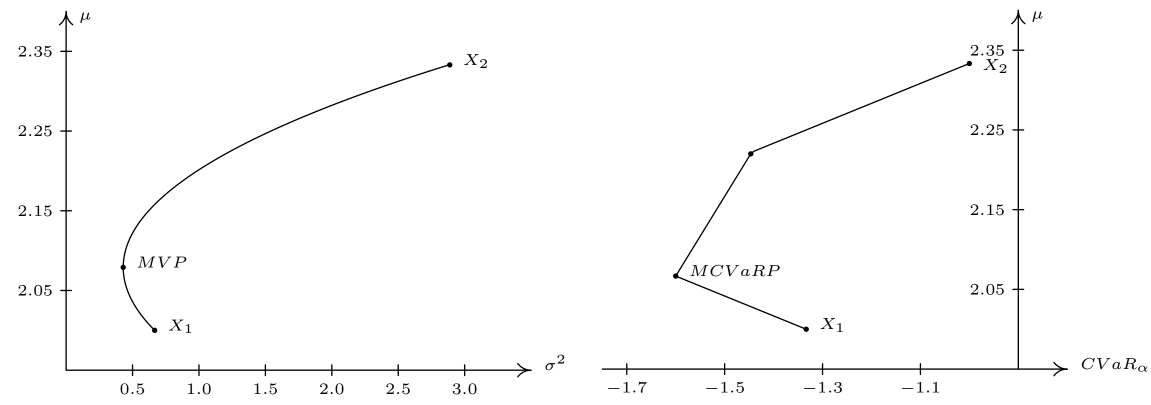

Figure 1: $(\mu, \sigma^2)_1$- versus $(\mu, CVaR_\alpha)_1$-boundary with two risky assets

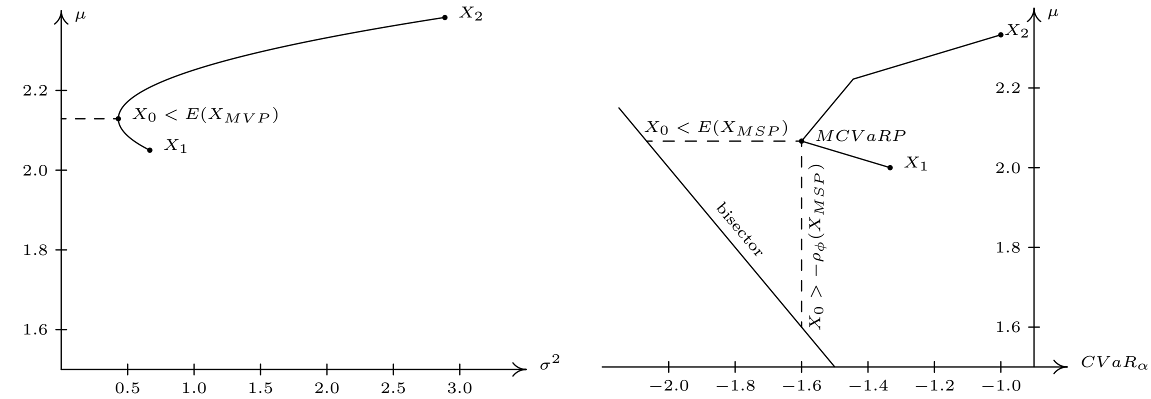

Figure 2: Locus of the risk free asset

$\gamma \in (0.34; 0.8]$ $(x_2 \leq x_1 \leq x_3)$, $\gamma \in (0.8; 1.5]$ $(x_1 \leq x_2 \leq x_3)$, and $\gamma \in (1.5; \infty]$ $(x_1 \leq x_3 \leq x_2)$. The corner positions $X_{0.34} = (3; 0.67; 3)'$, $X_{0.8} = (1.6; 1.6; 3)'$, and $X_{1.5} = (-0.5; 3; 3)'$ are characterized by having at least two identical state-dependent realizations. $\square$

3.3. One risk free and two risky assets

We continue portfolio selection by introducing a risk free asset. For (μ, σ^2)-preferences, we stay in line with the literature and assume that $X_0 < E(X_{MVP})$ to ensure that the risk free asset lies below the intersection of the asymptote of the $(\mu, \sigma^2)_1$-efficient frontier with the ordinate. In the (μ, ρ_ϕ)-framework, the risk free asset satisfies $X_0 = E(X_0) = -\rho_\phi(X_0)$ and lies on the bisector of the second quadrant. Therefore, we assume that $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$ (see Figure 2).

As the first step, the $(\mu, \rho)_2$-efficient frontiers for one risk free asset X_0 and only one risky asset $X_{\bar{\gamma}}$ are analyzed, i.e., the set of alternatives is $X_{\beta,\bar{\gamma}} = \beta \cdot X_{\bar{\gamma}} + (1 - \beta) \cdot X_0, \beta \geq 0$. Afterwards, we interpret the risky asset $X_{\bar{\gamma}}$ as a $(\mu, \rho)_1$-efficient portfolio that is composed of the risky basic assets.

Again, we briefly recall the traditional (μ, σ^2)-framework. The derivation of the $(\mu, \sigma^2)_2$-efficient frontier with respect to the set of alternatives $X_{\beta,\bar{\gamma}}, \beta \geq 0$ requires solving the portfolio's variance for the proportion β and plugging it into its expected return, which gives the well-known strictly concave (square root) function

$$E(X_{\beta,\bar{\gamma}}) = \frac{E(X_{\bar{\gamma}}) - X_0}{\sqrt{Var(X_{\bar{\gamma}})}} \cdot \sqrt{Var(X_{\beta,\bar{\gamma}})} + X_0. \tag{15}$$

Generally, any $(\mu, \sigma^2)_1$-efficient portfolio can serve as a risky asset $X_{\bar{\gamma}}$ in the above considerations. $(\mu, \sigma^2)_2$-efficient mean-variance combinations consist of the risk free asset X_0 and the $(\mu, \sigma^2)_1$-efficient portfolio X_{T,σ^2} that touches the parabola (11) at only one point, and thus is called *tangency portfolio* (see Figure 3).

Proposition 3.4. *Let $\mathcal{X}$ be as in Setting 2 and $X_0 < E(X_{MVP})$. Then*

1. *the $(\mu, \sigma^2)_2$-efficient frontier is a strictly concave curve through the risk free asset and the tangency portfolio;*
2. *the tangency portfolio is given by*

$$\gamma_{T,\sigma^2} = \frac{(E(X_2) - X_0) \cdot b - (E(X_1) - E(X_2)) \cdot c}{(E(X_1) - E(X_2)) \cdot b - (E(X_2) - X_0) \cdot a}. \tag{16}$$

The proof is straightforward and therefore omitted. Proposition 3.4 provides the well-known Tobin separation (Tobin (1958)): Any $(\mu, \sigma^2)_2$-efficient portfolio is a linear combination of the risk free asset and the tangency portfolio. An investor's individual risk aversion only affects the proportions of the initial wealth that are invested in these assets. Note that for any given parameters $E(X_1), E(X_2), Var(X_1), Var(X_2)$, and $corr(X_1, X_2) \in (-1, 1)$, there exists a corresponding value X_0, so that any $(\mu, \sigma^2)_1$-efficient portfolio can serve as the tangency portfolio, i.e., $\gamma_{T,\sigma^2} \in (-\infty, \gamma_{MVP})$.

A similar argument applies to the $(\mu, \rho_\phi)_2$-efficient frontier. As the set of alternatives $X_{\beta,\bar{\gamma}}, \beta \geq 0$ is comonotonic and spectral risk measures are comonotonic additive and translation invariant, we can solve the portfolio's spectral risk for the proportion β as

$$\begin{aligned} \rho_\phi(X_{\beta,\bar{\gamma}}) &= \rho_\phi(\beta \cdot X_{\bar{\gamma}} + (1-\beta) \cdot X_0) = \beta \cdot \rho_\phi(X_{\bar{\gamma}}) + (1-\beta) \cdot \rho_\phi(X_0) \\ \Rightarrow \beta &= \frac{\rho_\phi(X_{\beta,\bar{\gamma}}) - \rho_\phi(X_0)}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)} \end{aligned} \tag{17}$$

and substitute β into its expected return:

$$\begin{aligned} E(X_{\beta,\bar{\gamma}}) &= \beta \cdot E(X_{\bar{\gamma}}) + (1-\beta) \cdot X_0 \\ &= \frac{E(X_{\bar{\gamma}}) - X_0}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)} \cdot \rho_\phi(X_{\beta,\bar{\gamma}}) - \frac{E(X_{\bar{\gamma}}) - X_0}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)} \cdot \rho_\phi(X_0) + X_0. \end{aligned} \tag{18}$$

The portfolio's expected return is linearly increasing in its spectral risk.

Again, any $(\mu, \rho_\phi)_1$-efficient portfolio $X_{\bar{\gamma}}$ can serve as the risky asset. However, the only $(\mu, \rho_\phi)_2$-efficient combination consists of the risk free asset X_0 and the $(\mu, \rho_\phi)_1$-efficient portfolio, X_{T,ρ_ϕ}, where (18) is a tangent to the $(\mu, \rho_\phi)_1$-efficient frontier (tangency portfolio) (see Figure 3); Tobin separation still holds. Note that this result crucially depends on the assumption of

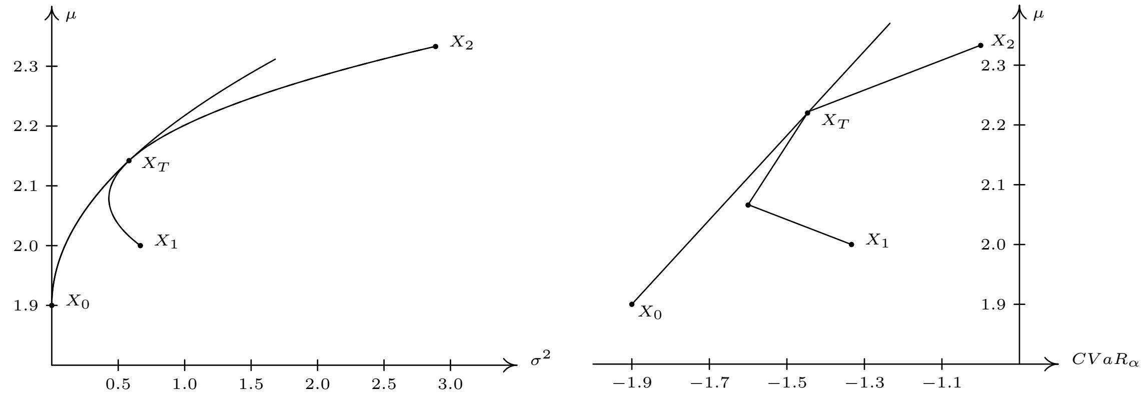

Figure 3: $(\mu, \sigma^2)_2$- versus $(\mu, CVaR_\alpha)_2$-boundary with a risk free and two risky assets

(μ, ρ_ϕ)-efficient risky basic assets; without this assumption, the $(\mu, \rho_\phi)_1$-efficient frontier might be empty and Tobin separation does not hold. The results are summarized in the following proposition.

Proposition 3.5. *Let $\mathcal{X}$ as in Setting 2 and $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$. Then*

1. *the $(\mu, \rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio;*

2. *the tangency portfolio γ_{T,ρ_ϕ} lies in the set $\{\gamma_{ij,1:k}, \ldots, \gamma_{ij,k:k}\}$.*

We continue the stylized Example 3.3 by adding a risk free asset.

Example 3.6. Let $X_0 = 1.9$ be the return of the risk free asset, which is added to the risky assets X_1 and X_2.

Figure 3 shows the $(\mu, \sigma^2)_2$-efficient frontier, which is a strictly concave curve through X_0 and X_{T,σ^2}. The tangency portfolio $\gamma_{T,\sigma^2} = 0.57$ is characterized by the state-dependent returns $X_T = (2.28; 1.15; 3)'$.

Further, Figure 3 shows the $(\mu, CVaR_\alpha)_2$-efficient frontier as a straight line through X_0 and $X_{T,CVaR_\alpha}$. The tangency portfolio $\gamma_{T,CVaR_\alpha} = 0.8$ with $X_{T,CVaR_\alpha} = (1.6; 1.6; 3)'$ is characterized by having at least two identical state-dependent realizations. □

3.4. Extensions

In order to keep the analysis simple, so far we have imposed two assumptions: (i) $m = 2$ risky assets, and (ii) (μ, ρ)-efficiency of these risky basic assets. We now show that relaxing the assumptions does not change the shape of the (μ, ρ)-efficient frontiers.

For the (μ, σ^2)-framework, it is well-known that the strict concavity of the (μ, σ^2)-efficient frontiers as well as Tobin separation hold in the absence of the above restrictions; (see the Appendix A for another proof).

In order to proceed with the (μ, ρ_ϕ)-framework, we start by omitting the (μ, ρ_ϕ)-efficiency of the risky basic assets, but still restrict their number to $m = 2$. The efficiency assumption ensures

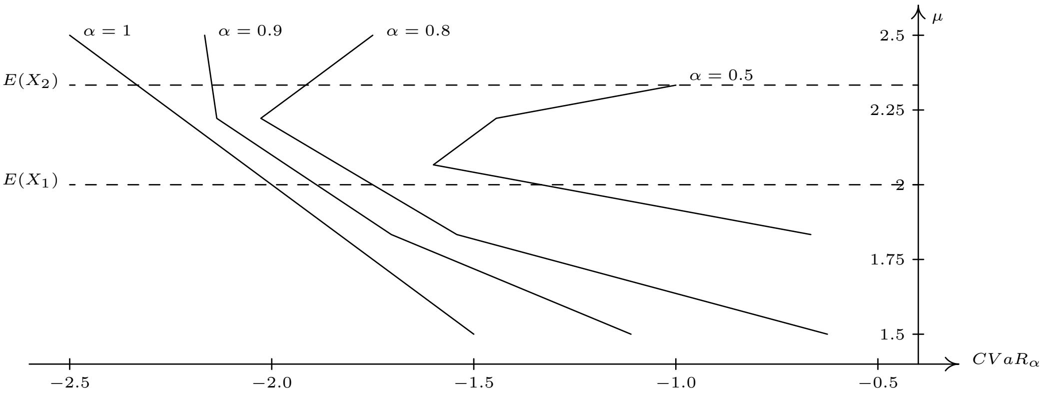

Figure 4: $(\mu, CVaR_\alpha)_1$-boundary without efficiency restriction

that the minimum-spectral risk portfolio exists and the $(\mu, \rho_\phi)_1$-efficient frontier is non-empty. The following stylized counter-example shows that this result no longer holds when relaxing the assumption.

Example 3.7. Let the risky assets be given as in Example 3.3. As spectral risk measures, the investor applies Conditional Value-at-Risk at the confidence level $\alpha_1 = 0.8$ and $\alpha_2 = 0.9$. In both cases, the asset X_1 is not $(\mu, CVaR_\alpha)$-efficient; we have $E(X_1) = 2 < 2.34 = E(X_2)$ and $CVaR_{0.8}(X_1) = -1.75 > -1.92 = CVaR_{0.8}(X_2)$ and $CVaR_{0.9}(X_1) = -1.89 > -2.15 = CVaR_{0.9}(X_2)$, respectively.

For $\alpha_1 = 0.8$, the minimum-spectral risk portfolio is given by $X_{0.34} = (3; 0.67, 3)$, and the $(\mu, CVaR_{\alpha_1})_1$-efficient frontier contains all portfolios $\gamma \in (-\infty; 0.34]$. For $\alpha_2 = 0.9$, the minimum-spectral risk portfolio does not exist. Instead, the $(\mu, CVaR_{\alpha_2})_1$-boundary is strictly decreasing. Figure 4 shows the corresponding $(\mu, CVaR_\alpha)_1$-boundaries. □

Note that the non-existence of the minimum-spectral risk portfolio is closely related to the property of translation invariance; due to

$$\rho_\phi(X) = \rho_\phi(X + E(X) - E(X)) = -E(X) + \rho_\phi(X - E(X)), \tag{19}$$

two separate effects can be identified when moving upward along the $(\mu, \rho_\phi)_1$-boundary. The first effect is captured by $-E(X) < 0$ (mean effect), and leads to a decrease in spectral risk. The second effect (deviation effect) refers to $\rho_\phi(X - E(X)) > 0$, and has been introduced as *deviation measure* by Rockafellar et al. (2006). The deviation effect leads to an increase in spectral risk. Depending on whether the mean effect outweighs the deviation effect, the $(\mu, \rho_\phi)_1$-efficient frontier is empty or non-empty.

These effects show that spectral risk measures exhibit both properties of location measures and deviation measures simultaneously. Especially, the former is a reasonable requirement for monetary and regulatory risk measurement ("risk"), as an increase in a financial position's mean should result in reduced solvency capital requirements. At the same time, the location property may lead to the non-existence of the minimum-spectral risk portfolio when applied to portfolio selection ("decision"). We summarize as follows.

Extension 3.8. *For $m = 2$ risky assets, the assumption of (μ, ρ_ϕ)-efficiency is a sufficient,* *although not necessary, condition for the existence of the minimum-spectral risk portfolio and the non-emptiness of the $(\mu, \rho_\phi)_1$-efficient frontier. For a risk free asset and $m = 2$ risky assets with $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$, the assumption of (μ, ρ_ϕ)-efficient risky basic assets is a sufficient, although not necessary, condition for Tobin separation.*

The sufficiency immediately follows from the relations $E(X_1) < E(X_2) \wedge \rho_\phi(X_1) < \rho_\phi(X_2)$ in connection with the convexity of spectral risk measures; Example 3.7 shows the non-necessity.

Considering $m \geq 2$ risky assets and omitting the efficiency restriction does not change the results from the two-asset framework either.

Extension 3.9. *For $m \geq 2$ risky assets and in the absence of efficiency restrictions,*

1. *if the $(\mu, \rho_\phi)_1$-efficient frontier is non-empty, it is a concave curve that is piecewise linear for comonotonic subsets of alternatives;*

2. *if the $(\mu, \rho_\phi)_1$-efficient frontier is non-empty and a risk free asset with $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$ exists, the $(\mu, \rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio.*

For the proof, see Benati (2003), Theorem 2, in connection with (13) and (18) for the linearity property.

Finally note that the linearity of the $(\mu, \rho_\phi)_2$-efficient frontier in Extension 3.9, part 2, is valid under arbitrary distributions, as the derivation in (17) and (18) only relies on the properties of the spectral risk measure, but does not impose any assumption on the underlying random variable X. Especially, the linearity preserves under the assumption of normally distributed returns, which is common in the theoretical literature on portfolio selection with Conditional Value-at-Risk and spectral risk measures as yet (e.g., Alexander/Baptista (2002), Alexander/Baptista (2004), De Giorgi (2002), Deng et al. (2009)). For this case, De Giorgi (2002), Section 5.1, has proved the following proposition, which is a special case of the more general Extension 3.9.

Proposition 3.10. *For one risk free asset and $m \geq 2$ risky assets with multivariate normally distributed returns, $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$, and in the absence of efficiency restrictions,*

1. *if the $(\mu, \rho_\phi)_1$-efficient frontier is non-empty, the $(\mu, \rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio;*

2. *if the $(\mu, \rho_\phi)_1$-efficient frontier is non-empty, the tangency portfolios in the (μ, σ^2)- and the (μ, ρ_ϕ)-framework coincide.*

Note that under the assumption of normally distributed returns the portfolios on the $(\mu, \sigma^2)_2$- and the $(\mu, \rho_\phi)_2$-efficient frontiers coincide, but are still different in shape: While the $(\mu, \sigma^2)_2$-efficient frontier is a strictly concave square root function, the $(\mu, \rho_\phi)_2$-efficient frontier is a straight line. As a consequence, the choice of optimal portfolios will also differ fundamentally under the variance and spectral risk measures, respectively.

4. Optimal portfolios

4.1. Determination of optimal portfolios

In order to establish our integrated portfolio selection approach, we now turn to the choice of optimal portfolios ("decision" context).

In the prevailing literature, optimal portfolios are chosen by fixing a certain level of expected return $\bar{\mu}$, and finding the corresponding portfolio on the (μ, ρ)-efficient frontier (e.g., ADAM ET AL. (2008), BASSETT ET AL. (2004), BENATI (2003), BERTSIMAS ET AL. (2004), DE GIORGI (2002), KROKHMAL ET AL. (2002), ROCKAFELLAR/URYASEV (2000)). Unlike this limited analysis, we apply a more general tradeoff analysis, which explicitly models an investors (μ, ρ)-preferences in the form of a (μ, ρ)-utility function.

In the (μ, σ^2)-framework, the limited analysis and the tradeoff analysis are equivalent approaches. Searching for the $(\mu, \sigma^2)_2$-efficient frontier requires solving

$$\begin{aligned} &\min_{\beta \geq 0, \gamma \in \mathbb{R}} \frac{1}{2} \cdot Var(X_{\beta,\gamma}) \\ &\text{s.t. } E(X_{\beta,\gamma}) = \bar{\mu} \geq E(X_{MVP}), \end{aligned} \tag{20}$$

which can be written in the tradeoff form

$$\max_{\beta \geq 0, \gamma \in \mathbb{R}} E(X_{\beta,\gamma}) - \frac{\bar{\lambda}}{2} \cdot Var(X_{\beta,\gamma}). \tag{21}$$

We refer to (21) as the *mean-variance utility function.* The two problems (20) and (21) with parameters $\bar{\mu}$ and $\bar{\lambda}$, respectively, are equivalent if and only if $\bar{\mu} = X_0 + \frac{(E(X_T,\sigma^2)-X_0)^2}{\bar{\lambda} \cdot Var(X_T,\sigma^2)}$ (e.g., STEINBACH (2001), Theorem 1.9). Due to this one-to-one-correspondence between $\bar{\mu}$ and $\bar{\lambda}$, one can chose between two equivalent approaches for optimal portfolio selection. As a first approach (limited analysis), an investor can fix a certain level of expected return $\bar{\mu}$. The optimal portfolio then is given by the corresponding portfolio on the $(\mu, \sigma^2)_2$-efficient frontier. As a second approach (tradeoff analysis), the same optimal portfolio obtains if the investor applies the indifference curve induced by the mean-variance utility function (21) with risk aversion $\bar{\lambda} = \frac{(E(X_T,\sigma^2)-X_0)^2}{(\bar{\mu}-X_0) \cdot Var(X_T,\sigma^2)}$ to the $(\mu, \sigma^2)_2$-efficient frontier.

From a decision theoretic perspective, the mean-variance utility function (21) follows from the assumptions of an expected utility maximizer with constant absolute risk aversion $\lambda = \bar{\lambda}$ and normally distributed returns (e.g., BAMBERG (1986), p. 20). These assumptions are well-established in portfolio selection due to their striking analytical advantages (e.g., ALEXANDER/BAPTISTA (2002), LINTNER (1969), SENTANA (2003)).

In the (μ, ρ_ϕ)-framework, the situation is fundamentally different. Searching for the $(\mu, \rho_\phi)_2$-efficient frontier requires solving

$$\begin{aligned} &\min_{\beta \geq 0, \gamma \in \mathbb{R}} \rho_\phi(X_{\beta,\gamma}) \\ &\text{s.t. } E(X_{\beta,\gamma}) = \bar{\mu} \geq E(X_{MSP}), \end{aligned} \tag{22}$$

which has a tradeoff version of the form

$$\max_{\beta\geq 0,\gamma\in\mathbb{R}}(1-\bar{\lambda})\cdot E(X_{\beta,\gamma})-\bar{\lambda}\cdot\rho_\phi(X_{\beta,\gamma}),\bar{\lambda}\in[0,1]. \tag{23}$$

For reasons stated below, we refer to (23) as *spectral utility function.* While the two problems (22) and (23) induce the same $(\mu,\rho_\phi)_2$-efficient frontier (e.g., Krokhmal et al. (2002), Theorem 3, Acerbi/Simonetti (2002), Proposition 4.2), we no longer observe a one-to-one correspondence between $\bar{\mu}$ and $\bar{\lambda}$. As Tobin separation holds in Setting 2, the relevant first-order condition of (23) is given by

$$\frac{\mathrm{d}(\cdot)}{\mathrm{d}\beta}=(E(X_{T,\rho_\phi})-X_0)-\bar{\lambda}\cdot(E(X_{T,\rho_\phi})+\rho_\phi(X_{T,\rho_\phi})), \tag{24}$$

and has

$$\bar{\lambda}=\frac{E(X_{T,\rho_\phi})-X_0}{E(X_{T,\rho_\phi})+\rho_\phi(X_{T,\rho_\phi})}\geq 0. \tag{25}$$

as a unique solution. Obviously, $\bar{\lambda}$ does not depend on $\bar{\mu}$, so an investor's specific (μ,ρ_ϕ)-preferences in the form of a spectral utility function with risk aversion $\bar{\lambda}$ do not imply a unique level of expected return $\bar{\mu}$. In other words, the limited analysis is not covered by the more general tradeoff analysis anymore. Hence, for the choice of an optimal portfolio it is no longer sufficient to fix a certain level of expected return $\bar{\mu}$ and to find the corresponding portfolio on the $(\mu,\rho_\phi)_2$-efficient frontier as within the limited analysis. Rather, this approach neglects the fact that certain levels of expected return are not under an investor's consideration if she maximizes a spectral utility function. It instead becomes necessary to apply the tradeoff analysis, which finds the optimal portfolio at the tangential point between the indifference curves induced by the spectral utility function (23) and the $(\mu,\rho_\phi)_2$-efficient frontier. This constitutes our integrated portfolio selection approach.

The spectral utility function (23) receives strong support also from a decision theoretic perspective. Besides the two components (negative) mean $-E(X)$ and the spectral risk measure $\rho_\phi(X)$, the negative convex combination $\pi_\phi(X):=-\rho_\phi(X)=(1-\lambda)\cdot E(X)-\lambda\cdot\rho_\phi(X),\lambda\in[0,1]$ satisfies (up to the algebraic sign) the properties of spectral risk measures as well (Acerbi (2002), Proposition 2.2). Therefore, the determination of the (μ,ρ_ϕ)-efficient frontiers and the consequent choice of optimal portfolios by maximizing a spectral utility function π_ϕ are based on one consistent axiomatic, and thus integrated, framework. However, this framework is still based on the underlying regulatory concept of diversification, which relates exclusively to the dependence structure.

Note that any negative spectral risk measure is a spectral utility function. Splitting up the risk spectrum ϕ into

$$\phi(p)=\phi(1)+(1-\phi(1))\cdot\hat{\phi}(p),\ \text{where}\ \hat{\phi}(p)=\frac{\phi(p)-\phi(1)}{1-\phi(1)},\phi(1)\in[0,1], \tag{26}$$

shows that the corresponding spectral utility function indeed becomes a reward-risk model of the form

$$\pi_\phi(X) = -\rho_\phi(X) = \phi(1) \cdot E(X) - (1 - \phi(1)) \cdot \rho_{\hat{\phi}}(X). \tag{27}$$

Therefore, the integrated portfolio selection approach is in line with Acerbi/Simonetti (2002), who note that "minimizing a Spectral Measure is already in some sense 'minimizing risks and maximizing returns at the same time'" (p. 10). However, these authors do not consider portfolio selection problems within an integrated framework as we do.

Based on the above argument we give the following definition, which implements the tradeoff analysis.

Definition 4.1. *A portfolio is called optimal if it maximizes a utility function π over a set of alternatives $\mathcal{X}$.*

The optimal portfolios are located where the indifference curves induced by the mean-variance utility function (21) and the spectral utility function (23) are tangent to the (μ, ρ)-efficient frontiers. As the analysis so far has been based on the (μ, σ^2)- and the (μ, ρ_ϕ)-plane, the induced indifference curves both are linearly increasing with slope $\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{\lambda}{2} \geq 0$ and $\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \frac{\lambda}{1-\lambda} \geq 0$, respectively. Note that for the mean-variance utility function and the spectral utility function, an investor's risk aversion increases with increasing λ, as the corresponding certainty equivalents, $\pi(X)$ and $\pi_\phi(X)$, decrease.

4.2. The mean-variance utility function and full diversification

As has been argued in Section 3, the $(\mu, \sigma^2)_2$-efficient frontier is a strictly concave curve. The marginal rate of transformation according to (15) is

$$\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{E(X_{T,\sigma^2}) - X_0}{2 \cdot \beta \cdot Var(X_{T,\sigma^2})} \in (0, \infty), \beta \in (0, \infty). \tag{28}$$

Similarly, the $(\mu, \sigma^2)_1$-efficient frontier corresponds to the strictly concave upper branch of a parabola. Its marginal rate of transformation according to (11) is given by

$$\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{E(X_1) - E(X_2)}{2 \cdot (\gamma \cdot a + b)} \in (0, \infty), \gamma \in (-\infty, \gamma_{MVP}). \tag{29}$$

Taking into account that the indifference curves of the mean-variance utility function (21) are linear, i.e., the marginal rate of substitution $\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{\lambda}{2}$ is constant, we immediately get the following proposition (see Figure 5).

Proposition 4.2. *Suppose an investor maximizes the mean-variance utility function (21) with respect to β and γ in Settings 1 and 2, respectively. Then*

$$\beta^* = \frac{E(X_{T,\sigma^2}) - X_0}{\lambda \cdot Var(X_{T,\sigma^2})}, \tag{30}$$

$$\gamma^* = \gamma_{MVP} - \frac{E(X_2) - E(X_1)}{\lambda \cdot a}. \tag{31}$$

We are now prepared to characterize the concept of diversification underlying the traditional (μ, σ^2)-framework. First, consider the case with a risk free asset. The set of the $(\mu, \sigma^2)_2$-efficient portfolios is given by $\beta \in [0, \infty)$, where $\beta = 0$ describes the risk free asset and $\beta = 1$ corresponds to the tangency portfolio. The observations are as follows.

– *Positive risk premium:* A positive risk premium is a necessary and sufficient condition for diversification, i.e., $\beta^* > 0 \Leftrightarrow E(X_{T,\sigma^2}) - X_0 > 0$.

– *Comonotonicity:* As the risk free asset and the tangency portfolio are comonotonic, diversification also obtains for comonotonic assets.

– *Efficient versus optimal portfolios:* Any $(\mu, \sigma^2)_2$-efficient portfolio $\beta \in [0, \infty)$ can be optimal if the risk aversion is chosen adequately as

$$\lambda(\beta^*) = \frac{E(X_{T,\sigma^2}) - X_0}{\beta^* \cdot Var(X_{T,\sigma^2})} \geq 0; \tag{32}$$

thus, the set of $(\mu, \sigma^2)_2$-efficient portfolios and the set of optimal portfolios coincide.

– *Comparative risk aversion:* The investment in the risk free asset is continuously increasing in the risk aversion λ, with $\lim_{\lambda \to 0} \beta^* = \infty$ and $\lim_{\lambda \to \infty} \beta^* = 0$.

Now consider the case without the risk free asset. The set of the $(\mu, \sigma^2)_1$-efficient portfolios is given by $\gamma \in (-\infty, \gamma_{MVP}]$, where $\gamma = 0$ corresponds to the risky asset X_2. The observations are as follows:

– *Positive excess return:* A positive excess return is a necessary and sufficient condition for diversification between the minimum-variance portfolio and the risky asset X_2, i.e., $\gamma^* < \gamma_{MVP} \Leftrightarrow E(X_2) - E(X_1) > 0$.

– *Comonotonicity:* As $a > 0$ and $b < 0$ for any correlation coefficient $corr(X_1, X_2) \in [-1, 1]$, diversification obtains for any dependence structure, and also for comonotonic risky assets.

– *Efficient versus optimal portfolios:* Any $(\mu, \sigma^2)_1$-efficient portfolio $\gamma \in (-\infty; \gamma_{MVP}]$ can be optimal if the risk aversion is adequately chosen as

$$\lambda(\gamma^*) = \frac{E(X_2) - E(X_1)}{(\gamma_{MVP} - \gamma^*) \cdot a} \geq 0; \tag{33}$$

thus, the set of $(\mu, \sigma^2)_1$-efficient portfolios and the set of optimal portfolios, for any correlation coefficient $corr(X_1, X_2) \in [-1, 1]$, coincide.

– *Comparative risk aversion:* The investment towards the minimum-variance portfolio is continuously increasing in the risk aversion λ, with $\lim_{\lambda \to 0} \gamma^* = \infty$ and $\lim_{\lambda \to \infty} \gamma^* = \gamma_{MVP}$.

The concept of diversification underlying the traditional (μ, σ^2)-framework is based on the tradeoff between an investor's risk aversion and a positive risk premium, i.e., on the tradeoff between reward and risk. This follows from the fact that in both Settings 1 and 2, a positive risk premium and excess return is a necessary and sufficient condition for diversification, $\beta^* > 0$ and

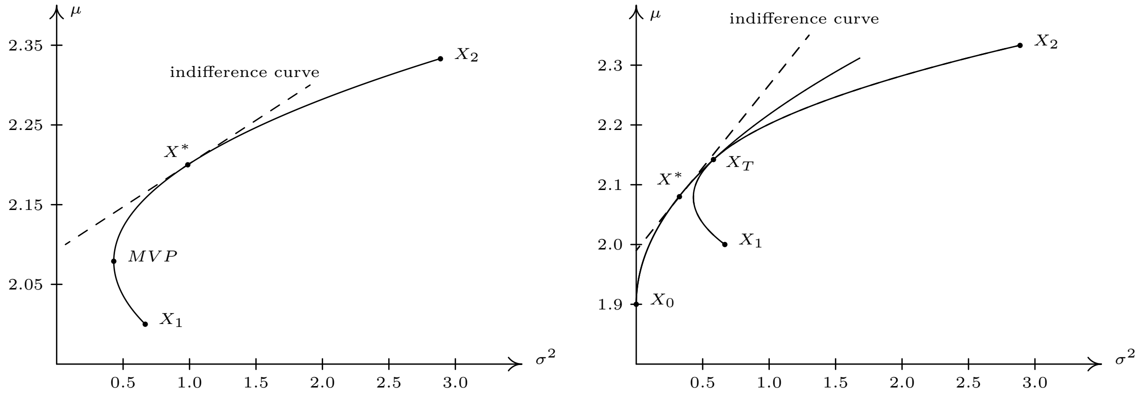

Figure 5: Choice of optimal portfolios with the mean-variance utility function

$\gamma^* < \gamma_{MVP}$, respectively. This result is independent of the dependence structure between the risky assets; it also holds for comonotonic risky assets, and the risk free asset and the tangency portfolio, respectively. The dependence structure affects this tradeoff indirectly by affecting the variance of the tangency portfolio and the minimum-variance portfolio, but it is not the origin of diversification. For these reasons, we refer to this kind of diversification as *full diversification.* Although full diversification under comonotonicity, and even for $corr(X_1, X_2) = 1$, may appear counter-intuitive at first sight, it is consistent with the tradeoff between reward and risk: Even if there is no additional diversification benefit from the dependence structure, an investor may prefer a (μ, σ^2)-efficient reward-risk profile that lies in the interior of the comonotonic assets' reward-risk profiles.

4.3. Spectral utility functions and limited diversification

We take the case of a risk free asset as a starting point, as it brings us to our main result. The $(\mu, \rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio. Its constant marginal rate of transformation according to (18) is given by

$$\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \frac{E(X_{T,\rho_\phi}) - X_0}{\rho_\phi(X_{T,\rho_\phi}) - \rho_\phi(X_0)} \geq 0. \tag{34}$$

As the indifference curves of the spectral utility function are linear as well, and the marginal rate of substitution $\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \frac{\lambda}{1-\lambda}$ is constant, we immediately obtain the following Proposition (see Figure 6).[6]

Proposition 4.3. *Suppose an investor maximizes a spectral utility function (23) with respect to β in Setting 2. Then*

$$\beta^* = \begin{cases} 0 & \textit{if } \frac{E(X_{T,\rho_\phi})-X_0}{\rho_\phi(X_{T,\rho_\phi})-\rho_\phi(X_0)} \geq \frac{\lambda}{1-\lambda} \\ \infty & \textit{else} \end{cases} . \tag{35}$$

In order to provide a more real-world interpretation, we impose the short-sale constraint $\beta \in [0, 1]$, and get $\beta^* \in \{0, 1\}$, i.e., the investor either invests in the risk free asset ($\beta = 0$) or the

[6] Without loss of generality, we assume that if the marginal rate of transformation and the marginal rate of substitution coincide, the investor decides for a corner position.

tangency portfolio ($\beta = 1$).

Recall from Section 2.1 and Proposition 2.3 that spectral risk measures and spectral utility functions are based on a regulatory concept of diversification. Within that concept, and unlike under the variance, it is no longer the tradeoff between reward and risk, that induces diversification. Instead, non-comonotonicity is a necessary condition for a positive diversification benefit. For that reason, we do not observe diversification between the risk free asset and the tangency portfolio, which are comonotonic. Spectral utility functions restrict diversification on the elementary level "risk free versus risky", as either the exclusive investment in the risk free asset or in the tangency portfolio remain as optimal solutions. We refer to this portfolio structure as *limited diversification.* Note that, under limited diversification, a risk averse investor may decide for the exclusive investment in the tangency portfolio though a risk free asset is available. On the other hand, a risk averse investor may also decide for the exclusive investment in the risk free asset though the tangency portfolio offers a positive risk premium. Both portfolio selection decisions appear as rather counter-intuitive when compared to the traditional (μ, σ^2)-framework. In more detail, we have the following observations:

– *Positive risk premium:* A positive risk premium is neither necessary nor sufficient for diversification, $\beta^* \in (0, 1)$.

– *Comonotonicity:* Diversification does not occur between a risk free and a risky asset, which are comonotonic.

– *Efficient versus optimal portfolios:* The set of $(\mu, \rho_\phi)_2$-efficient portfolios, $\beta \in [0, 1]$, does not coincide with the set of optimal portfolios, $\beta^* \in \{0, 1\}$, which only contains the corner positions.

– *Comparative risk aversion:* The optimal proportion β^* is not continuous in the risk aversion λ. Up to a certain degree of risk aversion, the exclusive investment in the tangency portfolio is optimal, and subsequently the optimum jumps towards the exclusive risk free investment. The underlying concept of risk aversion is consistent in the sense that a more risk averse investor demands a higher risk premium for switching to the risky investment (see (35)).

From a decision theoretic perspective, limited diversification is not a convincing result, as there is no reason for excluding other $(\mu, \rho_\phi)_2$-efficient portfolios from being optimal, merely because they belong to a comonotonic set of alternatives. This only means that both expected return and spectral risk increase linearly. It might well be possible, and is economically plausible, that an investor prefers a $(\mu, \rho_\phi)_2$-efficient reward-risk profile somewhere in the interior of a comonotonic set of alternatives.

Further, Extension 3.9 shows that the all-or-nothing decision holds in settings that are far more general than our Setting 2, as the $(\mu, \rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio also for $m \geq 2$ risky assets, and for arbitrary distributions of the risky asset-returns. Therefore, the all-or-nothing decision also obtains under the assumption of normally distributed returns, and thus provides an interesting result: Although the efficient frontiers in the (μ, σ^2)- and the (μ, ρ_ϕ)-framework coincide, the choice of optimal portfolios differs fundamentally between full diversification and limited diversification.

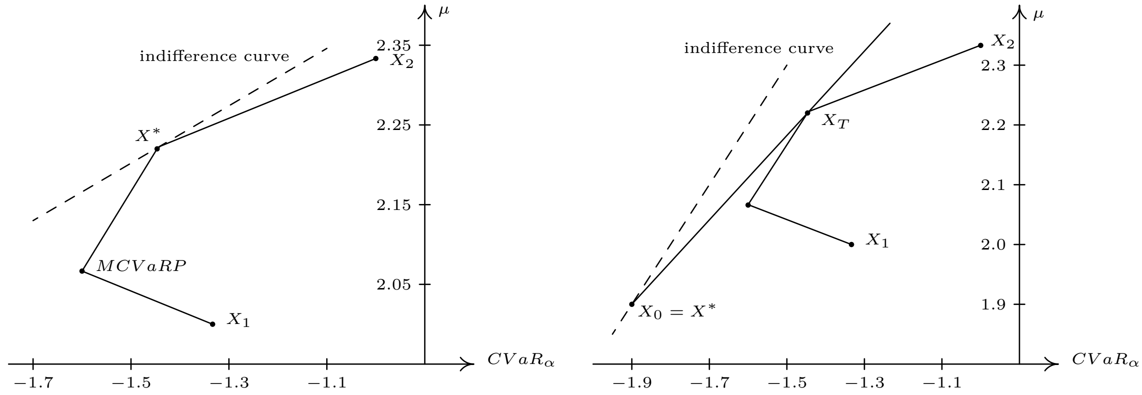

Figure 6: Choice of optimal portfolios with the spectral utility function

The case without the risk free asset yields similar results. The $(\mu, \rho_\phi)_1$-efficient frontier is a concave curve that for comonotonic subsets of alternatives is piecewise linear. For a comonotonic subset of alternatives $X_\gamma, \gamma \in [\gamma_d, \gamma_u]$, the constant marginal rate of transformation is given by (see (14)):

$$\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})}. \tag{36}$$

Together with the linear indifference curves of the spectral utility function, i.e., the marginal rate of substitution $\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \frac{\lambda}{1-\lambda}$ is constant, we immediately get the following result (see Figure 6).

Proposition 4.4. *Suppose an investor maximizes a spectral utility function (23) with respect to γ in Setting 1. Then*

$$\gamma^* \in \{-\infty, \gamma_{ij,1:k}, \ldots, \gamma_{ij,k:k}\}. \tag{37}$$

Diversification under spectral risk measures and risky assets thus provides similar results to the case with a risk free asset.

– *Positive excess return:* A positive excess return is neither necessary nor sufficient for diversification, $\gamma^* \in (-\infty, \gamma_{MSP})$.

– *Comonotonicity:* For comonotonic subsets of alternatives, the $(\mu, \rho_\phi)_1$-efficient frontier is a straight line, so that diversification within this subset does not occur. Reward-risk profiles that lie in the interior of a comonotonic subset of alternatives thus can never be optimal.

– *Efficient versus optimal portfolios:* The set of optimal portfolios is restricted to the $(\mu, \rho_\phi)_1$-efficient boundaries of the comonotonic subsets of alternatives, whereas the set of $(\mu, \rho_\phi)_1$-efficient portfolios also contains the interior portfolios. Therefore, the set of efficient portfolios and the set of optimal portfolios do not coincide, and only limited diversification prevails.

– *Comparative risk aversion:* The optimal proportion γ^* is not continuous in the risk aversion λ. With risk aversion increasing, the same proportion remains optimal until the portfolio jumps to the next corner position.

5. Discussion

The findings on limited diversification within our integrated framework have strong implications for the use of spectral risk measures and spectral utility functions. If an investor does not agree with either investing exclusively risk free or investing exclusively in risky assets, spectral utility functions are not a reasonable approach for choosing optimal portfolios. As a consequence of our integrated framework, the portfolio selection then cannot be based on (μ, ρ_ϕ)-efficient frontiers either. Therefore, the (μ, ρ_ϕ)-framework lacks a foundation, at least from a decision theoretic perspective.

Our results also raise doubts on the validity of the empirical findings on portfolio selection with Conditional Value-at-Risk and spectral risk measures. The relevant literature compares the composition of optimal portfolios in the (μ, σ^2)- and the (μ, ρ_ϕ)-framework based on a fixed level of expected return $\bar{\mu} \in [X_0, E(X_{T,\rho_\phi})]$. This limited analysis, however, is insufficient as it neglects the fact that an (μ, ρ_ϕ)-investor in our more general integrated framework would not choose expected returns in this interval, but always prefers one of the corner positions $\bar{\mu} = X_0$ or $\bar{\mu} = E(X_{T,\rho_\phi})$ to any interior portfolio. In other words, by fixing a certain level of expected return $\bar{\mu} \in (X_0, E(X_{T,\rho_\phi}))$ diversification is artificially enforced, although being absent naturally. Therefore, a non-optimal portfolio is regularly assumed to be optimal in these studies.

Moreover, experimental evidence is mixed. A first strand of literature shows that investors diversify between a risk free and a risky asset (e.g., Benartzi/Thaler (1999), Rapoport (1984)). In another strand of literature underdiversification is documented (e.g., Mitton/Vorkink (2007)). However, underdiversification and limited diversification, as in the present paper, are different concepts. Whereas underdiversification denotes the investment in only a few assets, limited diversification refers to the fact that only a few of the (μ, ρ_ϕ)-efficient portfolios are actually chosen by an investor; whether these portfolios consist of a few or a large number of assets is not subject to considerations of the present paper. On the other hand, the (μ, ρ_ϕ)-framework might be a possible explanation for the stock market participation puzzle. For example, Mankiw/Zeldes (1991) find that only 25% of investors hold stocks, which can easily be reconciled with a (μ, ρ_ϕ)-framework, but not with a (μ, σ^2)-framework.

We thus conclude that the use of spectral risk measures for portfolio selection appears inappropriate from a decision theoretic, an empirical and, partly, an experimental perspective. In a sense, already Markowitz (1952) raised serious doubts in this respect by stating that "diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim" (p. 77).

The all-or-nothing decisions under spectral risk measures also provide counter-intuitive results in that they may lead to out-of-equilibrium capital markets. If we assume a representative investor economy with a risk free asset, due to (35) it might be that only the risk free asset is demanded, and risky assets are not held. An equilibrium in the sub-market for risky assets thus is not established and they are not in zero net supply either. Conversely, it might also be that there is no demand and no need for the risk free asset at all.

While in the present literature on spectral risk measures these results so far have been overlooked, our criticism is well-known from the dual theory of choice, which provides the representation

$$\pi_D(X) = \int_0^1 F_X^*(p) \mathrm{d}v(p) \tag{38}$$

(e.g., Denneberg (1988), Roell (1987), Yaari (1987)). The dual utility function v coincides with the primitive function Φ of the risk spectrum. For the representation of risk aversion, v has to be concave with $v(0) = 0, v(1) = 1$, recovering the class of spectral risk measures and spectral utility functions, respectively. Within this framework, already Yaari (1987), Section 6, notes that the dual theory of choice tends towards all-or-nothing decisions instead of diversification. On the other hand, Hadar/Seo (1995) give conditions under which a dual investor diversifies between two (or more) risky assets. Nonetheless, they do not consider a risk free asset, and they do not refer to the portfolio structure itself as we do.

6. Conclusions

In this paper, we applied Conditional Value-a-Risk and, as its natural extension, spectral risk measures to some simple portfolio selection problems. Our approach differs from the previous literature in two respects. First, we use a finite state space approach to show that the efficient frontiers are piecewise linear in the case of risky assets, and linear in the case of an additional risk free asset. By contrast, in the traditional mean-variance framework the efficient frontiers are strictly concave in any case. Second, we show that choosing optimal portfolios by fixing a certain level of expected return and finding the corresponding portfolio on the efficient frontier (limited analysis), as is done in the relevant literature so far, provides misleading results. By applying the indifference curves induced by a spectral utility function (tradeoff analysis), which naturally arises both from a decision theoretic and an optimization perspective, we obtain fundamentally different optimal portfolios. If a risk free asset is available, diversification is never optimal. Likewise, in the case of risky assets we find only limited diversification. By contrast, the mean-variance framework shows full diversification in any case.

The reason is that spectral risk measures originally have been introduced for the assessment of solvency capital ("risk"). The underlying regulatory concept of diversification regards the dependence structure between the assets as the only source for positive diversification benefits. For the special case of perfect positive dependence (comonotonicity) the diversification benefit is zero, which is an adequate requirement for the assessment of solvency capital. Traditional mean-variance portfolio selection ("decision"), by contrast, is based on a fundamentally different concept of diversification that refers to the tradeoff between reward and risk, and that is only indirectly affected by the dependence structure between the assets. The incompatibility of these conflicting concepts of diversification produces limited diversification.

In formal terms, the concept of diversification underlying spectral risk measures (and spectral utility functions) is determined jointly by the properties of subadditivity (superadditivity) and comonotonic additivity. The relevant literature is focused on subadditivity only and omits considering comonotonic additivity, e.g., "sub-additivity is an essential property also in portfolio-optimization problems" (Acerbi/Tasche (2002a), p. 381). We indeed agree that subadditivity is an essential property for the assessment of solvency capital and for portfolio selection. However, we have shown that comonotonic additivity is essential only for the assessment of solvency capital ("risk"), but leads to paradoxical results if applied to portfolio selection ("decision").

Notwithstanding these findings, we do not have any doubt that an axiomatic framework is useful to avoid mathematical and contextual inconsistencies, and that it preserves investors from choosing a risk measure or utility function in an ad hoc manner. On the other hand, axiomatic approaches are by no means a universal concept that can be applied to any context regardless of the original context in which they have been developed. This view contrasts with that of Acerbi/Tasche (2002a) (p. 380), who state that "in our opinion speaking of non-coherent [by this, they basically mean what we call in the present paper "non-spectral"] measures of risk is (..) useless and dangerous. In our language, the adjective coherent is simply redundant." Instead, we have seen that it is insufficient to claim that axiomatic approaches are per se superior without taking account of the consequences from both an economic and a decision theoretic perspective. As a more appropriate alternative to spectral risk measures, and as an agenda for future research, we would instead propose convex risk measures, which do not require comonotonic additivity (e.g., Föllmer/Schied (2002)).

A. Extensions of the (μ, σ^2)-framework

Considering $m \geq 2$ risky assets and omitting the efficiency restriction yields the well-known mutual fund theorem (Merton (1972), Section 3): The $(\mu, \sigma^2)_1$-boundary can be generated by any two distinct $(\mu, \sigma^2)_1$-boundary portfolios. The $(\mu, \sigma^2)_1$-efficient frontier lies on the upper branch of the $(\mu, \sigma^2)_1$-boundary starting from the minimum-variance portfolio. For a rigorous analytical treatment see, for example, De Giorgi (2002), Section 4.1. Recall from Proposition 3.1 that the $(\mu, \sigma^2)_1$-efficient frontier for two risky assets is a strictly concave curve for any correlation coefficient $corr(X_1, X_2) \in [-1, 1]$. As the case of $m \geq 2$ risky assets is formally equivalent to the case of two distinct $(\mu, \sigma^2)_1$-boundary portfolios, we obtain the following extension.

Extension A.1. *For $m \geq 2$ risky assets and in the absence of the efficiency restriction, the $(\mu, \sigma^2)_1$-efficient frontier is a strictly concave curve that starts from the minimum-variance portfolio.*

Also, it is well-known that by additionally considering a risk free asset, Tobin separation from Proposition 3.4 still holds. For a full analytical treatment see again De Giorgi (2002), Section 5.1. We summarize as follows.

Extension A.2. *For a risk free asset and $m \geq 2$ risky assets with $E(X_0) < E(X_{MVP})$ and in the absence of the efficiency restriction, the $(\mu, \sigma^2)_2$-efficient frontier is a strictly concave curve through the risk free asset and the tangency portfolio.*

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