Pricing the CBOT T-Bonds Futures*

Ramzi Ben-Abdallah†
HEC Montréal, Canada

Hatem Ben-Ameur
CREF and GERAD, HEC Montréal, Canada

Michèle Breton
CREF and GERAD, HEC Montréal, Canada

January 22, 2007

Abstract

The aim of this paper is to propose a numerical method to price the Chicago Board of Trade Treasury-bond futures. This contract is one of the most traded in the world, largely because of its ability to hedge long term interest rate risk. The difficulty to price it arises from its multiple inter-dependent embedded delivery options, which can be exercised at various times and dates during the delivery month. We consider a continuous time model with a continuous underlying factor (the interest rate), moving according to a Markov diffusion process consistent with the no-arbitrage principle. We propose a model that can handle all the delivery rules embedded in the CBOT T-bond futures, interpreted here as an American-style interest rate derivative. Our pricing procedure is a backward numerical algorithm combining Dynamic Programming (DP), approximation by finite elements, and fixed point evaluation. Numerical illustrations are provided under the Vacisek and CIR models.

JEL Classification: C61; C63; G12; G13.
Keywords: Futures, asset pricing, dynamic programming.

*Research supported by NSERC Canada and IFM2.
†Ph.D. Candidate, HEC Montréal, 3000 chemin de la Côte Sainte-Catherine, Montreal, Canada, H3T 2A7; Tel: (514) 340-6000, Ext 2469; E-mail: ramzi.ben-abdallah@hec.ca.

1 Introduction

A *futures contract* is an agreement between two investors traded on an exchange to sell or to buy an *underlying asset* at some given time in the future, called the *delivery date*, for a given price, called the *futures price*. By convention, at the time the futures is written (the *inception date*), the futures price is known and sets the value for both parties to zero. A futures contract is marked to market once a day to eliminate counterparty risk. Precisely, at the end of each trading day, the futures contract is rewritten at a new *settlement price*, that is, the closing futures price, and the difference with the last settlement futures price is substracted (*resp. added*) from the short (*resp. long*) trader account.

The *Treasury Bond futures* traded on the Chicago Board of Trade (the CBOT T-bond futures in the sequel) is the most actively traded and widely used futures contract in the United States, largely because of its ability to hedge long term interest rate risk. It calls for the delivery of $100,000 of a long-term governmental bond. The *notional* or *reference bond* is a bond with a 6% coupon rate and a maturity of 20 years. Delivery months are March, June, September and December.

Since the notional bond is a hypothetical bond that is generally not traded in the market place, the short has the option to choose which bond to deliver among a deliverable set fixed by the CBOT. The actual delivery day within the delivery month is also at the option of the short. These two delivery privileges offered to the short trader are known as the *choosing option* (or *quality option*) and the *timing option*.

The choosing option allows the delivery of any governmental bond with at least 15 years to maturity or earliest call. To make the delivery fair for both parties, the price received by the short trader is adjusted according to the quality of the T-bond delivered. This adjustment is made via a set of *conversion factors* defined by the CBOT as the prices of the eligible T-bonds at the first delivery date under the assumption that interest rates for all maturities equal 6% par annum, compounded semiannually. The T-bond actually delivered by the short trader is called the *cheapest-to-deliver* (CTD).

The timing option allows the short trader to deliver early within a delivery month according to special features, that is, the *delivery sequence* and the *end-of-month delivery rule*. The delivery sequence consists of three consecutive business days: The *position day*, the *notice day*, and the *delivery day*. During the position day, the short trader can declare his intention to deliver until up to 8:00 p.m., while the CBOT closes at 2:00 p.m. (Central Standard Time). On the notice day, the short trader has until 5:00 p.m. to state which T-bond will be actually delivered. The delivery

then takes place before 10:00 a.m. of the delivery day, against a payment based on the settlement price of the position day (adjusted according to the conversion factor). Finally, during the last seven business days before maturity, trading on the T-bond futures contracts stops while delivery, based on the last settlement price, remains possible according to the delivery sequence. The so-called *wild card play* (or *end-of-the day option* or *six hours option*) and the *end-of-month option* refer respectively to the timing option during the three day delivery sequence and to the end-of-month rule.

The modeling and measurement of the delivery options implicit in T-bond futures contracts has been examined in the literature using different methods and leading to non consensual empirical results. In particular, the issues of the performance of the conversion factor system, the identification of cheapest-to-deliver bond as well as the valuation of the quality option embedded in Treasury-bond futures contracts have been the subject of a substantial volume of research. A first stream of papers deals with the so-called *conversion factor risk* and its impact on the market, propose alternative conversion systems and rules for the identification of the CTD (see for instance Livingston (1984), Kane and Marcus (1984), Jones (1985), Arak et al. (1986), Benninga and Wiener (1999) and Oviedo (2006)). A second stream of papers proposes valuation approaches for the quality option, which is considered to be the most important, assuming a flat term structure for interest rates. Four main methods are used: the exchange option pricing formula (Gay and Manaster (1984), Boyle (1989), Hemler (1990)), the buy-and-hold approach (Hemler (1990), Kane and Marcus (1986a), Hegde (1990)), the implied option value approach (Hegde (1988), Hemler (1990), Hegde (1990)) and the switching option method (Barnhill and Seale (1988a, 1988b), Hedge (1990)). A third stream of research, also concentrating on the quality option, takes into consideration the term structure and stochastic nature of interest rates (Ritchken and Sankarasubramanian (1992, 1995), Bick (1997), Carr and Chen (1997), Lacoste (2002)). While the quality option is assumed to be the most important, ignoring the other delivery options may lead to mispricing, and fail to suggest optimal delivery strategies. A last stream of research considers the timing option, either separately (Gay and Manaster (1986), Kane and Marcus (1986b)) or in conjunction with the quality option (Arak and Goodman (1987), Peck and Williams (1990), Boyle (1989), Gay and Manaster (1991), Nielsen and Ronn(1997), Chen and Yeh (2004), Hranaoiva et al. (2005)). It is worth mentioning that the papers in this last category use simplifying assumptions on the dynamics or the strategies. To date, no work has been presented regarding the identification of optimal exercise strategies in the CBOT T-bond futures trading and the pricing of the contract under stochastic interest rates when the interaction of all the delivery options is taken into

account. In fact, complexity arising from all the embedded inter-dependent delivery rules makes the contract computationally and analytically difficult to price.

The aim of this paper is to propose a model and a pricing algorithm that can handle explicitly and simultaneously all the delivery rules embedded in the CBOT T-bond futures, in a stochastic interest rate environment. To do so, we consider a continuous time model with a continuous underlying factor, the risk-free short term interest rate. We assume that this rate moves according to a Markov diffusion process that is consistent with the no-arbitrage principle. Our pricing procedure is a backward numerical algorithm combining Dynamic Programming (DP), approximation by finite elements, and fixed point evaluation. In this context, the DP value function is the value of the contract for the short trader (which is reset to 0 at the settlement dates) and the DP recursion is given by no-arbitrage pricing (Elliott and Kopp, 1999). Under a given assumption about the evolution of interest rates, the numerical procedure output may be summarized into three results. The first gives the theoretical futures prices at settlement dates. The second gives the delivery position strategy (deliver or not) on the position day. The third identifies the CTD on the notice day, given the futures price at the last settlement date. All three results are functions of time and current interest rate.

The paper is organized as follows. Section 2 presents the model and the Dynamic Programming formulation for the value of the contract. Section 3 describes in details the numerical procedure. Section 4 reports on numerical results obtained under both the Vasicek (1977) and Cox, Ingersoll and Ross (1985) (hereafter CIR) models for the short rate process. Section 5 is a conclusion.

2 Model and DP Formulation

2.1 Notation

We consider frictionless cash and T-bond futures markets in which trading takes place continuously. Denote

- $(c, M) \in \Theta$ an eligible T-bond with a principal of 1 dollar, a continuous coupon rate c, and a maturity M, where the set Θ of eligible bonds is known at the date the contract is written;

- f_{cM} the conversion factor corresponding to the T-bond (c, M), where the set $\{f_{cM} : (c, M) \in \Theta\}$ is known at the date the contract is written;

- $\{r_t\}$ a Markov process for the risk-free short term interest rate that is consistent with the no-arbitrage principle;
- $\rho(r,t,\tau)$ the discount factor over the period $[t,\tau]$, $\tau \geq t$, when $r_t = r$

$$\rho(r,t,\tau) = E\left[\exp(-\int_t^\tau r_u du) \mid r_t = r\right]; \tag{1}$$

- $p(t,c,M,r)$ the price at t of the eligible T-bond (c,M) when $r_t = r$,

$$p(t,c,M,r) = c\int_t^M \rho(r,t,u)du + \rho(r,t,M);$$

- $g_t(r)$ a given price for the T-bond futures at t when $r_t = r$;
- $g_t^*(r)$ the fair settlement price for the T-bond futures at t when $r_t = r$.

To be consistent with the CBOT delivery rules, we consider a sequence of motoring dates t_m^h where the lower index $m = 0,...,\overline{n}$ is computed in days from the date the contract is written and the upper index $h \in \{2,5,8\}$ indicates the time in hours within that day. Assuming that the contract is written at $t_0 = t_0^2$, we denote the marking to market dates by t_m^2 for $m = 0,\ldots,n$, where t_n represents the last trading date during the delivery month. We denote the delivery position dates by t_m^8 for $m = \underline{n},\ldots,\overline{n}$, where $t_{\underline{n}}$ and $t_{\overline{n}}$ are respectively the first and the last date of the delivery month, $0 < \underline{n} < n < \overline{n}$. Finally, the delivery notice dates are denoted t_m^5 for $m = \underline{n}+1,\ldots,\overline{n}+1$. Our choice of monitoring dates is justified by the fact that, within the delivery month, it is better for the short trader to wait until 8:00 p.m. each day to decide whether to take or not a delivery position. Moreover, we assume that the delivery notice date coincides with the actual delivery, since this does not change the (expected) value of the contract.

Our DP model determines the value of the contract for the short trader at each monitoring date, as a function of the interest rate at the current and last settlement dates, assuming that the short trader behaves optimally. We obtain the fair settlement price by making the value of the contract null for both parties at the settlement dates.

The contract is evaluated by backward recursion in three distinct periods: The end-of-the-month period, where no trading takes place, but delivery is still possible $(m = n,...\overline{n})$, the beginning of the delivery month where trading and delivery are both possible $(m = \underline{n},...,n)$, and the period before the delivery month, where no action is taken by the short trader, but the settlement price is adjusted every day $(m = 0,...,\underline{n})$.

2.2 End-of-the-month Period

Recall that during the last seven business days before maturity, trading on the T-bond futures contracts stops while delivery remains possible, based on the settlement price at the last settlement date, indexed by m'. If an intention to deliver is issued at the delivery position date t_m^8, for $m = n, ...\overline{n}$, we define the *expected exercise value* $v_m^e(r', r)$ and the *actual exercise value* $v_m^a(r', r)$ for the short trader, as a function of the interest rate at the last settlement date, denoted r', and at the current date, denoted r, as follows:

$$v_m^e(r', r) = E\left[\left(v_m^a\left(r', r_{t_{m+1}^5}\right) e^{-\int_{t_m^8}^{t_{m+1}^5} r_u du}\right) \mid r_{t_m^8} = r\right], \tag{2}$$

$$v_m^a(r', r) = \max_{(c,M)\in\Theta} \left\{g_{m'}(r') f_{cM} - p\left(t_{m+1}^5, c, M, r\right)\right\}, \tag{3}$$

where $m' = n$.

Otherwise, if the short trader decides not to deliver at t_m^8, for $m = n, ..., \overline{n}$, we define the *holding value* $v_m^h(r', r)$, which is computed by no-arbitrage to be the expected value of the future potentialities of the contract and given by (5) below. The short trader will of course issue an intention to deliver at (t_m^8, r', r) if and only if

$$v_m^e(r', r) > v_m^h(r', r).$$

The *value function* for the short trader at t_m^8, for $m = n, ..., \overline{n}$, is thus defined recursively by:

$$v_m^8(r', r) = \max\left\{v_m^e(r', r), v_m^h(r', r)\right\} \tag{4}$$

$$v_m^h(r', r) = E\left[v_{m+1}^8\left(r', r_{t_{m+1}^8}\right) e^{-\int_{t_m^8}^{t_{m+1}^8} r_u du} \mid r_{t_m^8} = r\right] \tag{5}$$

$$v_{\overline{n}}^8(r', r) = v_{\overline{n}}^e(r', r), \tag{6}$$

and the *settlement value* for the short trader at $(t_{m'}^2, r_{t_{m'}^2})$ is the expected discounted value at $t_{m'}^8$:

$$v_{m'}^2(r') = E\left[v_{m'}^8\left(r', r_{t_{m'}^8}\right) e^{-\int_{t_{m'}^2}^{t_{m'}^8} r_u du} \mid r_{t_{m'}^2} = r'\right], \tag{7}$$

where $m' = n$. Notice that equations (2) - (7) define a mapping from the space of functions $g_{m'} : \mathbb{R} \to \mathbb{R}$ to the space of functions $v_{m'}^2 : \mathbb{R} \to \mathbb{R}$, but we did not make this

dependency on $g_{m'}$ explicit to alleviate the notation. Moreover, the settlement value at m' and r' is defined for any settlement price function $g_{m'}(r')$, constant during the end-of-the-month period, which can be written

$$v_{m'}^{2}\left(r'\right) = F_{r'm'}(g)$$

where $F_{r'm'}$:$\mathbb{R}\rightarrow\mathbb{R}$ is a function defined by the Dynamic Program (2) - (7) with $g = g_{m'}(r')$ and $m' = n$.

However, the settlement price at $t_{m'}^{2}$ should be selected so that the value to both parties is 0, taking into account the timing and quality options. Consequently, the fair settlement price at $t_{m'}^{2}$, denoted $g_{m'}^{*}(r)$, is a function of the settlement date interest rate such that:

$$v_{m'}^{2}\left(r'\right) = F_{r'm'}(g_{m'}^{*}(r')) = 0 \text{ for all } r', \tag{8}$$

where $m' = n$.

2.3 Delivery Month

During the delivery month ($m = \underline{n}, ..., n-1$), the value of the contract for the short trader may be evaluated in the same manner as in the previous section, but the holding value must account for the interim payments at the marking to market dates. Thus, the exercise value functions $v_{m}^{e}\left(r', r\right)$ and $v_{m}^{a}\left(r', r\right)$ at respectively t_{m}^{8} and t_{m}^{5}, for $m = \underline{n}, n-1$ are given by (2)-(3), where $m' = m$. The holding value at t_{m}^{8}, however, accounts for the interim payment at the next marking to market date, that is,

$$\begin{aligned} v_{m}^{h}\left(r', r\right) &= E\left[\left(g_{m}(r') - g_{m+1}^{*}\left(r_{t_{m+1}^{2}}\right)\right) e^{-\int_{t_{m}^{8}}^{t_{m+1}^{2}} r_{u}du} \right. \\ &\quad \left. + \, v_{m+1}^{2}\left(r_{t_{m+1}^{2}}\right) e^{-\int_{t_{m}^{8}}^{t_{m+1}^{2}} r_{u}du} \mid r_{t_{m}^{8}} = r\right] \\ &= E\left[\left(g_{m}(r') - g_{m+1}^{*}\left(r_{t_{m+1}^{2}}\right)\right) e^{-\int_{t_{m}^{8}}^{t_{m+1}^{2}} r_{u}du} \mid r_{t_{m}^{8}} = r\right], \end{aligned} \tag{9}$$

since the settlement value at $m+1$ is the null function for a fair settlement price. The value function at t_{m}^{8} and t_{m}^{2} is then given by (4) and (7), with $m' = m$. Finally, for each mark-to-market date t_{m}^{2}, $m = \underline{n}-1, ..., n-1$, the settlement price function $g_{m'}^{*}(r)$ is such that (8) is verified, with $m' = m$.

2.4 Initial period

Within the time period $\left[t_0, t^2_{\underline{n}-1}\right]$, delivery is not possible, so that the value of the contract for the short trader only involves taking into account the interim payments in the marking to market account. The value function at t^2_m, for $m = 0, ..., \underline{n} - 1$, is thus given by

$$v^2_m(r) = E\left[\left(g_m(r) - g^*_{m+1}\left(r_{t^2_{m+1}}\right)\right) e^{-\int_{t^2_m}^{t^2_{m+1}} r_u du} \mid r_{t^2_m} = r\right],$$

where g^*_m is such that (8) is satisfied for $m' = m$, $m = 0, ...\underline{n} - 1$. Therefore, the successive settlement prices can be obtained by the recursive relation

$$g^*_m(r) = \frac{E\left[g^*_{m+1}\left(r_{t^2_{m+1}}\right) e^{-\int_{t^2_m}^{t^2_{m+1}} r_u du} \mid r_{t^2_m} = r\right]}{\rho(r, t^2_m, t^2_{m+1})} \text{ for all } r, m = 0, ...\underline{n} - 1. \quad (10)$$

3 Dynamic Programming Procedure

Equations (2)-(10) define a dynamic program which can be used to find the fair settlement prices and the optimal timing and choosing strategies for the short trader by backward induction. This dynamic program is defined on the state space $\{(r', r) : r' \geq 0,\ r \geq 0\}$ and does not admit a closed-form solution, even for the most simple case where the interest rate is assumed to be constant. In this section, we describe a numerical procedure for the solution of this dynamic program. Three specific numerical problems must be addressed:

- the optimization in (3) which involves the price of the eligible bonds according to the underlying interest rate model,
- the computation of the expectations in (2), (5), (7), (9) and (10) of functions which are analytically intractable,
- the determination of the root of (8).

The numerical procedures consists in

- finding the CTD by an appropriate search over the eligible set according to the properties of the bond prices,

- approximating the functions $v_m^8(g,\cdot)$ and $g_m^*(\cdot)$ by expectations of linear finite elements interpolation functions,
- finding the fair settlement price as a fixed point of a contraction mapping.

3.1 Optimization Procedure

Finding the CTD at m, r', r consists in solving the following:

$$v_m^a(r',r) = \max_{(c,M)\in\Theta} \left\{ g_{m'}(r') f_{cM} - \left(c \int_{t_{m+1}^5}^{M} \rho\left(r, t_{m+1}^5, u\right) du + \rho\left(r, t_{m+1}^5, M\right)\right) \right\}$$

where the finite set of eligible bonds and their conversion factors are fixed at the signature of the contract and $g_{m'}(r')$ is given. The function to be maximized is linear in c, so that the optimal coupon is extremal and given by either $\underline{c} \equiv \min c$ or $\overline{c} \equiv \max c$. If an analytic expression for $\rho(r,t,\tau)$ is known, it is straightforward to check the properties of the projections of the function to optimize on $c = \overline{c}$ and $c = \underline{c}$. If these are convex, simple inspection of the extremal maturities will yield the CTD. If either one is concave, a line search for fixed $\overline{c}$ and/or $\underline{c}$ can be performed. Otherwise, since the number of eligible bonds is fixed, an enumeration of all eligible bonds with extremal coupons will yield the CTD and the value of $v_m^a(r',r)$. Notice that, while the value of $v_m^a(r',r)$ can be obtained with as much precision as the price of a given bond for any r', r and $g_{m'}$, it cannot be expressed in closed form.

3.2 Interpolation Procedure

The interpolation procedure consists in approximating an analytically intractable function, the value of which is known at a finite number of points, by a piecewise linear continuous function.

Let $\mathcal{G} = \{a_1, \ldots, a_q\}$ be a grid defined on the set of interest rates, with the convention that $a_0 = -\infty$ and $a_{q+1} = +\infty$. Given a function $h_j : \mathcal{G} \to \mathbb{R}$, the interpolation function $\widehat{h}_j : \mathbb{R} \to \mathbb{R}$ is given by:

$$\widehat{h}_j(r) = \sum_{i=0}^{q} (\alpha_i + \beta_i r)\, I(a_i \le r < a_{i+1}), \text{ for all } r \in \mathbb{R},$$

where the function I is the indicator function and the coefficients α_i and β_i are obtained by matching $\widehat{h}$ and h on $\mathcal{G}$, that is

$$\alpha_i = \frac{a_{i+1}h(a_i) - a_i h(a_{i+1})}{a_{i+1} - a_i},$$
$$\beta_i = \frac{h(a_{i+1}) - h(a_i)}{a_{i+1} - a_i}, i = 1, ...q-1,$$

and $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\alpha_q = \alpha_{q-1}$, $\beta_q = \beta_{q-1}$.

3.3 Expectations of Interpolation Functions

Define the *transition parameters*

$$A_{k,i}^{\tau-t} \equiv E\left[I\left(a_i \leq r_\tau < a_{i+1}\right) e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \tag{11}$$

and

$$B_{k,i}^{\tau-t} \equiv E\left[r_\tau I\left(a_i \leq r_\tau < a_{i+1}\right) e^{-\int_t^\tau r_u du} \mid r_t = a_k\right], \tag{12}$$

where $t_0 \leq t \leq \tau$, $k = 0, \ldots, q$, and $i = 0, \ldots, q$.

We assume that these transition parameters and the discount factor $\rho(r, t, \tau)$ can be obtained with precision from the dynamics of $\{r_t,\, t \geq t_0\}$. Notice that for several dynamics of the interest rates, closed-form solutions exist for the transition parameters and discount factor, as discussed in Ben-Ameur et al (2007). Examples include Vasicek (1977), CIR (1985), and Hull and White (1990). Closed form formulas for the transition parameters and discount factor for the Vasicek and in the CIR model are recalled in the Appendix.

Given an interpolation function $\widehat{h}_j : \mathbb{R} \to \mathbb{R}$, the expected value at t and $r_t = a_k$ of a future payoff $\widehat{h}_j$ at τ is given by:

$$\begin{aligned}
\widetilde{h}_j\left(t, \tau, a_k\right) &\equiv E\left[\widehat{h}\left(r_\tau\right) e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \\
&= E\left[\sum_{i=0}^{q} \left(\alpha_i + \beta_i r_\tau\right) I\left(a_i \leq r_\tau < a_{i+1}\right) e^{-\int_t^\tau r_u du} \mid r_t = a_k\right] \\
&= \sum_{i=0}^{q} \alpha_i A_{k,i}^{\tau-t} + \beta_i B_{k,i}^{\tau-t} \text{ for all } a_k \in \mathcal{G}.
\end{aligned}$$

3.4 Root Finding Procedure

At a given r' and m', the root finding procedure consists in finding a constant g such that

$$v_{m'}^{2}\left(r'\right) = F_{r'm'}(g) = 0$$

where $F_{r'm'}$ is defined by the Dynamic Program (2) - (7), (9) with $g = g_{m'}(r')$ Consider two settlement prices g_1 and g_2 such that $g_1 < g_2$. Since

$$\max_{(c,M)\in\Theta} \left\{ g_1 f_{cM} - p\left(t_{m+1}^{5}, c, M, r\right) \right\} < \max_{(c,M)\in\Theta} \left\{ g_2 f_{cM} - p\left(t_{m+1}^{5}, c, M, r\right) \right\}$$

for all r and m, it is easy to show that $F_{r'm'}$ is strictly monotone in g. Moreover,

$$\lim_{g\to-\infty} F_{r'm'}(g) = -\infty$$
$$\lim_{g\to+\infty} F_{r'm'}(g) = +\infty.$$

Therefore, $F_{r'm'}(\cdot)$ admits a unique root.

Consider the following successive approximation scheme:

$$\begin{aligned} g^{1} &= g^{0} - F_{r'm'}\left(g^{0}\right) \\ g^{k+1} &= g^{k} - F_{r'm'}\left(g^{k}\right) \frac{g^{k} - g^{k-1}}{F_{r'm'}\left(g^{k}\right) - F_{r'm'}\left(g^{k-1}\right)}, \qquad k > 1 \end{aligned}$$

where g^0 is given.

Now, since $F_{r'm'}(\cdot)$ is strictly increasing, this Quasi-Newton successive approximation scheme will converge to the unique root from any starting point (for example, a good starting point is

$$g^{0} = \min_{(c,M)\in\Theta} \left\{ \frac{c\int_{T}^{M} \rho(r, t_{m'}, u)du + \rho(r, t_{m'}, M)}{\rho(r, t_{m'}, T) f_{cM}} \right\} \tag{13}$$

which is the price at $t_{m'}$ of a forward contract maturing at an appropriately chosen T with a choosing option on the same basket Θ). Moreover, since the number of exercise strategies is finite, it can be shown that there exists a neighborhood of the root where $F_{r'm'}$ is linear in g, so that this approximation scheme will converge in a finite number of steps.

3.5 Algorithm

The algorithm consists in solving the dynamic program (2)-(10) by backward induction from the last delivery position date $t_{\frac{8}{n}}$ on the grid $\mathcal{G}$.

In each of the three periods spanning the contract, the main loop of the algorithm consists in iteratively finding, from an initial guess, the fair settlement price at the settlement dates, as a function of the current spot interest rate, considering all the delivery options. This is realized by applying, at a given position date, the root finding procedure on all points of $\mathcal{G}$, and then applying the interpolation procedure to obtain the price as a continuous function of the interest rate. The fair price at a settlement date is then obtained by applying the expectation procedure over the time interval between the settlement and the position dates.

The inner loop of the algorithm consists in obtaining, for a given settlement price function, the value of the contract for the short trader at a given position date, considering all the delivery options, as a function of the interest rate at the last settlement date and the current interest rate. This is realized by applying, on all points of $\mathcal{G} \times \mathcal{G}$, the optimization procedure to find the CTD and the actual exercise value on the grid. The interpolation procedure is then applied to obtain a continuous function, and the expectation procedure is applied on the time interval between the position and the notice dates, yielding the exercise value at the position date. This is compared with the holding value, which is known on the grid points. The optimal value function at the notice date is then interpolated and the expectation procedure is applied between either two successive notice dates (during the end-of-the-month period) or the last settlement and current notice dates. This yields the value for the short seller at the settlement date as a function of the interest rate, which is null if the settlement price is fair.

The detailed algorithm is provided in the Appendix.

4 Numerical Illustration

In our numerical experiments, the finite set of deliverable bonds contains 62 bonds with maturity ranging from 15 to 30 years in steps of 6 months. Since only the bonds with extremal coupon rates are optimal to deliver, we consider only two coupon rates corresponding to the highest and lowest coupon rates in the current CBOT set of deliverable bonds, namely $\overline{c} = 8.125\%$ and $\underline{c} = 5.25\%$. The inception date is chosen to be three months prior to the first day of the delivery month.

We apply our dynamic programming procedure to obtain futures prices at the inception date under both the Vasicek and CIR term structure models, using the

closed-form formulas (17)-(19) or (20)-(22) for the discount factor and transition parameters. Table 1 below gives the (risk neutral) parameter values used in the numerical experiments.

Table 1: Input data

	Deterministic	Vasicek	CIR
$\overline{r}$	0.06	0.098397028	0.133976855
κ	10^{-6}	0.44178462	0.14294371
σ	10^{-6}	0.13264223	0.38757496

The interest rates grid points $a_1, ..., a_q$ are selected to be equally spaced with $a_0 = -\infty$, $a_1 = \overline{r} - 8\sqrt{\frac{\sigma^2}{2\kappa}}$, $a_q = \overline{r} + 8\sqrt{\frac{\sigma^2}{2\kappa}}$ for the Vasicek model, while $a_1 = 0$ and $a_q = 1$ for the CIR model. The number of grid points is denoted N.

We first give the precise definitions adopted here for the timing options. Definitions of implicit delivery options are not uniform throughout the literature and one must be cautious in comparing results across studies. In this paper, the *timing option* gives the short trader the right to deliver on any day during the delivery month (also called the accrued interest option). The delivery month is divided into two periods; in the first 15 business days, the futures market is open, while in the last 7 business days, it is closed. The *wild card option* (WC) (also called the six hours option or the end-of-the day option) refers to the right, during the first 15 days of the delivery month, to wait until 8:00 p.m. (6 hours after the closing of the futures market) to decide about taking or not a delivery position. The *end-of-the month option* (EOM) refers to the right to deliver on any day during the last 7 business days of the delivery month, during which the futures market is closed while the bond market is open.

We disentangle the individual effects of each implicit option by pricing various futures contracts embedding different combinations of these options. The price of the quality option, for instance, is computed as the difference between the price of a futures contract offering all the embedded delivery options and the same contract allowing only one bond to be delivered (here, the notional bond). The price of all the embedded options is obtained by comparing the price of the *straight futures contract* (offering no options) with the price of the contract with all embedded options. The values of the timing options are obtained similarly.

The Dynamic Programming procedure provides the price of the futures and the embedded options, as a function of the current spot interest rate, on any day prior to expiration.

4.1 Convergence

We first examine the convergence properties of the DP procedure. In order to do so, we report on two numerical experiments. In the first one, we use a set of parameters defining a nearly deterministic interest rate of 6%, for which the closed-form forward price is 1 (forward and future prices are equal when rates are deterministic). The relative error in the price of the contract on the inception date is recorded as the number of grid points is increased from 40 to 640. In the second experiment, we use the Vasicek model for the stochastic interest rate. Since in that case we do not have an exact solution, we report on the difference between two successive values of the futures price when the number of points of the grid is doubled, from 75 to 600.

Parameter values are those in Table 1. Results are reported in Tables 2 and 3 in the Appendix. Figure 1 below represents the log of the error as a function of the log of the distance between grid points and the error as a function of the log of computation time.

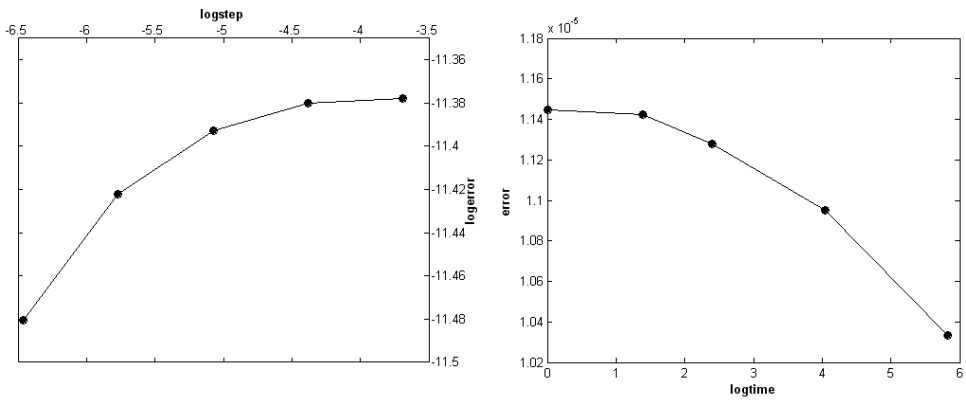

Figure 1: Convergence of the DP futures prices (r=6%)

Figure 2 plots, for various levels of interest rates at the inception date, the relative difference between two successive prices as a function of the grid size and computation time in the Vasicek experiment.

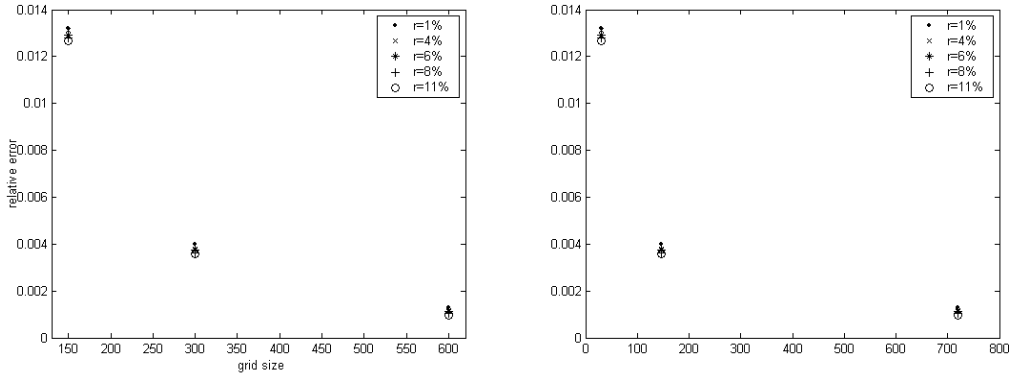

Figure 2: Convergence of the DP futures prices (Vasicek)

4.2 Futures and Option Prices

We now report on futures prices at the inception date for levels of interest rates ranging from 1% to 20%, and for various combinations of embedded options, from which we obtain the value of the different options. Parameter values are given in Table 1. Tables 4 and 5 in the Appendix present the contract prices and option values for the Vasicek dynamics, while Tables 6 and 7 do so for the CIR dynamics. Figure 3 compares the prices of the straight futures contract and the contract with all its embedded options and Figure 4 compares the values of the embedded options as a function of the interest rate. Futures prices are seen to be negatively related to interest rates, which is to be expected since the T-bond futures contract is an interest rate derivative with a T-bond as underlying security.

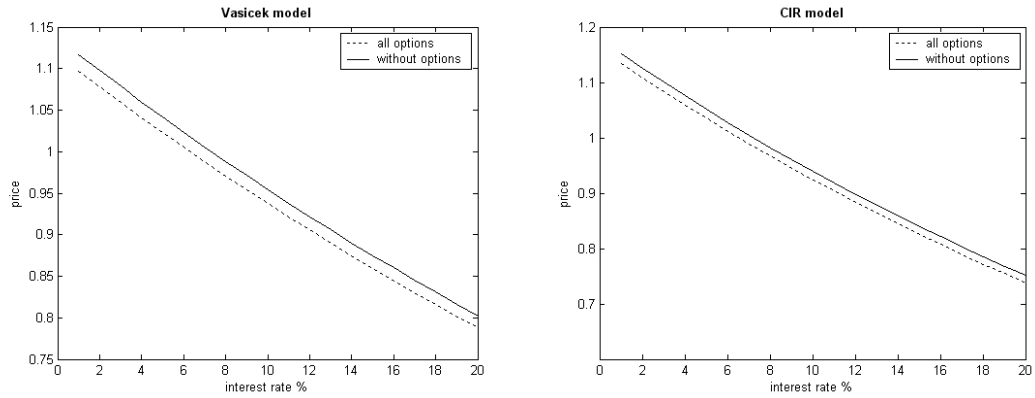

Figure 3: Futures prices vs interest rate

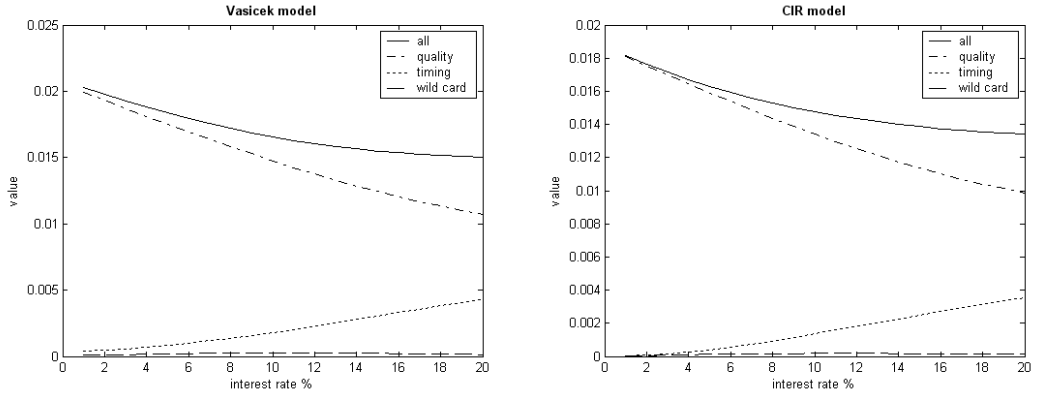

Figure 4: Options values vs interest rates

From Tables 5 and 7, we observe that for the Vasicek (*resp.* CIR) interest rate dynamics, the quality option is worth an average 1.48 (*resp.* 1.35) percentage points of par and represents approximately 88% (*resp.* 89%) of the price of all the options together. The value of the timing option is relatively very small, averaging 0.2 (*resp.* 0.15) percentage points of par. These results are consistent with previous studies, where the quality option is reported to be much more valuable than the timing option, with average values in the range of our results. For both dynamics, the value of the Wild Card option is very small, while the value of the end-of-the month option is found to be approximately nil for all interest rate levels and is therefore not reported in our results.

The value of the quality option is observed to be negatively related to interest rates, while the converse is true for the timing option, with an overall effect dominated by the quality option. The fact that the quality option is negatively related to the interest rate can be explained by the fact that the basket of eligible bonds becomes less valuable with rising interest rates. The fact that the timing option is positively related to the interest rate can be explained by the fact that the proceeds of early exercise can be invested at a higher rate.

In Tables 8 to 13 in the Appendix, we report on the evolution, during the delivery month, of the futures price and quality and timing options values under both the Vasicek and the CIR dynamics. The timing option value is observed to decrease with elapsed time (see Figure 5), which is due to the fact that the opportunities to deliver early are reducing as maturity approaches. On the other hand, the value of the quality option and the futures price do not exhibit a specific relation with respect to time to maturity.

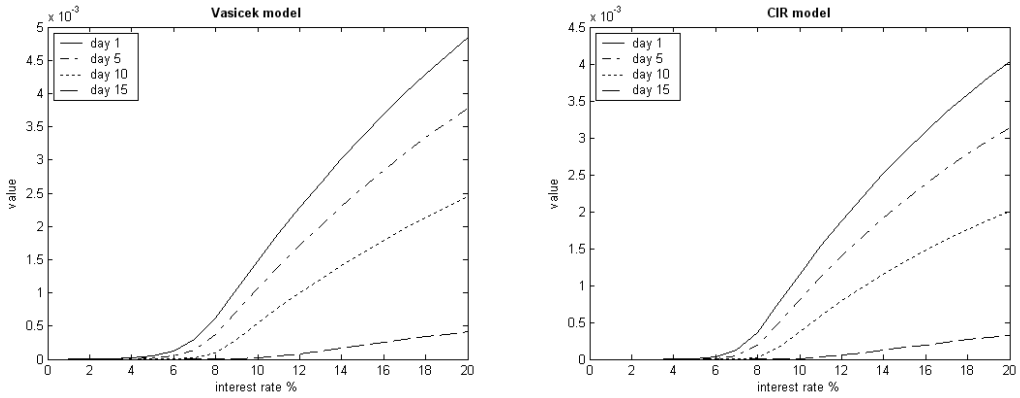

Figure 5: Evolution of the value of the timing option through the delivery month

4.3 Optimal delivery strategy

To illustrate the evolution of the optimal delivery decision and the associated change in the CTD through the delivery month, we report the optimal exercise strategy for days 1, 10, 15, 18, 20 and 22 of the delivery month. The decision is represented by a binary variable equal to 1 if delivery is optimal and 0 otherwise and the CTD is identified by the pair (c, M). Notice that the decision during the wild card period depends on the level of interest rates at 2:00 p.m. and 8:00 p.m. on the same day,

while the decision during the end-of-the month period depends on the interest rate at 2:00 p.m. on the last futures trading day and at 8:00 p.m. on the notice day. Tables 14 and 15 in the Appendix present the results for the Vasicek and CIR dynamics in the specific case where the spot rate is 6% at the settlement date.

We notice that, during the first 15 days of the delivery month, the timing option is exercised optimally whenever interest rates rise during the period between 2:00 p.m. and 8:00 p.m., while it is generally not exercised during the end-of-the month (unless interest rates reach a very high level). It is also interesting to notice that the CTD may have interior maturity, especially for high levels of interest rates. Moreover, bonds with both minimal coupon rates and minimal maturities may be optimal to deliver (which is not observed for deterministic rates).

Finally, Table 16 presents the optimal delivery strategy on the 15th day of the delivery month for various combinations of interest rates at 2:00 p.m. and 8:00 p.m.

4.4 Sensitivity of futures and option prices

In this last section, we perform sensitivity analyses of the option values to the parameters of the interest rate models. Results are presented for the Vasicek model; sensitivity under the CIR model is qualitatively similar.

Figure 6 presents the impact of a variation of the mean reversion speed when $\overline{r} = 0.03$ and $\sigma = 0.1$.

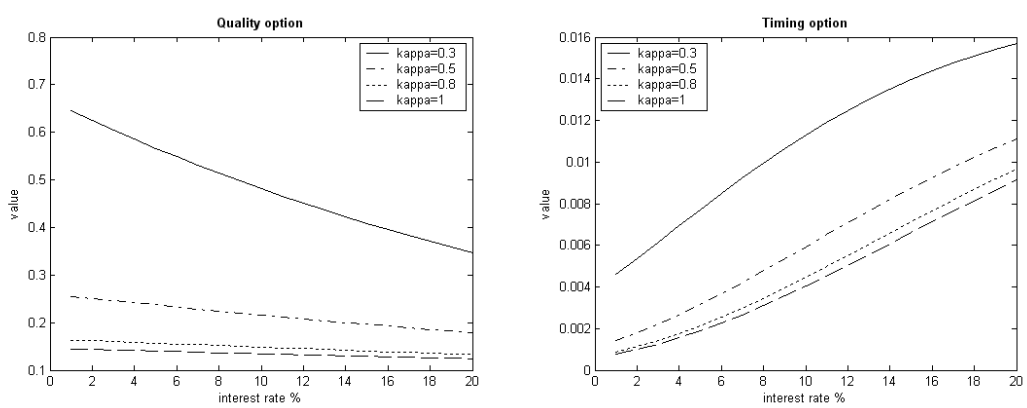

Figure 6: Options values sensitivities to κ (Vasicek)

We observe that both the quality and timing options values are negatively affected by the mean reversion speed. This is consistent with the observation of Chen et al.

(1999) about the quality option value in the Japanese futures market under the Hull and White dynamics. The authors argue that an increase in the mean reversion rate, which determines the relative volatilities of long and short rates, dampens out short term rate movements quickly and therefore reduces the long term volatility, which is positively related to the quality option value.

Figure 7 presents the impact of a variation of the long term mean when $\kappa = 0.3$ and $\sigma = 0.1$ and shows a negative relation between option values and long term mean.

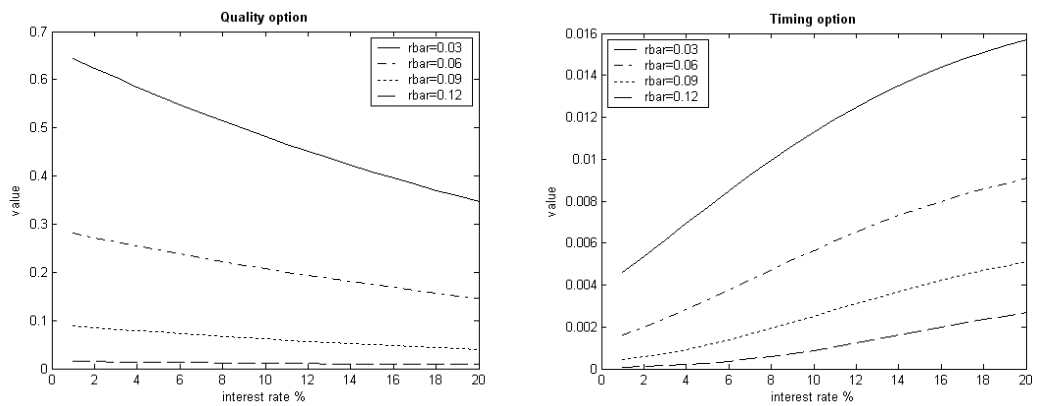

Figure 7: Options values sensitivities to $\overline{r}$ (Vasicek)

Figures 8 and 9 present the impact of a variation in volatility when $\overline{r} = 0.03$ and $\kappa = 0.3$. As expected, the relation between futures prices and options values and volatility is observed to be positive. Chen et al. (1999) find similar results for the quality option in the Japanese futures market.

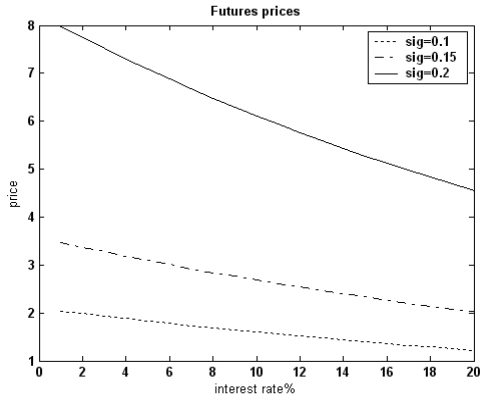

Figure 8: Futures price sensitivity to σ (Vasicek)

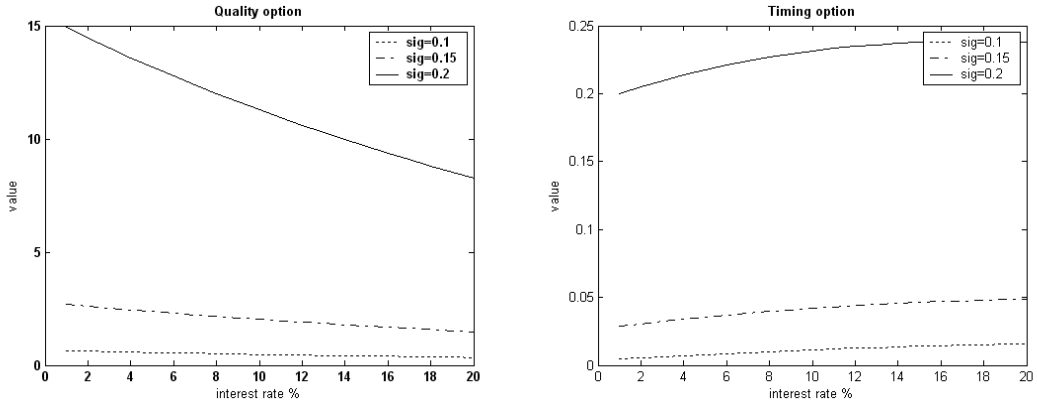

Figure 9: Options values sensitivity to σ (Vasicek)

5 Conclusion

This paper presents an efficient numerical method for the identification of optimal exercise strategies for the CBOT T-bond futures, and for the pricing of the contract, under stochastic interest rate dynamics, and accounting for the interaction of all the inter-dependent delivery options. This numerical algorithm, which combines dynamic programming, finite elements approximation, analytical integration and fixed point evaluation, is to our knowledge the first to tackle all the complexities of the CBOT futures contract in a stochastic interest rate framework. The numerical illustrations provided here use the Vasicek and CIR models, but the method is

flexible and can be used with any specification for the interest rate dynamics, provided the transition parameters and discount factor can be obtained or approximated efficiently.

Our numerical investigations show convergence and consistency. An extension to two-factor or stochastic volatility models (such as GARCH for example) should be the object of further research.

6 Appendix

6.1 Transition parameters

We give, for the Vasicek (1977) and CIR (1985) models, the closed-form formulas for the transition parameters $A_{k,i}^{\delta}$ and $B_{k,i}^{\delta}$ defined respectively in (11) and (12) as well as for the discount factor $\rho(r,t,t+\delta)$ defined in (1). For both models, the derivation of these closed-forms starts from the distribution of the random vector

$$\left(r_{t+\delta}, \int_t^{t+\delta} r_u du\right) \tag{14}$$

conditional on the value of r_t, for $0 \leq t \leq t+\delta$. For proofs and more details about the derivation of these closed-forms we refer to Ben-Ameur et al (2007).

6.1.1 The Vasicek model

Under the risk-neutral probability measure, the interest rate process is the solution to the following stochastic differential equation

$$dr_t = \kappa(\overline{r} - r_t)dt + \sigma dB_t, \quad \text{for } t \geq 0,$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion, κ is the mean reversion speed, $\overline{r}$ is the long term mean and σ is the volatility. For the Vasicek model, the distribution of the random vector (**??**) conditional on $r_t = r$ is bivariate normal with mean

$$\mu(r,\delta) = (\mu_1(r,\delta), \mu_2(r,\delta)) = (\overline{r} + e^{-\kappa\delta}(r-\overline{r}), \overline{r}\delta + \frac{1-e^{-\kappa\delta}}{\kappa}(r-\overline{r})) \tag{15}$$

and covariance matrix

$$\sum(\delta) = \begin{bmatrix} \sigma_1^2(\delta) & \sigma_{12}(\delta) \\ \sigma_{21}(\delta) & \sigma_2^2(\delta) \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa\delta}\right) & \frac{\sigma^2}{2\kappa^2}\left(1-2e^{-\kappa\delta}+e^{-2\kappa\delta}\right) \\ \sigma_{21} & \frac{\sigma^2}{2\kappa^3}\left(-3+2\kappa\delta+4e^{-\kappa\delta}-e^{-2\kappa\delta}\right) \end{bmatrix}. \tag{16}$$

The discount factor and the transition parameters are then given by

$$\rho(r,t,t+\delta) = \exp(-\mu_2(r,\delta) + \sigma_2^2(\delta)/2), \tag{17}$$

$$A_{k,i}^{\delta} = e^{-(\mu_2(a_k,\delta)+\sigma_2^2(\delta)/2)}\left[\Phi(x_{k,i}) - \Phi(x_{k,i-1})\right], \tag{18}$$

and

$$\begin{aligned} B_{k,i}^{\delta} &= e^{-(\mu_2(a_k,\delta)+\sigma_2^2(\delta)/2)}\left[(\mu_1(a_k,\delta) - \sigma_{12}(\delta))(\Phi(x_{k,i}) - \Phi(x_{k,i-1})) \right. \\ &\qquad \left. -\sigma_1(\delta)\left(e^{-x_{k,i}^2} - e^{-x_{k,i-1}^2}\right)/\sqrt{2\pi}\right], \end{aligned} \tag{19}$$

where

$$\begin{aligned} x_{k,i} &= (a_i - \mu_1(a_k,\delta) + \sigma_{12}(\delta))/\sigma_1 \text{ for } i = 0,...,q, \\ x_{k,-1} &= -\infty \end{aligned}$$

and Φ is the standard normal distribution function.

6.1.2 The CIR model

Under the risk-neutral probability measure, the interest rate process is the solution to the following stochastic differential equation

$$dr_t = \kappa(\overline{r} - r_t)dt + \sigma\sqrt{r_t}dB_t, \quad \text{for } t \geq 0.$$

For the CIR model, the distribution of the random vector (**??**) conditional on $r_t = r$ is characterized by its Laplace transform:

$$\begin{aligned} & E\left[\exp(-\omega\int_t^{t+\delta} r_u du - \upsilon r_{t+\delta}) \mid r_t = r\right] \\ = \; & \exp(X(\delta,\omega,\upsilon) - rY(\delta,\omega,\upsilon)), \end{aligned}$$

where

$$\begin{aligned} X(\delta,\omega,\upsilon) &= \frac{2\kappa\overline{r}}{\sigma^2}\log\left[\frac{2\gamma(\omega)e^{(\gamma(\omega)+\kappa)\delta/2}}{(\upsilon\sigma^2+\gamma(\omega)+\kappa)(e^{\gamma(\omega)\delta}-1)+2\gamma(\omega)}\right], \\ Y(\delta,\omega,\upsilon) &= \frac{\upsilon(\gamma(\omega)+\kappa+e^{\gamma(\omega)\delta}(\gamma(\omega)-\kappa))+2\omega(e^{\gamma(\omega)\delta}-1)}{(\upsilon\sigma^2+\gamma(\omega)+\kappa)(e^{\gamma(\omega)\delta}-1)+2\gamma(\omega)} \text{ and} \\ \gamma(\omega) &= \sqrt{\kappa^2+2\omega\sigma^2}. \end{aligned}$$

For the CIR model, the discount factor and the transition parameters are given by

$$\rho(r,t,t+\delta) = \exp(X(\delta,1,0) - rY(\delta,1,0)), \tag{20}$$

$$A^{\delta}_{k,i} = \rho(a_k,t,t+\delta)\sum_{u=0}^{\infty} e^{-\lambda_\kappa/2\frac{(\lambda_\kappa/2)^u}{u!}}\left[F_{d+2u}(\frac{a_{i+1}}{\eta}) - F_{d+2u}(\frac{a_i}{\eta})\right], \tag{21}$$

and

$$\begin{aligned} B^{\delta}_{k,i} &= \rho(a_k,t,t+\delta)\eta\sum_{u=0}^{\infty} e^{-\lambda_\kappa/2\frac{(\lambda_\kappa/2)^u}{u!}}\left[-2(a_{i+1}f_{d+2u}(\frac{a_{i+1}}{\eta}) - a_i f_{d+2u}(\frac{a_i}{\eta})\right. \\ &\qquad \left. +(d+2u)(F_{d+2u}(\frac{a_{i+1}}{\eta}) - F_{d+2u}(\frac{a_i}{\eta}))\right], \end{aligned} \tag{22}$$

where F_{d+2u} and f_{d+2u} are the distribution and the density functions of a chi-square random variable with $d+2u$ degrees of freedom,

$$\begin{aligned} \gamma &= \sqrt{\kappa^2+2\sigma^2}, \\ \eta &= \frac{\sigma^2 e^{\gamma\delta}-1}{2(\gamma+\kappa)(e^{\gamma\delta}-1)+2\gamma}, \\ d &= \frac{4\kappa\overline{r}}{\sigma^2} \text{ and} \\ \lambda_k &= \frac{8\gamma^2 e^{\gamma\delta} a_k}{\sigma^2\left[(\gamma+\kappa)(e^{\gamma\delta}-1)+2\gamma\right](e^{\gamma\delta}-1)}. \end{aligned}$$

6.2 Algorithm

1. **Initialization:**

   Define $\mathcal{G}$. Define ε. Set $\widetilde{v}^h_{\overline{n}}(a_{k'},a_k) = 0$ for all $a_{k'}, a_k \in \mathcal{G}$

2. **Step 1:** (end-of-the-month, $m = n, ..., \overline{n}$)

   2.1 Set $m = \overline{n}$.

   2.2 Set $k' = 1$.

   2.3 Set $g_n(a_{k'})$ using (13) with $T = t^5_{\overline{n}+1}$ and $m' = t^2_n$.

   2.4 Apply the optimization procedure at $(m, a_{k'}, a_k)$ for all $a_k \in \mathcal{G}$ yielding

$$\widetilde{v}^a_m(a_{k'},a_k) = v^a_m(a_{k'},a_k).$$

2.5 Apply the interpolation procedure, setting $h(a_k) = \widetilde{v}_m^a(a_{k'}, a_k)$, $a_k \in \mathcal{G}$, yielding

$$\widehat{v}_m^a\left(a_{k'}, r\right) = \widehat{h}(r).$$

2.6 Apply the expectation procedure to $\widehat{h}(r) = \widehat{v}_m^a\left(a_{k'}, r\right)$ at $t = t_m^8$ and $\tau = t_{m+1}^5$ for all $a_k \in \mathcal{G}$, yielding

$$\widetilde{v}_m^e\left(a_{k'}, a_k\right) = \widetilde{h}\left(t_m^8, t_{m+1}^5, a_k\right) \quad \text{for all} \quad a_k \in \mathcal{G}.$$

2.7 Compute

$$\widetilde{v}_m\left(a_{k'}, a_k\right) = \max\left[\widetilde{v}_m^e\left(a_{k'}, a_k\right), \widetilde{v}_m^h\left(a_{k'}, a_k\right)\right] \quad \text{for all } a_k \in \mathcal{G}$$

and apply the interpolation procedure, setting $h(a_k) = \widetilde{v}_m(a_{k'}, a_k)$, $a_k \in \mathcal{G}$, yielding

$$\widehat{v}_m\left(a_{k'}, r\right) = \widehat{h}(r).$$

2.8 While $m \geq n$, apply the expectation procedure to $\widehat{h}(r) = \widehat{v}_m\left(a_{k'}, r\right)$ at $t = t_{m-1}^8$ and $\tau = t_m^8$ for all $a_k \in \mathcal{G}$, yielding

$$\widetilde{v}_{m-1}^h\left(a_{k'}, a_k\right) = \widetilde{h}_2\left(t_{m-1}^8, t_m^8, a_k\right) \quad \text{for all } a_k \in \mathcal{G},$$

set $m = m - 1$ and go to step 2.4,

Else, apply the expectation procedure to $\widehat{h}(r) = \widehat{v}_n\left(a_{k'}, r\right)$ at $t = t_n^2$ and $\tau = t_n^8$, yielding

$$\widetilde{v}_n^2\left(a_{k'}\right) = \widetilde{h}\left(t_n^2, t_n^8, a_{k'}\right).$$

2.9 While $\left|\widetilde{v}_n\left(a_{k'}\right)\right| > \varepsilon$, apply the root finding procedure to update $g_n\left(a_{k'}\right)$ and go to step 2.4,

Else, set $\widetilde{g}_n^*\left(a_{k'}\right) = g_n\left(a_{k'}\right)$.

2.10 While $k' < q$, set $k' = k' + 1$ and go to step 2.3.

2.11 Apply the interpolation procedure, setting $h(a_k) = \widetilde{g}_n^*\left(a_k\right)$, $a_k \in \mathcal{G}$, yielding

$$\widehat{g}_n^*\left(r\right) = \widehat{h}(r),$$

and apply the expectation procedure to $\widehat{h}(r)$ at $t = t_{n-1}^8$ and $\tau = t_n^2$, for all $a_k \in \mathcal{G}$, yielding

$$\widetilde{g}_{n-1}^8\left(a_k\right) = \widetilde{h}\left(t_{n-1}^8, t_n^2, a_k\right) = \text{ for all } \ a_k \in \mathcal{G}.$$

3. **Step 2** (delivery month, $m = \underline{n}, ..., n-1$)

3.1 Set $m = n - 1$.

3.2 Set $k' = 1$.

3.3 Set $g_m(a_{k'})$ using (13) with $T = t^5_{m+1}$and $m' = t^2_m$.

3.4 Apply the optimization procedure at $(m, a_{k'}, a_k)$ for all $a_k \in \mathcal{G}$ as in step 2.4, yielding $\widetilde{v}^a_m(a_{k'}, a_k) = v^a_m(a_{k'}, a_k)$.

3.5 Apply the interpolation and expectation procedures at $t = t^8_m$ and $\tau = t^5_{m+1}$ as in steps 2.5 and 2.6, setting $h(a_k) = \widetilde{v}^a_m(a_{k'}, a_k)$, $a_k \in \mathcal{G}$, yielding $\widetilde{v}^e_m(a_{k'}, a_k) = \widetilde{h}\left(t^8_m, t^5_{m+1}, a_k\right)$ for all $a_k \in \mathcal{G}$.

3.6 Using (9), compute

$$\widetilde{v}^h_m(a_{k'}, a_k) = g_m(a_{k'})\,\rho\left(a_k, t^8_m, t^2_{m+1}\right) - \widetilde{g}^8_m(a_k), \text{ for all } a_k \in \mathcal{G}.$$

3.7 Compute

$$\widetilde{v}_m(a_{k'}, a_k) = \max\left[\widetilde{v}^e_m(a_{k'}, a_k), \widetilde{v}^h_m(a_{k'}, a_k)\right] \quad \text{for all } a_k \in \mathcal{G}$$

and apply the interpolation procedure as in step 2.7, setting $h(a_k) = \widetilde{v}_m(a_{k'}, a_k)$, $a_k \in \mathcal{G}$, yielding $\widehat{v}_m(a_{k'}, r) = \widehat{h}(r)$.

3.8 Apply the expectation procedure to $\widehat{h}(r) = \widehat{v}_m(a_{k'}, r)$ at $t = t^2_m$ and $\tau = t^8_m$ as in step 2.8, yielding $\widetilde{v}^2_m(a_{k'}) = \widetilde{h}(t^2_m, t^8_m, a_{k'})$.

3.9 While $|\widetilde{v}_m(a_{k'})| > \varepsilon$, apply the root finding procedure to update $g_m(a_{k'})$ and go to step 3.4,

Else set $\widetilde{g}^*_m(a_{k'}) = g_m(a_{k'})$.

3.10 While $k' < q$, set $k' = k' + 1$ and go to step 3.3.

3.11 Apply the interpolation procedure as in step 2.11, setting $h(a_k) = \widetilde{g}^*_m(a_k)$, $a_k \in \mathcal{G}$, yielding $\widehat{g}^*_m(r) = \widehat{h}(r)$.

3.12 While $m \geq \underline{n}$, apply the expectation procedure to $\widehat{h}(r)$ as in step 2.11 at $t = t^8_{m-1}$ and $\tau = t^2_m$, for all $a_k \in \mathcal{G}$, yielding $\widetilde{g}^8_{m-1}(a_k) = \widetilde{h}\left(t^8_{m-1}, t^2_m, a_k\right)$ for all $a_k \in \mathcal{G}$, set $m = m - 1$ and go to step 3-2.

Else, apply the expectation procedure to $\widehat{h}(r)$ as in step 2.11 at $t = t^2_{m-1}$ and $\tau = t^2_m$, for all $a_k \in \mathcal{G}$, yielding $\widetilde{g}^2_{m-1}(a_k) = \widetilde{h}\left(t^2_{m-1}, t^2_m, a_k\right)$ for all $a_k \in \mathcal{G}$.

4. **Step 3** (before the delivery month, $m = -1, \ldots, \underline{n} - 1$)

   4.1 Set $m = \underline{n} - 1$.

   4.2 Using (10), compute

$$\widetilde{g}_m^* (a_k) = \frac{\widetilde{g}_m^2 (a_k)}{\rho(a_k, t_m^2, t_{m+1}^2)}.$$

   4.3 Apply the interpolation procedure as in step 3.11, setting $h(a_k) = \widetilde{g}_m^* (a_k)$, $a_k \in \mathcal{G}$, yielding $\widehat{g}_m^* (r) = \widehat{h}(r)$.

   Apply the expectation procedure to $\widehat{h}(r)$ as in step 3.12 at $t = t_{m-1}^2$ and $\tau = t_m^2$, for all $a_k \in \mathcal{G}$, yielding $\widetilde{g}_{m-1}^2 (a_k) = \widetilde{h} \left(t_{m-1}^2, t_m^2, a_k\right)$ for all $a_k \in \mathcal{G}$.

   4.4 While $m \geq -1$, set $m = m - 1$ and go to step 4.2.

6.3 Numerical Results Tables

Table 2: Convergence of the DP futures prices (r=6%)

N	Futures price	CPU (sec.)
40	0.999988554424208	1
80	0.999988578106129	2
160	0.999988723848566	11
320	0.999989050184833	57
640	0.999989668027805	338

Table 3: Convergence of the DP futures prices (Vasicek)

	N			
r (%)	75	150	300	600
1	1.117557	1.102783	1.098359	1.096947
2	1.097853	1.083635	1.079358	1.077998
3	1.078868	1.06484	1.060669	1.059384
4	1.060213	1.046297	1.042331	1.041097
5	1.041558	1.028163	1.024308	1.023124
6	1.023569	1.010356	1.006578	1.005465
7	1.0059	0.992787	0.989176	0.988111
8	0.988232	0.975609	0.972072	0.97105
9	0.97118	0.958733	0.955246	0.954285
10	0.95444	0.942081	0.938726	0.937804
11	0.937699	0.9258	0.92249	0.921598
12	0.92153	0.9098	0.906516	0.90567
20	0.801306	0.79126	0.788245	0.787505
CPU (sec.)	3	30	146	720

Table 4: Prices of futures contracts (Vasicek)

r (%)	All options	No options	No Quality	No EOM	No WC
1	1.096947	1.117247	1.116872	1.096947	1.097044
2	1.077998	1.097775	1.097306	1.077998	1.078118
3	1.059384	1.078663	1.078085	1.059385	1.059528
4	1.041097	1.059906	1.059202	1.041098	1.041265
5	1.023124	1.041488	1.040641	1.023124	1.023313
6	1.005465	1.023414	1.022406	1.005466	1.005674
7	0.988111	1.005672	1.004486	0.988112	0.988337
8	0.97105	0.988251	0.986872	0.971051	0.971289
9	0.954285	0.971155	0.969568	0.954285	0.954534
10	0.937804	0.954373	0.952564	0.937805	0.938059
11	0.921598	0.937894	0.935851	0.921599	0.921854
12	0.90567	0.921724	0.919438	0.905671	0.905923
20	0.787505	0.802563	0.798247	0.787506	0.787634
Average	0.934282	0.951179	0.949092	0.934282	0.934479

Table 5: Value of delivery options (Vasicek)				
r (%)	All options	Quality	Timing	Wild Card
1	0.020301	0.019926	0.000375	9.7e−05
2	0.019777	0.019308	0.000469	0.00012
3	0.019279	0.018701	0.000578	0.000144
4	0.018809	0.018104	0.000704	0.000168
5	0.018365	0.017517	0.000847	0.000189
6	0.017949	0.016941	0.001008	0.000209
7	0.017561	0.016375	0.001185	0.000226
8	0.017201	0.015822	0.001379	0.000239
9	0.016871	0.015283	0.001588	0.000249
10	0.016569	0.01476	0.00181	0.000255
11	0.016297	0.014254	0.002043	0.000256
12	0.016053	0.013767	0.002286	0.000253
20	0.015058	0.010742	0.004316	0.000129
Average	0.016897	0.014811	0.002087	0.000197

Table 6: Prices of futures contracts (CIR)					
r (%)	All options	No options	No Quality	No EOM	No WC
1	1.133371	1.151521	1.151492	1.133371	1.133397
2	1.107959	1.12558	1.1255	1.107959	1.10801
3	1.083116	1.100257	1.100099	1.083116	1.083191
4	1.058838	1.075541	1.075277	1.058838	1.058936
5	1.035116	1.051418	1.051021	1.035117	1.035236
6	1.011937	1.027872	1.027319	1.011938	1.012075
7	0.989289	1.004889	1.004157	0.98929	0.989443
8	0.967159	0.982454	0.981526	0.967161	0.967325
9	0.945535	0.960551	0.959413	0.945537	0.945709
10	0.924407	0.939172	0.937814	0.924409	0.924587
11	0.903762	0.918301	0.916716	0.903764	0.903944
12	0.88359	0.897927	0.896112	0.883592	0.883772
20	0.737978	0.751386	0.747853	0.737979	0.738115
Average	0.921991	0.937053	0.935476	0.921992	0.922132

Table 7: Value of delivery options (CIR)

r (%)	All options	Quality	Timing	Wild Card
1	0.01815	0.018121	2.9e−05	2.6e−05
2	0.017621	0.017541	8e−05	5.1e−05
3	0.017141	0.016983	0.000158	7.5e−05
4	0.016703	0.016439	0.000264	9.8e−05
5	0.016302	0.015905	0.000397	0.00012
6	0.015935	0.015382	0.000553	0.000138
7	0.0156	0.014868	0.000732	0.000154
8	0.015295	0.014367	0.000928	0.000166
9	0.015016	0.013878	0.001138	0.000174
10	0.014765	0.013407	0.001358	0.00018
11	0.014539	0.012954	0.001585	0.000182
12	0.014337	0.012522	0.001815	0.000182
20	0.013408	0.009875	0.003533	0.000137
Average	0.015061	0.013484	0.001577	0.000141

Table 8: Evolution of the futures contract price during the delivery month (Vasicek)

r (%)	Inception	Day 1	Day 5	Day 10	Day 15
1	1.096947	1.107754	1.108219	1.108809	1.109536
2	1.077998	1.086479	1.086840	1.087298	1.087855
3	1.059384	1.065630	1.065896	1.066227	1.066626
4	1.041097	1.045194	1.045371	1.045586	1.045839
5	1.023124	1.025144	1.025243	1.025352	1.025469
6	1.005465	1.005466	1.005500	1.005521	1.005502
7	0.988111	0.986156	0.986141	0.986088	0.985948
8	0.97105	0.967218	0.967162	0.967043	0.966790
9	0.954285	0.948665	0.948571	0.948389	0.948032
10	0.937804	0.930480	0.930354	0.930110	0.929638
11	0.921598	0.912639	0.912499	0.912195	0.911610
12	0.90567	0.895093	0.895000	0.894648	0.893959
20	0.787505	0.765590	0.765590	0.765590	0.764891
Average	0.934282	0.927090	0.927125	0.927090	0.926703

Table 9: Evolution of the value of the Quality option during the delivery month (Vasicek)

r (%)	Inception	Day 1	Day 5	Day 10	Day 15
1	0.019926	0.020240	0.020236	0.020222	0.020186
2	0.019308	0.019607	0.019593	0.019567	0.019527
3	0.018701	0.018997	0.018972	0.018935	0.018886
4	0.018104	0.018409	0.018377	0.018326	0.018263
5	0.017517	0.017838	0.017804	0.017742	0.017662
6	0.016941	0.017276	0.017254	0.017190	0.017101
7	0.016375	0.016669	0.016676	0.016649	0.016555
8	0.015822	0.015933	0.015987	0.016071	0.016024
9	0.015283	0.015101	0.015195	0.015376	0.015505
10	0.01476	0.014297	0.014423	0.014667	0.015017
11	0.014254	0.013540	0.013680	0.013984	0.014540
12	0.013767	0.012887	0.012980	0.013332	0.014067
20	0.010742	0.009821	0.009821	0.009821	0.010935
Average	0.014811	0.014375	0.014403	0.014506	0.015042

Table 10: Evolution of the value of the Timing option during the delivery month (Vasicek)

r (%)	Inception	Day 1	Day 5	Day 10	Day 15
1	0.000375	1e−06	0	0	0
2	0.000469	1e−06	0	0	0
3	0.000578	6e−06	1e−06	0	0
4	0.000704	1.8e−05	4e−06	0	0
5	0.000847	5.1e−05	1.5e−05	1e−06	0
6	0.001008	0.000129	4.9e−05	4e−06	0
7	0.001185	0.000298	0.000145	2e−05	1e−06
8	0.001379	0.000622	0.000373	9.1e−05	3e−06
9	0.001588	0.001058	0.000723	0.000299	9e−06
10	0.00181	0.001486	0.001069	0.000542	2.2e−05
11	0.002043	0.001895	0.0014	0.000775	4.5e−05
12	0.002286	0.002285	0.001715	0.000997	7.8e−05
20	0.004316	0.004843	0.003779	0.002443	0.000415
Average	0.002087	0.001914	0.001451	0.000879	0.000117

Table 11: Evolution of the futures contract price during the delivery month (CIR)

r (%)	Inception	Day 1	Day 5	Day 10	Day 15
1	1.133371	1.144027	1.144522	1.145144	1.145904
2	1.107959	1.116384	1.116778	1.117277	1.1179
3	1.083116	1.089348	1.089635	1.090003	1.090485
4	1.058838	1.06295	1.063139	1.063383	1.063717
5	1.035116	1.037198	1.037304	1.037439	1.037623
6	1.011937	1.012085	1.012116	1.012152	1.012189
7	0.989289	0.987604	0.987567	0.987505	0.987409
8	0.967159	0.963746	0.963647	0.963495	0.963269
9	0.945535	0.940494	0.940341	0.940103	0.939746
10	0.924407	0.917834	0.917635	0.917311	0.916837
11	0.903762	0.895744	0.895508	0.895102	0.894514
12	0.88359	0.87421	0.873947	0.873464	0.872772
20	0.737978	0.719414	0.719413	0.719148	0.717873
Average	0.921991	0.915982	0.915954	0.915742	0.915314

Table 12: Evolution of the value of the Quality option during the delivery month (CIR)

r (%)	Inception	Day 1	Day 5	Day 10	Day 15
1	0.018121	0.018238	0.018234	0.018228	0.01821
2	0.017541	0.01756	0.017545	0.017521	0.017469
3	0.016983	0.017006	0.016991	0.016964	0.016892
4	0.016439	0.016529	0.016511	0.016479	0.0164
5	0.015905	0.016096	0.016067	0.016023	0.015945
6	0.015382	0.015678	0.01565	0.015594	0.015523
7	0.014868	0.015218	0.015225	0.01519	0.015121
8	0.014367	0.01462	0.014708	0.014781	0.014736
9	0.013878	0.013882	0.014035	0.014271	0.01437
10	0.013407	0.013169	0.013368	0.013692	0.014007
11	0.012954	0.012497	0.012733	0.013139	0.013656
12	0.012522	0.011865	0.012128	0.012611	0.013305
20	0.009875	0.00895	0.008951	0.009216	0.010814
Average	0.013484	0.013068	0.013165	0.013453	0.014019

Table 13: Evolution of the value of the Timing option during the delivery month (CIR)

r (%)	Inception	Day 1	Day 5	Day 10	Day 15
1	2.9e−05	0	0	0	0
2	8e−05	0	0	0	0
3	0.000158	0	0	0	0
4	0.000264	1e−06	0	0	0
5	0.000397	5e−06	1e−06	0	0
6	0.000553	2.9e−05	7e−06	1e−06	0
7	0.000732	0.000119	4.2e−05	3e−06	0
8	0.000928	0.000364	0.000185	2.3e−05	1e−06
9	0.001138	0.000766	0.000497	0.00016	4e−06
10	0.001358	0.001157	0.000815	0.000381	1.2e−05
11	0.001585	0.001527	0.001114	0.000592	2.8e−05
12	0.001815	0.001876	0.001396	0.000791	5.4e−05
20	0.003533	0.004033	0.003136	0.002009	0.000325
Average	0.001577	0.001563	0.001181	0.000708	8.915e−05

Table 14: Evolution of the optimal strategy (Vasicek)

	$r^{2p.m.} = 6\%$						
$r^{8p.m.}(\%)$	day 1	day 5	day 10	day 15	day 18	day 20	day 22
1	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	1 $(5;15)$
3	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	1 $(5;15)$
5	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	1 $(5;15)$
6	0 $(8;15)$	0 $(8;15)$	0 $(8;15)$	0 $(8;15)$	0 $(8;15)$	0 $(8;15)$	1 $(8;15)$
7	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$
8	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$
10	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$
12	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$
20	1 $(8;17.5)$	1 $(8;17.5)$	1 $(8;17.5)$	1 $(8;17.5)$	1 $(8;17.5)$	1 $(8;18)$	1 $(8;18)$

Table 15: Evolution of the optimal strategy (CIR)

	$r^{2p.m.} = 6\%$						
$r^{8p.m.}(\%)$	day 1	day 5	day 10	day 15	day 18	day 20	day 22
1	0 $(5;15)$	0 $(5;15)$	1 $(5;15)$	1 $(5;15)$	0 $(5;15)$	0 $(5;15)$	1 $(5;15)$
3	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	1 $(5;15)$
5	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	0 $(5;15)$	1 $(5;15)$
6	0 $(8;15)$	0 $(8;15)$	0 $(8;15)$	0 $(8;15)$	0 $(8;15)$	0 $(8;15)$	1 $(8;15)$
7	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	0 $(8;15)$	0 $(8;15)$	1 $(8;15)$
8	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	0 $(8;15)$	0 $(8;15)$	1 $(8;15)$
10	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	0 $(8;15)$	0 $(8;15)$	1 $(8;15)$
12	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$	1 $(8;15)$
20	1 $(8;30)$	1 $(8;30)$	1 $(8;30)$	1 $(8;30)$	1 $(8;30)$	1 $(8;30)$	1 $(8;30)$

Table 16: Optimal delivery strategy on the 15th day of the delivery month

	$r^{8p.m.}(\%)$												
$r^{2p.m.}(\%)$	1	2	3	4	5	6	7	8	9	10	12	14	20
1	0	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	1	1	1	1	1	1	1	1	1	1	1
3	0	0	0	1	1	1	1	1	1	1	1	1	1
4	0	0	0	0	1	1	1	1	1	1	1	1	1
5	0	0	0	0	0	1	1	1	1	1	1	1	1
6	1	0	0	0	0	0	1	1	1	1	1	1	1
7	1	1	1	0	0	0	0	1	1	1	1	1	1
8	1	1	1	1	0	0	0	0	1	1	1	1	1
9	1	1	1	1	1	0	0	0	0	1	1	1	1
10	1	1	1	1	1	1	0	0	0	0	1	1	1
12	1	1	1	1	1	1	1	1	1	0	0	1	1
14	1	1	1	1	1	1	1	1	1	1	0	0	1
20	1	1	1	1	1	1	1	1	1	1	1	1	0

References

[1] Arak, M., L.S. Goodman, and S. Ross, "The Cheapest to Deliver Bond on the Treasury Bond Futures Contract," *Advances in Futures and Options Research*, 1 (1986), 49-74.

[2] Arak, M. and L.S. Goodman, "Treasury Bond Futures: Valuing the Delivery Options," *Journal of Futures Markets*, 7 (1987), 269-286.

[3] Barnhill, T. and W. Seale, "Optimal Exercise of the Switching Option in Treasury Bond Arbitrages," *Journal of Futures Markets*, 8 (1988a), 517-532.

[4] Barnhill, T. and W. Seale, "Valuation of the Treasury Bond Quality and Switching Options in Both Perfect and Noisy Markets," Working Paper, School of Government and Business Administration, George Washington University, (1988b).

[5] Ben-Ameur, H., M. Breton, L. Karoui, and P. L'Écuyer, Pricing Call and Put Options Embedded in Bonds, to appear in *Journal of Economic Dynamics and Control*.

[6] Benninga, S. and Z. Wiener, "An Investigation of the Cheapest-to-Deliver on Treasury Bond Futures Contracts," *Journal of Computational Finance*, 2 (1999), 39-55.

[7] Bick, A., "Two Closed-form Formulas for the Futures Price in the Presence of a Quality Option," *European Finance Review*, 1 (1997), 81-104.

[8] Boyle, P., "The Quality Option and Timing Option in Futures Contract", *Journal of Finance*, 44 (1989), 101-113.

[9] Carr, P. and R. Chen, "Valuing Bond Futures and the Quality Option," Working Paper, Morgan Stanley and Rutgers University, 1997.

[10] Chen, R.R., J.H. Chou, and B.H. Lin, "Pricing the Quality Option in Japanese Government Bond Futures," *Applied Financial Economics*, 9 (1999), 51-65.

[11] Chen, R.R. and S.K. Yeh, Analytical Bounds for Treasury Bond Futures Prices, Working Paper, Rutgers University, (2004).

[12] Cox, J.C, J.E. Ingersoll, and S.A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53 (1985), 385-407.

[13] Gay, G. and S. Manaster, "The Quality Option Implicit in Futures Contracts," *Journal of Financial Economics*, 13 (1984), 353-370.

[14] Gay, G. and S. Manaster, "Implicit Delivery Options and Optimal Delivery Strategies for Financial Futures Contracts," *Journal of Financial Economics*, 16 (1986), 41-72.

[15] Gay, G. and S. Manaster, "Equilibrium Treasury Bond Futures Pricing in the Presence of Implicit Delivery Options," *Journal of Futures Markets*, 11 (1981), 613-623.

[16] Hedge, S.P., "An Empirical Analysis of Implicit Delivery Options in the Treasury Bond Futures Contract", *Journal of Banking and Finance*, 12 (1988), 469-492.

[17] Hedge, S.P., "An Ex-post Valuation of the Quality Option Implicit in the Treasury Bond Futures Contract," *Journal of Banking & Finance*, 14 (1990a), 741-760.

[18] Hemler, M.J., "The Quality Delivery Option in Treasury Bond Futures Contracts," *Journal of Finance,* 45 (1990), 1565-1586.

[19] Hranaoiva, J., R.A. Jarrow, and W.G. Tomek, "Estimating the Value of Delivery Options in Futures Contracts," *Journal of Financial Research*, 28 (2005), 363-383.

[20] Hull, J.C. and A. White, "Pricing Interest-Rate Derivative Securities," *Review of Financial Studies*, 3 (1990), 573-592.

[21] Jarrow, R.A. and G.S. Oldfield, "Forward Contracts and Futures Contracts," *Journal of Financial Economics*, 9 (1981), 373-382.

[22] Jones, S., "Conversion Factor Risk in Treasury Bond Futures: A Comment," *Journal of Futures Markets*, 5 (1985), 115-119.

[23] Kane, A. and A.J. Marcus, "Conversion Factor Risk and Hedging in the Treasury Bond Futures Market," *Journal of Futures Markets*, 4 (1984), 55-64.

[24] Kane, A. and A.J. Marcus, "The Quality Option in the Treasury Bond Futures Market: An Empirical Assessment," *Journal of Futures Markets*, 6 (1986a), 231-248.

[25] Kane, A. and A.J. Marcus, "Valuation and Optimal Exercise of Wild Card Option in The Treasury Bond Futures Market," *Journal of Futures Markets*, 41 (1986b), 195-207.

[26] Lacoste, V., "Choix de la Moins Chère à Livrer: un Raccourci Utile," *Finance*, 23 (2002), 77-92.

[27] Livingston, M., "The Cheapest Deliverable Bond for the CBOT Treasury Bond Futures Contract," *Journal of Futures Markets*, 4 (1984), 161-172.

[28] Livingston, M., "The Delivery Option on Forward Contracts," *Journal of Financial and Quantitative Analysis*, 22 (1987), 79-87.

[29] Nielson, S.S. and E.I. Ronn, A Two-Factor Model for the Valuation of the T-Bond Futures Contract's Embedded Options, In Advances in Fixed Income Valuation Modeling and Risk Management, Edited by F.J. Fabozzi, New Hope, 1997, Pennsylvania.

[30] Oviedo, R.A., "Improving the Design of Treasury-Bond Futures Contracts," *The Journal of Business*, 79 (2006), 1293-1315.

[31] Peck, A. and J. Williams, The Timing Option in Commodity Futures Contracts, Unpublished Manuscript, Standford University, 1990.

[32] Ritchken, P. and L. Sankarasubramanian, "Pricing the Quality Option in Treasury Bond Futures," *Mathematical Finance*, 2 (1992), 197-214.

[33] Ritchken, P. and L. Sankarasubramanian, "A Multifactor Model of the Quality Option in Treasury Futures Contracts," *The Journal of Financial Research*, 18 (1995), 261-279.

[34] Vasicek, O., "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5 (1977), 177-188.