Asset Pricing Implications of Housing in General Equilibrium.
Job Market Paper

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Abstract

This paper asks two questions. First, we seek to develop a dynamic general equilibrium model with aggregate uncertainty where the agents face a binary choice. We learn that a standard overlapping-generations model in combination with a randomization scheme is sufficient to overcome the nonconvexities caused by binary choices. With our newly developed modeling technique in hand, we turn our attention to an analysis of the housing markets. Real estate transactions generate large fixed costs, and one can interpret fixed costs as simply a binary choice. For our second question, we ask whether these fixed costs, which are micro-level phenomena, can affect macroeconomic variables such as relative prices and the rewards to risk-taking. We learn that while fixed costs dramatically affect individual behavior, the aggregate impact of these discrete choices is relatively muted. Since the primary asset of the old is real estate, the answers to these questions are important for our understanding of the impact of the demographic wave on the economy.

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1 Introduction.

1.1 Overview.

We solve a general equilibrium model of the housing markets where the agents must pay a fixed transaction cost to adjust their housing stock. We care about fixed costs because they are the reason why large and infrequent adjustments characterize the dynamics of the housing market. Only after we have an equilibrium model that incorporates fixed costs can we ask about their macroeconomic consequences. We will show that one can naturally address questions involving binary choices (such as fixed costs) within the structure of an overlapping-generations (OLG) model.

Fixed costs are a type of nonconvexity that could lead to nonexistence of competitive equilibria, even in static models. At least as early as Starr (1969), economists have recognized that moving from a finite number of agents to an infinite number of agents may restore equilibrium in nonconvex environments. Nevertheless, the mechanism involved, assigning identical agents to different bundles, does not extend in a straightforward way to the dynamic case. The novel feature of this paper is to show that the OLG model is well suited to handle nonconvexities such as transaction costs.

Suppose we initialize a dynamic model with a continuum of agents, all of whom are the same type. When fixed costs are present, the agent's optimal policy may not be unique, so heterogeneity could develop endogenously as different agents may make different choices. When agents live forever, dynamic models may become intractable as the number of types of agents grows without bound. An OLG structure restricts type-proliferation, restoring tractability. Generations begin with one type, they segment into many types, but eventually each generation exits the economic stage and removes some heterogeneity from the model.

Heterogeneity is a desirable feature of an economic model since it is plainly a characteristic of the economy. Unfortunately, heterogeneity makes solving dynamic models (which respect correct expectations) dramatically more difficult. In correct expectations[1] models, prices are unknown functions of the underlying state space, which often includes the distribution of wealth. Solving for these unknown functions in practice is difficult, and

[1] Following Radner (1982) we reserve the term "Rational Expectations" for models where the current market price increases the information set of the agents. In our model the current market price is a deterministic mapping from the current state; agents learn nothing from observing the current spot price.

these difficulties may rise to the level of infeasibility with as little as three or four types of agents. If ever the principle or parsimony binds, it binds here.

More generally, one could easily adapt the methods used in this paper to focus on some other interesting nonconvex choice. Leaving aside the details, we overcome the nonconvexity created by fixed costs by allowing either the agents or the auctioneer to randomize the decision of whether or not to pay the fixed adjustment cost. Since agents randomize this binary decision, one type of agent may become two types of agents. By restricting agent's lives to only two periods we limit the amount of heterogeneity that can develop and keep the problem within the realm of numerical feasibility. The tractability of this environment appears only to depend on the OLG structure and a nonconvexity that one could describe as a binary choice.

Any life-cycle model of owner-occupied housing inescapably leads to a discussion of the reverse mortgage market. Our model is no exception, and our paper adds to the theoretical literature suggesting the need for such a market. Consider the housing financing options facing an agent in her last period of life. If she does not borrow against the value of her home, then, in the absence of a bequest motive, she leaves an unintended bequest. She would therefore pay any price (she is not making a marginal decision) for the opportunity to extract some of her housing wealth to fund current consumption. In addition, reverse mortgages allow the old to (optimally) shift all of the risk in value of their homes to younger investors.

	Net Worth ($ Median, 000's)	Home Value ($ Median, 000's)	Home Ownership Rate	Home Value/ Net Worth
65-74	176.3	129.0	82.5%	73%
74+	151.4	111.0	76.2%	73%
Source: SCF 2001, Federal Reserve Board of Governors.				

Figure 1:

We close this introduction with a reminder of the size and importance of the real estate market in the US. Consider that the stock household real estate is worth about $15.3 trillion, while the capitalization of the stock market is about $13.3 trillion and the flow GDP is about

$11.0 trillion.[2] Figure 1 shows that the 65+ demographic has about 73% of their wealth invested in real estate. While this calculation is somewhat suspect since it relies of the ratio of median values, we do not think it is qualitatively misleading. According to Census projections those age 65+ will rise from 35 million people (12.4% of population) in 2000 to 66 million (19.6% of population) by the year 2030. Real estate is a large part of the economy, it is the primary asset of the old, and the population of older Americans is about to double.

1.2 Literature Review.

Asset Pricing and Portfolio Choice. The OLG model[3] is not widely used in asset pricing papers. Instead, economists have traditionally used recursive equilibrium models with infinitely lived agents. This is not surprising for at least two important and related reasons. First, one can characterize the solution to the individual programming problem relatively easily using the Euler equations (see for example §4.5 of Stokey, Lucas, and Prescott (1989)). Imperfections resulting from life-cycle considerations, which are often incorporated into OLG models, may invalidate the simple use of the Euler equations. Second, empirical evaluation of an asset pricing model usually occurs at a quarterly frequency. For an OLG model this seems to imply the need for at least 200 generations, a situation well beyond the realm of computational feasibility. These challenges, however, do not imply that one cannot derive new asset pricing insights from OLG models. For example, the paper by Constantinides, Donaldson, and Mehra (2002) incorporates borrowing constrained younger generations into a stochastic OLG model. The constraints on the young reduce the supply of bonds, raising bond prices, and hence lower the equilibrium rate of interest. Since the young have no ability to use leverage in the equity market, the equilibrium rate of return on risky assets increases. Another important contribution comes from Geanakoplos,

[2] The real estate estimate is from the Federal Reserve Board of Governors' (Flow of Funds) 6/10/2004 estimate of the value of household real estate at the end of the first quarter 2004. Equity value is the market capitalization of the Russell 3000 at its rebalance on July 1, 2004. The GDP number is from the BEA's 8/27/2004 estimate of nominal GDP for 2003.

[3] The OLG model has a long history in economics, and a full review is not required here. We do, however, feel obliged to at least footnote a few key papers. The handbook treatment of the OLG model is contained in Geanakoplos and Polemarchakis (1991). Karl Shell contributed a number of early, and insightful, works. For example, his note on the "double infinity of commodities and consumers" (Shell (1971)) and his group of papers including Balasko and Shell (1981) clarified some important issues on the structure of OLG models.

Magill, and Quinzii (2003) where demographic fluctuations create predictable variation in asset prices.

Since housing is the major intersection between an agent's consumption and investment decisions, it is a natural starting point for consumption-based explanations of excess returns. Piazzesi, Schneider, and Tuzel (2003) use housing as a factor in an empirical asset pricing model. In a cross-sectional regression of stock returns where Fama and French (1995) use a factor for the size of firms, Piazzesi, Schneider, and Tuzel (2003) use a factor for the share of personal expenditures devoted to housing. They find that this "expenditure share" does help explain differences in returns of portfolios of stocks. Although their empirical model is loosely based on equilibrium arguments, the authors do not solve an equilibrium model. If housing is actually a cause of excess returns, and not merely correlated with more fundamental sources of risk, then a general equilibrium model should also yield this result.

Recursive Macroeconomics. One feature of the OLG model we are highlighting throughout this paper is its ability to limit heterogeneity. In order to make clear why controlling heterogeneity is desirable, we turn next to a short description of the recursive models and, importantly, their numerical solutions. The fundamental concept in recursive macroeconomics is the notion of the state variable. A vector of state variables summarizes all the significant information about the economy and is the unique gateway to dynamic analysis. If we want our stochastic economy to be recursive in the state variables, we must assume that uncertainty is driven by a Markov process. A stationary Markov equilibrium is a pricing function and a policy function such that the economy satisfies agent optimization, market clearing, and correct expectations. The pricing function maps the current position of the economy into current prices, while the policy function describes the evolution of the state of the system.

Solving these models is difficult since the unknowns are *functions* that one must numerically approximate. The dimension of the state vector essentially determines whether a given model is tractable. If the dimension is low, say two or less, then numerical methods are likely to give positive results. The complexity of the problem rises exponentially in the number of state variables and the literature commonly refers to this problem as the curse of dimensionality.[4] In general, prices depend on the distribution of agents (say along the wealth space). The number of states variables is then likely to equal the number of types of agents. Therefore, one cannot compute solutions to models that generate a growing number of types with current technology. We will see in the next section that one can solve models with many types of agents if one dispenses either with correct expectations or with aggregate uncertainty.

Incorporating Discrete Choices. Many authors have incorporated binary choices into equilibrium models and we will broadly categorize the approaches into two styles. We will discuss the specifics papers below, but first we want sketch the big picture to show exactly where our paper extends the research frontier. In one approach, exemplified by Hansen (1985), all agents randomize over the binary choice, but the outcome of the lottery has no dynamic implications for the individual agents. That is, the outcome of the lottery does not affect the state of the agent next period. A model of this type that begins with a representative agent remains a model with a representative agent. Researchers using this approach can almost forget about the discrete choice and conduct a standard representative agent analysis. Most importantly, the analysis remains feasible in the presence of aggregate uncertainty. In short, this method applies to models with binary choices and aggregate uncertainty as long as the outcomes of the lotteries do not have dynamic implications.

In the other approach, exemplified by Caplin and Spulber (1987), agents take deterministic actions over a binary choice and these actions do have dynamic implications. That is, the binary choice has a discontinuous impact on current-period actions and on the state of the agent next period. Models of this type are initialized with agents distributed along an interval (the interpretation of this interval is model-dependent). The necessary next steps in this class of models are to eliminate all aggregate uncertainty and then to assume or show that the distribution of agents remains unchanged over time. Idiosyncratic shocks may cause agents to change positions along the interval, but the distribution of agents never changes. The microeconomic dynamics have no aggregate impact and prices are constant over time. In short, this method applies to models with binary choices but *without* aggregate uncertainty as long as the dynamic implications of the discrete choices lead to a

[4] See Judd (1998) for more information on numerical approximations. A recent paper using high-dimensional interpolation techniques is.Krueger and Kubler (2004)

distribution of the agents that is constant over time.

The innovative aspect of our methodology is that we are able to solve for equilibrium where lotteries have dynamic implications in a setting with aggregate uncertainty. The cost of incorporating dynamic implications and aggregate uncertainty is paid in state variables. Prices become a function of the non-constant distribution of agents. Our model remains tractable only because we use the OLG structure to limit the number of types of agents. The curse of dimensionality constrains the complexity of the model. We stress, however, that our solution respects correct expectations.

Standard Walrasian equilibrium existence proofs rely on fixed-point theorems, which, in turn, require optimal policies that are (at least) convex and defined over a convex set. By construction, a consumption set with discrete choices is not convex, and the optimal policy may not be convex-valued. An auctioneer can convexify the excess demand correspondence of a model populated by a continuum of agents by making assignments on behalf of the agents.[5] For example, suppose we have a unit mass of agents each demanding either zero or one unit of a product in fixed supply of one-half unit. Individual agent optimization will not necessarily lead to market clearing. In addition to announcing prices, the auctioneer must coordinate the plans of the agents so that half of the agents consume zero units and half of the agents consume one unit. We can view this process as auctioneer-level randomization.

Equilibrium may also exist in nonconvex environments if we allow a continuum of agents to choose lotteries over consumption bundles. Consider, for example, a model where the individual programming problem is entirely standard except for a binary choice. Instead of selecting, say, action 0 or action 1, the researcher can convexify the choice set by allowing agents to choose the probability, λ, of taking action 0. Unfortunately, nothing implies that the agent's problem is concave in λ, and thus the optimal choice set may still fail to be convex. If the agent's optimal policy is concave in λ, standard existence arguments again apply.

Prescott and Townsend (1984a, 1984b) developed the lottery framework, and Prescott and Shell (2002) have reviewed some more recent developments. In a lottery framework, agents may purchase random consumption bundles and the value of a commodity is linear in its delivery probability. With a continuum of agents of each type, we can construct

[5] For a quick textbook discussion of this topic see Mas-Colell, Whinston, and Green (1995) §17.1.

a randomizing device such that the actual expenditures of each type equal the expected expenditures of each type of agent, thus random choices made by individuals do not introduce aggregate uncertainty. In a lottery framework, agents pay for all random bundles with certainty knowing that actual delivery is uncertain. In contrast, mixed strategies refer to a policy where the agent's choice is still random, but she pays the full cost of only the bundle that the market actually delivers to her. The Prescott-Townsend approach is not necessarily about discrete choices, but we specialize their model to handle fixed costs, which is a focus of this paper.

As an example, suppose we have an economy populated by a unit mass of agents with identical linear utility functions defined over quantities of single good called c whose price we normalize to 1. We assume the world lasts one period and we endow each agent with one unit of c. Consider a lottery equilibrium where each agent purchases 2 units of c with probability $1/2$. Each agent pays his entire endowment to receive this random bundle, and, given linearity of the utility functions, this is a utility maximizing plan. Market clearing is satisfied, as half of the population will consume two units and the other half will consume zero units.

How should we interpret random contracts? We do not even have anecdotal evidence of people signing random contracts in the residential real estate market. This does not necessarily imply that a model with random contracts will not give an approximation to aggregate behavior. We might suppose unobserved characteristics of the agents partly determine actions that appear random to the economic researcher. In the specific context of the housing market, we might interpret random contracts as a search model where agents only have a probability of selling their home each period. Whatever the interpretation, the model we construct will imply that agents actually want to sign random contracts. The most important reason for using lotteries is that they allow us to solve for equilibrium in models where we may otherwise have no solutions.

The model of Hansen (1985) permits lotteries over an indivisible labor supply decision by a continuum of infinitely lived agents. That is, each agent initially has the option of either working full time or of being unemployed in a given period. Hansen convexifies the discrete choice by allowing the agents to specify the probability with which they work in exchange for a deterministic wage. The equilibrium wage depends on the aggregate labor supply, but the wages paid to an individual worker do not depend on whether the lottery selects that particular worker for employment in a given period. The only consequence of the result of the lottery is a change in current period utility as the employed agents suffer the disutility of work. This behavior is not akin to mixed strategy, as the agent strictly prefers not to work (she is paid anyway). All workers sign the same contracts, receive the same wages, and make the same investments. Therefore, at the start of the next period, all agents are again identical. Heterogeneity does not proliferate inside the model and distribution of wealth is not a state variable of Hansen's economy.

An oft-cited paper that incorporates fixed costs in equilibrium is Caplin and Spulber (1987). Their context is price-setting firms in the presence of menu costs. In their model, fixed menu costs make it optimal for all firms to follow (S, s) pricing rules. By abstracting from macroeconomic uncertainty, postulating continuous money growth, and imposing a uniform distribution of agents along the relative price space, the authors show how money can be neutral even in the presence of menu costs. In equilibrium, the distribution of prices remains fixed and firms simply change positions in the relative price space. These techniques do not extend to the case of aggregate uncertainty, as the distribution of firms would no longer be constant.

A recent paper that incorporates fixed costs within a general equilibrium model of housing is Gruber and Martin (2003). From the agent's perspective, our model is similar to their model in the sense that the agents enjoy utility over nondurable and durable goods. Their focus is on precautionary saving and the distribution of wealth rather than on asset pricing. A key modeling difference between our paper and Gruber and Martin (2003) is that they populate their model with a continuum of immortal agents endowed with uninsurable, idiosyncratic productivity. With no other sources of uncertainty, their assumptions remove all macroeconomic fluctuations and they are able to focus on the steady state distribution of agents. Computing this unknown stationary distribution is their main numerical accomplishment. The Caplin and Spulber (1987) and the Gruber and Martin (2003) construct continuous aggregate excess demand by aggregating over sufficiently dispersed agents. Since prices depend of this distribution, their models only remain tractable when this distribution remains unchanged.

We want to close this subsection by mentioning an alternative method for solving models with many heterogeneous agents. Suppose agents believe that heterogeneity affects prices only through the moments of its distribution. The basis of this approach, developed in Krusell and Smith (1998) and den Haan (1997), is to assume that only a few moments (a small finite number of scalars) are a good summary of an arbitrary distribution (an infinite-dimensional object). This methodology has one undeniable benefit; it is feasible. That is, one can solve the models using standard numerical methods as the number of state variables have been reduced from infinity down to (hopefully) one or two. On the negative side, we are not sure whether the generated solutions are approximate solutions to a model with boundedly-rational agents or approximate solutions to a model populated by agents with perfect conditional foresight.[6] An analysis of computational errors is not likely to resolve this question. An accurate computation of equilibrium in a bounded-rationality model does not imply that the computed equilibrium actually respects correct expectations. We stress, however, that this approach is computationally feasible and Khan and Thomas (2003) apply this methodology to model where firms must pay a firm-specific fixed cost to invest. Heterogeneity in firms develops endogenously in their model and the distribution of capital becomes a state variable that matters to prices only through its distributional moments. Their interesting findings are that fixed costs have a large effect upon firm behavior, but relatively little effect upon the macroeconomy.

Reverse Mortgages. Since "you can't take it with you," people have every incentive to exhaust their wealth prior to death. How can one exhaust their owner-occupied housing? Since we exclude rental markets from our analysis, agents borrow against the value of their homes in order to boost consumption when old. Homes prices are not deterministic, so the agent wants to short an asset whose value fluctuates with the value of their home. If only a riskless asset were available, the agent would borrow until the value her repayments are just equal to value of her home in the "worst" state of the world. In other states of nature, she would leave an unintended bequest. These powerful forces, however, have not led to the widespread use of reverse mortgages. The well-written survey by Caplin (2002) pegged cumulative reverse mortgage issuance at only 50,000, but recent reports[7] indicate substantial growth with at least 18,000 closing in 2003 and 36,000 closings in 2004. Mayer and Simons (1994) conservatively estimate the potential size of the (growing) market at 1.3 million reverse mortgages, while Rasmussen, Megbolugbe, and Morgan (1995) estimate

[6]Krueger and Kubler (2004) demonstrate cases where the Krusell and Smith methodolgy fails.
[7]See Kelly (2004). His article was based on HUD numbers for fiscal years 2003 and 2004.

the potential market size at 3.7 million reverse mortgages. Among the many reasons Caplin cites for the gap between the large number of people who would benefit from a reverse mortgage and the actual number of closings are transactions costs and uncertain impacts on taxes and eligibility for government transfer programs.

The importance of reverse mortgages in our model is somewhat determined by the absence of both bequest motives and rental markets. We noted earlier that the home ownership rate for households 65 years and older is about eighty-percent. Regardless of the reasons why the old own their homes, this market structure seems unlikely to change anytime soon and we think the owner-occupied model is the most relevant starting point for analysis. Economist need to provide policy advice for situations where the bequest motive is not active. Consider the viewpoint contained in Dynan, Skinner, and Zeldes (2002) where the bequest motive is only "operational" in certain states of the world. If the old have plenty of resources and insurance to both fund retirement expenditures (including healthcare costs) and leave generous bequests, then the retirement of the baby-boom generation is not as an important economic issue. The more serious (and likely) scenario is that advances in healthcare technology have increased both life expectancy and the cost of living longer. The current baby-boom generation may not have the resources to fund their future consumption paths, especially medical and long-term care costs.

More generally, one should put the analysis of reverse mortgages in the context of a larger portfolio choice problem. Agents should choose their exposure to housing price risk as if it were just another asset class. Investment in owner-occupied housing requires owner occupation. The last sentence is not a tautology; it is a central ingredient in a proper economic model of housing. In particular, one cannot easily separate the consumption services from owner-occupied housing from its characteristics as a pure investment. This stands in stark contrast to the equity market where one can invest only in Pepsi but drink only Coke. Many papers, most recently including Caplin, Gordon, and Joye (2004) and Shiller (2004), have called for the development of an asset or insurance product to protect homeowners against idiosyncratic fluctuations in the value of real estate.[8]

[8] The recent increase in home prices has heighted interest in these issues. See, for example, Gallin (2004), Case and Shiller (2004) and McCarthy and Peach (2004).

Summary of Zero Transaction Costs Case. The methodological focus of this paper is to describe a model of housing with fixed transaction costs. Since a model of housing necessitates the use of durable goods, we summarize the features of an OLG model with durable goods.[9] One notable difference in the durable goods model compared with a nondurable goods model with agents who live two periods is that a collateralized bond market may exist in equilibrium, as agents may borrow against the equity in their homes. Adding durable goods to an OLG model without aggregate uncertainty and without population growth does not change some important standard results. For example, the competitive equilibrium may not be Pareto optimal. Inefficient equilibria are characterized by negative interest rates, and a transfer scheme from the young to the old in every period can increase welfare in that case. Both steady state and inflationary monetary equilibria exist when the competitive equilibrium without money is not Pareto optimal.

2 $\mathcal{E}_1$: No Aggregate Uncertainty, Auctioneer-Level Randomization.

2.1 Maintained Assumptions.

In this section we detail a model without aggregate uncertainty, as this allows us to introduce the structure of our model while avoiding some complications. We postpone our comments on asset pricing until section 3. In Economy $\mathcal{E}_1$ time is discrete, it begins in the finite past, and it extends into the indefinite future. There are markets for two goods in positive net supply and a collateralized inside bond market with endogenous net interest rate r. One of the goods is nondurable while the other is a durable good (homes) that depreciates at rate $\delta \in (0, 1)$. Default is not permitted in any state of the world, and the inside bond market exists only because homes function as collateral. We assume that the nondurable good serves as the numeraire and agents can exchange durable goods for nondurable goods at price p. The economy is populated by a continuum of three types of economically active agents at any moment in time, and the measure of each type is normalized to unity. The agents in these overlapping generations are referred to as the young, the old, and the estates. The estates are simply the collection of assets owned by the generation that was

[9]The analysis is available from the author upon request.

previously old. The reason we need to introduce the estates is that the old agents must own their homes in the last period of life. Markets for rental housing do not exist and the agent cannot sell her home excluding the service flow housing. In contrast, she can sell her financial assets *ex-dividend,* but, again, consumption of owner-occupied housing requires owner occupation.

We assume for simplicity that the young are endowed with all of the nondurable good and an amount of durable good that is just enough to replace aggregate depreciation from last period. There are no production choices. Endowing the young with this amount of durable good ensures that the aggregate quantity of durable good remains constant, thus eliminating a potential state variable. The aggregate endowment of the nondurable good is denoted ω_c and the aggregate amount of durable good in the economy is denoted ω_k. While a more complete model would incorporate production, we do not believe it is essential for making our basic points. To the degree that land is a fixed factor, abstracting from production is the correct assumption. Furthermore, a production technology should also reflect some of the institutional details that make housing capital unique such as locational characteristics (local public services), minimum sizes (zoning laws), and irreversibilities. Of course, fixed costs at the level of the agent will have no effects if consumers can hire firms to make costless adjustments on their behalf.

The old have only the assets they purchased when young. In equilibrium, the older generation's assets will consist of savings in the form of bonds and ownership of some amount of durable goods. In equilibrium, the estates will consist of a positive amount of housing and a short position in bonds with a net value of zero. We invite to reader to look at Figure 2 for a graphical view of the timing conventions.

The important institutional detail is a fixed transaction cost that must be paid by the old if they sell their stock of durable good. In particular, if the old sell any amount of housing in the spot market they must 'destroy' $\tau \geq 0$ units of the nondurable good for each unit of the durable good that they bring to market. We will also assume free disposal, so we know in equilibrium $\tau \leq p$. The transaction cost is therefore a technical constraint on the economy and transaction costs are not received by another sector as income. When the old agent sells her durable stock, she is also simultaneously purchasing a new level of stock. There is no additional transaction cost associated with this purchase. The transaction cost will generate two types of agents: those who hold when old and those who sell when old.

In equilibrium, the choices of the young will depend on whether they plan to adjust their housing stock when old.

All agents have the same utility function over lifetime consumption patterns. We will let c represent nondurable consumption, k represent durable consumption, and θ^B represent bond purchases. We will use the subscripts 0 and 1 for consumption in the young and old period, respectively, and the subscripts h and s for the types of agents who hold and sell when old, respectively. Each agent's utility is a logarithmic transformation of a Cobb-Douglass consumption index with parameter $\alpha \in (0, 1)$, with time subjectively discounted at rate β. We assume the service flow from the ownership of the durable good is directly proportional to the stock.

$$\begin{aligned} \text{Utility if Hold:} \quad & U(c_{0h}, k_{0h}, c_{1h}, k_{1h}) = \log\left(c_{0h}^{\alpha} k_{0h}^{1-\alpha}\right) + \beta \log\left(c_{1h}^{\alpha} k_{1h}^{1-\alpha}\right) \\ \text{Utility if Sell:} \quad & U(c_{0s}, k_{0s}, c_{1s}, k_{1s}) = \log\left(c_{0s}^{\alpha} k_{0s}^{1-\alpha}\right) + \beta \log\left(c_{1s}^{\alpha} k_{1s}^{1-\alpha}\right) \end{aligned} \tag{1}$$

We now introduce a lottery framework by allowing agents to randomize over the choice of holding versus selling their stock of housing, and in the case where the agents are indifferent over these actions, we allow the auctioneer to choose among optimal plans on behalf of the agents. Let λ be the probability that the agent holds the durable good. The choice variable λ can take any value in the unit interval, but we will show that in Economy $\mathcal{E}_1$ the optimal policies are always corners with respect to λ.

2.2 Agent Optimization.

Program for the Old. The old agent will choose the probability of making a transaction in the durable goods market and the prices of the conditional consumption bundles are linear in this lottery. The assumption of a continuum of consumers assures that λ introduces no aggregate uncertainty and that the deliveries expected by individuals are realized in aggregate. The old agent begins this period with a level $\bar{k}$ of durable goods and A_1 of cash, which is the value of the bonds she bought last period. Her program is

$$
\begin{aligned}
& V^1\left(\bar{k}, A_1\right) = \max_{c_{1h}, k_{1h}, \theta^B_{1h}, c_{1s}, k_{1s}, \theta^B_{1s}, \lambda} \lambda \log\left(c^{\alpha}_{1h} k^{1-\alpha}_{1h}\right) + (1-\lambda)\log\left(c^{\alpha}_{1s} k^{1-\alpha}_{1s}\right) \\
& \hfill \textit{(objective function)} \\
& \text{SUBJECT TO:} \\
& \lambda\left(c_{1h} + \theta^B_{1h}\right) + (1-\lambda)\left(c_{1s} + pk_{1s} + \theta^B_{1s}\right) \leq A_1 + (1-\lambda)(p-\tau)\bar{k} \\
& \hfill \textit{(budget feasibility when old)} \\
& k_{1h} = \bar{k} \\
& \hfill \textit{(definition of holding the durable good)} \\
& 0 \leq \left(\lambda k_{1h} + (1-\lambda) k_{1s}\right)(1-\delta) p + (1+r)\left(\lambda\theta^B_{1h} + (1-\lambda)\theta^B_{1s}\right) \\
& \hfill \textit{(budget feasibility of the estate)} \\
& \lambda \in [0,1] \\
& \hfill \textit{(definition of a lottery)}
\end{aligned}
\tag{2}
$$

Using $k_{1h} = \bar{k}$, and combining the two budget feasibility conditions (thus eliminating bond purchases, $\left(\lambda\theta^B_{1h} + (1-\lambda)\theta^B_{1s}\right)$, from the program) we have the following simplified program.

$$
\begin{aligned}
& V^1\left(\bar{k}, A_1\right) = \max_{\substack{c_{1h}, c_{1s}, k_{1s} \\ \lambda \in [0,1]}} \lambda \log\left(c^{\alpha}_{1h} \bar{k}^{1-\alpha}\right) + (1-\lambda)\log\left(c^{\alpha}_{1s} k^{1-\alpha}_{1s}\right) \\
& \hfill \textit{(objective function)} \\
& \text{SUBJECT TO:} \\
& \lambda c_{1h} + (1-\lambda) c_{1s} + (1-\lambda)\tfrac{r+\delta}{1+r} pk_{1s} \leq \\
& \hfill A_1 + \left((1-\lambda)(p-\tau) + \lambda\tfrac{1-\delta}{1+r} p\right)\bar{k} \\
& \hfill \textit{(budget feasibility when old)}
\end{aligned}
\tag{3}
$$

We show later that in the steady-state equilibrium agents will only choose $\lambda \in \{0,1\}$. Suppose, for now, we take λ as an arbitrary parameter in the unit interval. We can then make a clear interpretation of the budget constraint. The right hand side of the inequality represents the agent's resources: cash (A_1), expected revenues from sales of durable goods today ($(1-\lambda)(p-\tau)\bar{k}$), and the expected present discounted value of sales of (then depreciated) durable goods tomorrow ($\lambda\frac{1-\delta}{1+r}p\bar{k}$). The left had side of the inequality represents expenditures on nondurable consumption conditional on holding, nondurable consumption conditional on selling, and durable consumption conditional on selling with the “prices” λ, $(1-\lambda)$, and $(1-\lambda)\frac{r+\delta}{1+r}p$, respectively. Taking λ as given and defining $I \equiv A_1 + \left((1-\lambda)(p-\tau) + \lambda\frac{1-\delta}{1+r}p\right)\bar{k}$, the solution to the above program is

$$\begin{aligned} c_{1h} &= I\frac{\alpha}{\alpha\lambda+(1-\lambda)} \\ c_{1s} &= I\frac{\alpha}{\alpha\lambda+(1-\lambda)} \\ k_{1s} &= I\frac{1-\alpha}{\alpha\lambda+(1-\lambda)} / \left(\frac{r+\delta}{1+r}p\right) \end{aligned} \tag{4}$$

The policies call for spending a constant fraction of income (suitably defined) on each of the commodities. The solution for λ is not available in closed-form, but numerical routines can compute its value instantaneously. For given values of $\bar{k}$ and A_1 the program, concentrated in λ, is concave. The values of $\bar{k}$ and A_1, however, are *not* fixed from the perspective of the young agent, they are determined by her optimal policy, interest rates, and depreciation. The result of this flexibility is that the young agent's problem is actually convex in λ.

Program for the Young. When the agent is young, she has an endowment with a value of $A_0 = \omega_c + p\delta\omega_k$. As we assume the young agent is not subject to transaction costs, her division of wealth between nondurable and durable goods is irrelevant; all that matters is its total value. Let $V^1\left(\bar{k}, A_1\right)$ be the objective function when old and let (c_0, k_0) represent, for now, the choices when young. Once we solve the optimal choice of λ (it will be zero or one) we will specialize the notation to reflect the polar cases where the agent deterministically holds or sells.

$$\begin{aligned} & V^0(A_0) = \max_{c_0,k_0,\theta_0^B} \log\left(c_0^{\alpha}k_0^{1-\alpha}\right) + \beta V^1\left(\bar{k}, A_1\right) \\ & \qquad\qquad\qquad\qquad \textit{(objective function)} \\ & \text{SUBJECT TO:} \\ & c_0 + pk_0 + \theta_0^B \leq A_0 \\ & \qquad\qquad\qquad\qquad \textit{(budget feasibility when young)} \\ & A_1 = (1+r)\,\theta_0^B \\ & \qquad\qquad\qquad\qquad \textit{(evolution of financial wealth)} \\ & \bar{k} = (1-\delta)\,k_0 \\ & \qquad\qquad\qquad\qquad \textit{(evolution of housing stock)} \end{aligned} \tag{5}$$

Using the definition of $V^1(k, A_1)$ along with the constraints defined in Equation 2, we can write the young agent's problem as

$$\begin{aligned}
&V^0(A_0) = \max_{\substack{c_0, k_0, c_{1h}, c_{1s}, k_{1s} \\ \lambda \in [0,1]}} \log\left(c_0^\alpha\right) + (1+\beta\lambda)\log\left(k_0^{1-\alpha}\right) \\
&+\beta\lambda\log\left(c_{1h}^\alpha\right) + \beta(1-\lambda)\log\left(c_{1s}^\alpha\right) + \beta(1-\lambda)\log\left(k_{1s}^{1-\alpha}\right) + (1-\alpha)\beta\lambda\log(1-\delta) \\
&\hspace{30em}\textit{(objective function)} \\
&\text{SUBJECT TO:} \\
&c_0 + \left(p - (1-\lambda)\tfrac{(1-\delta)}{(1+r)}(p-\tau) - \lambda\tfrac{(1-\delta)^2}{(1+r)^2}p\right)k_0 \\
&\qquad + \tfrac{\lambda}{1+r}c_{1h} + \tfrac{1-\lambda}{1+r}c_{1s} + (1-\lambda)\tfrac{r+\delta}{(1+r)^2}pk_{1s} \le A_0 \\
&\hspace{30em}\textit{(budget feasibility)}
\end{aligned} \tag{6}$$

Since we do not have a closed-form solution for λ, we temporarily take it as given. Straightforward computations yield the following optimal choices as a function of the value of the young agent's endowment, A_0. The optimal policies are written in a form so that we see, as expected, that the agent spends a constant fraction of her wealth on each good regardless of the price. The quantity purchased adjusts with price to maintain budget balance. The price as seen by the young agent is shown after the "/" sign.

$$\begin{aligned}
c_0 &= A_0 \tfrac{\alpha}{\alpha+\alpha\beta+(1-\alpha)(1+\beta\lambda)+(1-\lambda)(1-\alpha)\beta} / 1 \\
c_{1s} &= A_0 \tfrac{\alpha\beta}{\alpha+\alpha\beta+(1-\alpha)(1+\beta\lambda)+(1-\lambda)(1-\alpha)\beta} / \tfrac{1}{1+r} \\
c_{1h} &= A_0 \tfrac{\alpha\beta}{\alpha+\alpha\beta+(1-\alpha)(1+\beta\lambda)+(1-\lambda)(1-\alpha)\beta} / \tfrac{1}{1+r} \\
k_0 &= A_0 \tfrac{(1-\alpha)(1+\beta\lambda)}{\alpha+\alpha\beta+(1-\alpha)(1+\beta\lambda)+(1-\lambda)(1-\alpha)\beta} / \left(p - (1-\lambda)\tfrac{(1-\delta)}{(1+r)}(p-\tau) - \lambda\tfrac{(1-\delta)^2}{(1+r)^2}p\right) \\
k_{1s} &= A_0 \tfrac{(1-\lambda)(1-\alpha)\beta}{\alpha+\alpha\beta+(1-\alpha)(1+\beta\lambda)+(1-\lambda)(1-\alpha)\beta} / \left((1-\lambda)\tfrac{r+\delta}{(1+r)^2}p\right)
\end{aligned} \tag{7}$$

In addition to the choices the agent makes when young (c_0 and k_0), Equation 7 shows the choices she will make next period (c_{1h}, c_{1s}, and k_{1s}). Plugging these choices back into the objective function we could arrive at nearly a closed-form expression for the value function. The only choice parameter over which we have not optimized is λ. We can show analytically that the remaining optimization problem maximizes a convex function (convex in λ) over the unit interval and therefore the optimal policy will be a corner solution. Agents in this model will never choose proper lotteries, although they might be indifferent between holding and selling the durable good when old. The excess demand function of an agent will not be continuous in prices, as demand will shift when the agent switches from holding to selling the durable good. Thus, allowing for randomization at the level of the agent does not produce a model where equilibrium exists. The reason we allowed the parameter λ to take values in $[0, 1]$ is that we did not want to assume the agents would only choose $\lambda \in \{0, 1\}$.

Suppose we allow the auctioneer to make an assignment of hold or sell on behalf of the agents. The auctioneer may only choose actions that are among the optimal policies of the agents. If, at the posted prices, the utility of holding is greater than the utility of selling, then the auctioneer must assign all agents the hold bundle. In the case where the utility of holding and selling are equal, we allow the auctioneer to assign a fraction, γ, of the agents to hold the durable good and a fraction $(1 - \gamma)$ to sell the durable good. To be clear, λ is one of the agent's choices and we have shown $\lambda \in \{0, 1\}$. In contrast, γ is the fraction of agents assigned by the auctioneer to hold the durable good when old. We can see clearly from Equation 7 that the choice of λ (made when old) affects the optimal choices when young, so an interior choice of γ will imply the existence of two different types of young agents.

2.3 Steady-State Equilibrium.

We will search for a steady-state equilibrium, which we define as a constant relative price and a constant interest rate such that when agents optimize, taking these prices as given, all markets clear. Agents correctly anticipate these constant prices. The model we have constructed is dynamic, but we have not yet described the initial conditions of this system. We suppose that at the start of time there a generation who is initially old. We endow the initial old with the same division of wealth that future older generations will have in equilibrium. The old agents who plan to hold will have one combination of cash and durable goods and the old agents who plan to sell will have a different combination of cash and durable goods. Thus, these initial conditions are endogenous to the system and are defined by Equation 8.

Endowment for the Initial Old. The system begins with a generation that is old in the initial period and the endowment of that generation satisfies

$$
\begin{aligned}
\bar{k}_h &= (1-\delta)\, k_{0h} \\
A_h &= (1+r)\, \theta^B_{0h} \\
\bar{k}_s &= (1-\delta)\, k_{0s} \\
A_s &= (1+r)\, \theta^B_{0s}
\end{aligned}
\tag{8}
$$

The pair $\left(\bar{k}_h, A_h\right)$ is the endowment of durable goods and cash for the agent who will hold her stock, while the pair $\left(\bar{k}_s, A_s\right)$ is the endowment of durable goods and cash for the agent who will sell her durable good. The values on the right hand side of Equation 8 are the equilibrium interest rate and choices of the young agent. These initial conditions are an important component of the steady-state solution. When the initial conditions are defined in this way, the economy begins in the steady state and prices are therefore constant.[10] We assume that all of the durable goods that are not owned by the young or the old are owned by the estates. In equilibrium, the old will have, through their bond holdings, a claim to the entire durable stocked owned by the estates. The value of this claim is called cash because the agent need not pay a transaction cost when spending these resources.

Market Clearing Conditions. Given the market clearing interest rate and relative price, consumption levels satisfy the following conditions. The variable γ, chosen by the auctioneer, is the fraction of the agents who hold the durable good when old.

$$
\begin{aligned}
\omega_c &= \gamma\ (c_{0h} + c_{1h}) + (1-\gamma)\,(c_{0s} + c_{1s}) + \tau\ (1-\delta)\,(\gamma\, k_{1h} + (1-\gamma)\,(k_{0s} + k_{1s})) \\
&\qquad \textit{(market clearing in the nondurable goods market)} \\
\omega_k &= \gamma\ (k_{0h} + k_{1h}) + (1-\gamma)\,(k_{0s} + k_{1s}) \\
&\qquad \textit{(market clearing in the durable goods market)} \\
0 &= \gamma\ \left(\theta^B_{0h} + \theta^B_{1h}\right) + (1-\gamma)\left(\theta^B_{0s} + \theta^B_{1s}\right) \\
&\qquad \textit{(market clearing in the bond market)}
\end{aligned}
\tag{9}
$$

[10] We speculate that other initial conditions would converge to the same steady state. However, further analysis of the no-aggregate-uncertainty case is not likely to yield additional economic insight. In Section 3 we analyze a more general case, and in that model the endowment of the initial old is not fixed.

If Walras' law holds, one of the market clearing conditions is redundant[11]. The reason durable goods enter the market clearing condition in the nondurable goods market is that transaction costs, based on sales of durable goods, are modeled as a destruction of the nondurable good.

Equilibrium γ. Let V^{Hold} be the value function for the agent who always holds and V^{Sell} be the value function for the agent who always sells. The parameter γ, which the auctioneer chooses, is not arbitrary, and $\gamma \in [0, 1]$ must satisfy

$$0 = \max\left[0, \left(V^{Hold} - V^{Sell}\right) * (1 - \gamma)\right] + \max\left[0, \left(V^{Sell} - V^{Hold}\right) * \gamma\right] \tag{10}$$

If holding is preferred to selling then $\gamma = 1$ and all agents hold and conversely, if selling is preferable to holding then $\gamma = 0$ and all agents hold. The auctioneer may choose any $\gamma \in [0, 1]$ when $V^{Hold} = V^{Sell}$. Just to be clear, the parameter λ is a randomization parameter that the *agent* chooses, and we have shown the agent always chooses $\lambda \in \{0, 1\}$. The auctioneer may choose an interior γ only in the case when the agents are indifferent to $\lambda = 0$ or $\lambda = 1$.

Existence of Equilibrium. Suppose γ is fixed and agents mechanically follow the prescription of the auctioneer. That is, a fraction of the agents, γ, sell the durable good when old and a fraction $(1 - \gamma)$ do not. With this restriction, the OLG model is entirely standard and existence is assured. The next issue is whether a γ satisfying Equation 10 exists. Here we can consider some simple cases. If $V^{Hold} \geq V^{Sell}$ at $\gamma = 0$, then $\gamma = 0$ is an equilibrium. If $V^{Sell} \geq V^{Hold}$ at $\gamma = 1$, then $\gamma = 1$ is an equilibrium. Otherwise, by continuity of the value function, there exists an equilibrium point in $(0, 1)$ such that $V^{Sell} = V^{Hold}$.

2.4 A Numerical Example.

We turn next to a numerical solution of Economy $\mathcal{E}_1$ in order to evaluate the impact of transaction costs on the macroeconomy. For example, the level of the transaction costs can affect the level of interest rates and the relative price. We will present, in graphical form, a

[11]We only consider Pareto optimal cases (the interest rate is greater than the population gowth of zero), so Walras' law holds for the numerical case we consider.

sequence of comparative statics. For a given set of exogenous parameters we will vary the transaction cost and trace the equilibrium prices and choices in the economy. We turn next to a description of the values used for the exogenous parameters.

$\alpha = 0.85$ The coefficient on nondurable goods in the Cobb-Douglass aggregator α is set to 0.85. Just for this paragraph, consider the following thought experiment where a hypothetical agent consumes rental housing, S, other goods, N, and has the utility function $N^{\alpha}S^{1-\alpha}$. If units were normalized so that the relative price is one then the optimal choices of N and S would satisfy $\frac{\alpha}{1-\alpha} = \frac{N}{S}$. We can observe the ratio N/S in the national accounts and if we look at the ratio in a base year, by construction, the relative price is equal one and real and nominal values coincide. In 2001personal consumption expenditures excluding housing services were \$5.981 trillion while consumption of housing services were \$1.073 trillion[12], and so the value of α is computed. Fine, but the model constructed in this paper is not one of renting but rather one of purchasing the homes that yield a service flow. The Bureau of Economic Analysis produces an account of fixed assets that includes the stock of residential housing. Suppose instead of renting the entire housing stock for \$1.073 trillion the agent purchased the entire housing stock on margin, held it for one year, enjoyed the housing services and then sold. Renting and margin purchase followed by resale deliver the same commodity in the same states of the world, so their costs should be equal. The margin cost of a dollar at gross interest rate R is $(R-1)/R$. In 2001 the average adjustable-rate mortgage rate averaged[13] 5.83%, implying a margin cost of 5.51¢ per dollar. Therefore the cost of purchasing the entire housing stock, valued at \$11.493 trillion[14], on margin is a mere \$633 billion. This is substantially less than the \$1073 billion our hypothetical aggregate agent actually paid to rent the housing stock. No arbitrage intuition suggests the difference stems from costs associated with ownership such as depreciation, taxes, and insurance. Let δ be the sum of those costs. Then δ solves $11.493 * \left(\frac{1.0583-1}{1.0583} + \delta\right) = 1.073$, so the 'depreciation' δ is should be set to 0.0383.

[12]National Income and Product Accounts Table 2.3.5. Personal Consumption Expenditures by Major Type of Product. As of 7/30/2004.

[13]1-Year Adjustable-Rate Mortgages. Source: http://www.freddiemac.com.

[14]Fixed Asset Table. Table 1.1. Current-Cost Net Stock of Fixed Assets and Consumer Durable Goods, 1925-2002 Billions of dollars; yearend estimates. As of 7/30/2004.

Annualized $\delta = 0.005$**, Annualized** $\beta = 0.95$ The data certainly suggest a much higher depreciation rate, but the competitive equilibrium in that case is not a Pareto optimal. We do not want to use this model as an explanation of government transfer programs or as a rational for money, so we set the depreciation rate low enough to ensure a Pareto optimal competitive equilibrium. When we simulate economy $\mathcal{E}_2$ we will use the more realistic level of depreciation implied by the national accounts. The subjective discount factor, β, is set to the value used in Heaton and Lucas (1996), among other papers.

$\omega_c = 10, \omega_k = 19.2$ The average aggregate amount of nondurable good is normalized to 10 units, and in 2001 the ratio of the housing stock to personal consumption expenditures excluding housing was $11.493/5.981 = 1.92$. We use the same ratio of housing stock to nondurable goods.

$\tau \in [0, 0.025]$ We range the transaction cost from zero to just beyond the level where the older generation does not engage in any transactions in the durable goods market.

Optimal Consumption Plans. The numerical solution to this program is described in Appendix A. Given that the optimal choices are known in closed-form (conditional on λ), the numerical solution is relatively easy to compute and is highly accurate. Figure 3 shows the equilibrium choices of all four types of agents in the model (young or old and sell or hold dimensions). We note that the equilibrium interest rate is such that $(1 + r)\,\beta < 1$, so the optimal consumption profile is declining with age. As the utility function is separable in nondurable and durable goods, the hold or sell decision does not affect nondurable consumption. When the agent sells, she consumes a relatively large amount of durable when young and reduces her stock when old. When the agent holds, she can only reduce her stock through the passive effects of depreciation. The inability to optimally smooth leads her to consume less durable good when young and more durable good when old. Figure 3 also shows some unusual kinks that merit further explanation. The graphs represent a sequence of solutions with one solution for each value of the transaction cost parameter, so points with different τ-axis values represent different economies. The kinks in the graphs represent responses to different equilibrium prices and interest rates.

Equilibrium Price, Interest Rate, and "Randomization" Parameter. Figure 4 shows that, when transaction costs are low, all agents sell the durable good when old. As transaction costs rise, the rate of return earned by saving via durable goods declines and the young agents respond by shifting their portfolios toward bonds. This causes the price of durable goods and the interest rate to fall. Eventually, transaction costs become high enough that some agents begin to hold their stock of durable goods when old, and these agents consume more durable good when old. This creates a rise in the price of durable goods and also a much larger supply of bonds (more collateral now held by the old) that helps push up interest rates.

Lessons. In Economy $\mathcal{E}_1$ individual agents will never randomize their individual consumption plans. Equilibrium exists and is computable as long as we allow the auctioneer to assign different consumption plans to different agents. Heterogeneity is a useful modeling tool since it may help overcome nonconvexities, and the OLG framework is useful since it allows the economic researcher to control the amount of heterogeneity that enters the model. In the next section we will show how to extend the model the case of aggregate uncertainty.

3 $\mathcal{E}_2$: Aggregate Uncertainty, Agent-Level Randomization.

3.1 Maintained Assumptions.

An important question we can address in a model with aggregate uncertainty is whether fixed transaction costs affect the risk-reward profile of assets. Is it possible that individual agents would prefer to randomize their consumption in the presence of fixed costs and aggregate uncertainty? We will compare our results from the fixed transaction costs to a model of convex transaction costs. We turn next to developing a model with which we can answer these questions.

Here is how Economy $\mathcal{E}_2$ compares with Economy $\mathcal{E}_1$. The form of the utility function along with the time and generational structure remains the same. The old agent will now pay a fixed or quadratic adjustment cost for changing her holdings of the durable good. If the old agent begins the period with $\bar{k}_z$ units of durable good and consumes k_{z1s} units of the durable good while old with probability $(1-\lambda)$, then she pays a transaction cost of

$$\text{Transaction Cost} = (1-\lambda_z)\left(\tau\bar{k}_z + \upsilon\left(\bar{k}_z - k_{z1s}\right)^2\right) \tag{11}$$

Where τ indexes the level of the fixed costs and υ indexes the convex cost. In our numerical work we will set either τ or υ to zero. It is important to make this comparison since many economists have assumed, for tractability, that all transaction costs are convex. Transaction costs are again modeled as a destruction of nondurable good.

Aggregate uncertainty enters the model by the introduction of a "fruit tree" in positive net supply. This tree yields a random dividend of nondurable goods each period. We require that the dividend assume only a finite number, J, of values. Before introducing the notation for the price of the fruit tree we note that our solution will be Markov in the relevant state variable, denoted z. The state variable will indicate the current exogenous state (current dividend) and the current endogenous state summarized by the distribution of the durable good. When the state is z the amount of the dividend is denoted d_z and the price of the fruit tree in state z is denoted q_z^F. Shares in the fruit tree are infinitely divisible, and the number of fruit tree shares purchased is denoted θ^F, with a subscript 0 or 1 to indicate the young or old agent, respectively. Transitions among the exogenous states are given by the exogenous transition matrix $\pi_{z\hat{z}}$.

The aggregate stock of the durable good is still ω_k and $\delta * \omega_k$ is endowed to the younger generation. The young still receive an endowment of ω_c of nondurable goods, but the aggregate endowment of nondurable goods is uncertain and its value in state z is $\omega_c + d_z$. As we noted in Section 2, endowing the young with durable goods is not an optimal assumption, but it does reduce the number of state variables.

Just as in the case without aggregate uncertainty, default is not permitted and borrowing is only possible because the agents fully collateralize their debts with their homes. We now allow the agents to make trades in two zero-net-supply assets. The first asset is a bond, with quantities denoted θ^B, that sells for price 1 in any state z and pays $(1+r_z)$ next period. This asset is riskless in the sense that its payoff does not vary with realization of uncertainty next period. The second asset pays $(1-\delta)\,p_{\hat{z}}$ for each unit purchased at price q_z^M in the previous period. The parameter δ is the depreciation rate of durable goods and $p_{\hat{z}}$ is the price durable good next period given the realization of state $\hat{z}$. We will refer to this asset as a reverse mortgage and denote the quantity purchased by θ^M. Reverse mortgages are in zero-net-supply and in equilibrium the old will issue the reverse mortgage security.

Before moving on with the algebra of this model, we want to again defend our assumptions about the bequest motive and the existence of a reverse mortgage market. In equilibrium, the reverse mortgage security will trade, while the riskless bond will not. Since reverse mortgages are not widely used in the US, we want to give the reader a flavor for how to remove reverse mortgages from the model. One easy alteration is to drop reverse mortgages and allow for unintended bequests where we assign the (weakly positive) value of the estates to the endowment of the young. This assumption would re-open the riskless bond market and shift some wealth to the young generation, but little else. Turning back to the bequest motive, the reason that reverse mortgage market is active is not caused by the absence of bequests. If the agent had a positive bequest motive, she would still choose an optimal portfolio through which to save and it seems unlikely that this portfolio would consist of entirely riskless bonds.

3.2 Agent Optimization.

Program of the Old. We begin at the end of the agent's life and impose the condition that the estate of the agent must have non-negative wealth. In the model without aggregate uncertainty this did not create much of a problem as the agent simply borrows against the (deterministic) future value of her home. In the presence of aggregate uncertainty the agent knows only the distribution of the future value of her home. Suppose risk-free bonds are the only asset in the economy. Her optimal policy in that case is to borrow up until her future debt is equal to the minimum over all future values of her home. The reverse mortgage security allows the agent to borrow against the entire future value of her home since her repayment schedule varies directly with the value of the home. In equilibrium the old will sell all of their house-price risk to the younger generation. As a consequence, numeraire bond trading will shut down as the old will always have demand zero and the interest rate will equilibrate the younger generation's demand to zero. The younger generation must also purchase the entire fruit tree, as the old have no reason to save for the future.

As in Economy $\mathcal{E}_1$, the old agent holds the durable good with probability λ. Unlike Economy $\mathcal{E}_1$, however, the old agents may optimally choose a nondegenerate lottery. Her stock of durable goods at the beginning of the period is $\bar{k}_z$ and her financial assets are A_z, her program is

$$
\begin{aligned}
V^1\left(\bar{k}_z, A_z\right) = & \max_{\substack{c_{z1h}, k_{z1h}, c_{z1s}, k_{z1s}, \\ \theta^M_{z1}, \lambda_z}} \lambda_z \log\left(c^{\alpha}_{z1h} k^{1-\alpha}_{z1h}\right) + (1-\lambda_z) \log\left(c^{\alpha}_{z1s} k^{1-\alpha}_{z1s}\right) \\
& \hfill \textit{(objective function)} \\
& \text{SUBJECT TO:} \\
& \lambda_z c_{z1h} + (1-\lambda_z)\left(c_{z1s} + p_z k_{z1s}\right) + q^M_z \theta^M_{z1} \leq \\
& \qquad A_z + (1-\lambda_z)\left(\left(p_z - \tau\right)\bar{k}_z - \upsilon\left(\bar{k}_z - k_{z1s}\right)^2\right) \\
& \hfill \textit{(budget feasibility when old)} \\
& k_{z1h} = \bar{k}_z \\
& \hfill \textit{(definition of holding)} \\
& 0 \leq \left(\lambda_z k_{z1h} + (1-\lambda_z) k_{z1s}\right)(1-\delta)\, p_{\hat{z}} + (1-\delta)\, p_{\hat{z}} \theta^M_{z1} \\
& \hfill \textit{(budget feasibility of the estate)} \\
& \lambda \in [0,1] \\
& \hfill \textit{(definition of a lottery)}
\end{aligned}
\tag{12}
$$

Prices are linear in the lotteries so, for example, to purchase one unit of c with probability λ_z costs $\lambda_z * 1 * 1$ (probability times price times quantity). Using $k_{z1h} = \bar{k}_z$, and combining the two budget feasibility conditions (thus eliminating reverse mortgage purchases, θ^M_{z1}, from the program) we arrive at the following simplified program.

$$
\begin{aligned}
V^1\left(\bar{k}_z, A_z\right) = & \max_{\substack{c_{z1h}, c_{z1s}, k_{z1s} \\ \lambda_z \in [0,1]}} \lambda_z \log\left(c^{\alpha}_{z1h} \bar{k}^{1-\alpha}_{z}\right) + (1-\lambda_z) \log\left(c^{\alpha}_{z1s} k^{1-\alpha}_{z1s}\right) \\
& \hfill \textit{(objective function)} \\
& \text{SUBJECT TO:} \\
& \lambda_z c_{z1h} + (1-\lambda_z)\left(c_{z1s} + \left(p_z - q^M_z\right) k_{z1s} + \upsilon\left(\bar{k}_z - k_{z1s}\right)^2\right) \leq \\
& \qquad A_z + (1-\lambda_z)\left(p_z - \tau\right)\bar{k}_z + \lambda_z q^M_z \bar{k}_z \\
& \hfill \textit{(budget feasibility)}
\end{aligned}
\tag{13}
$$

As in Section 2, we will not have a closed-form solution for the optimal value of λ. We solve for the optimal policies of the other choices conditional on λ. We are then able to show that the young agent's objective function, concentrated in λ, is *concave*. The important implication of concavity is that individual agents may optimally randomize their consumption bundle. Solving the above program we find that $c_{z1h} = c_{z1s}$ and solving for k_{z1s} in terms of c_{z1s} leads to the following quadratic equation.

$$[2\upsilon]\, k_{z1s}^2 + \left[\left(p_z - q_z^M\right) - 2\upsilon \bar{k}_z\right] k_{z1s} - \left[(1-\alpha)\, c_{z1s}/\alpha\right] = 0 \tag{14}$$

The solutions for the optimal policies in the $\upsilon = 0$ case (no convex costs) are

$$\begin{aligned} c_{z1s} &= \tfrac{\alpha}{(1-\lambda_z)(1-\alpha)+\alpha} I \\ c_{z1h} &= \tfrac{\alpha}{(1-\lambda_z)(1-\alpha)+\alpha} I \\ k_{z1s} &= \tfrac{1-\alpha}{(1-\lambda_z)(1-\alpha)+\alpha} I \tfrac{1}{p_z - q_z^M} \end{aligned} \tag{15}$$

Where $I = A_z + (1-\lambda_z)(p_z - \tau)\,\bar{k}_z + \lambda_z q_z^M \bar{k}_z$. For $\upsilon > 0$ and $\tau = 0$, the above quadratic has one positive real root, it is

$$k_{z1s} = \frac{-\left(\left(p_z - q_z^M\right) - 2\upsilon \bar{k}_z\right) + \sqrt{\left(\left(p_z - q_z^M\right) - 2\upsilon \bar{k}_z\right)^2 + 8\upsilon\,(1-\alpha)\, c_{z1s}/\alpha}}{4\upsilon} \tag{16}$$

In the case of $\upsilon > 0$, the optimal policy for the consumption of c_{z1s} is given by the following equation

$$\begin{aligned} I &= c_{z1s} \\ &+ (1-\lambda_z)\left(p_z - q_z^M\right) \frac{-\left(\left(p_z - q_z^M\right) - 2\upsilon \bar{k}_z\right) + \sqrt{\left(\left(p_z - q_z^M\right) - 2\upsilon \bar{k}_z\right)^2 + 8\upsilon\,(1-\alpha)\, c_{z1s}/\alpha}}{4\upsilon} \\ &+ (1-\lambda_z)\,\upsilon \left(\bar{k}_z - \frac{-\left(\left(p_z - q_z^M\right) - 2\upsilon \bar{k}_z\right) + \sqrt{\left(\left(p_z - q_z^M\right) - 2\upsilon \bar{k}_z\right)^2 + 8\upsilon\,(1-\alpha)\, c_{z1s}/\alpha}}{4\upsilon}\right)^2 \end{aligned} \tag{17}$$

For $\upsilon > 0$ the equation is a quadratic[15] in c_{z1s} and we need to spend a moment analyzing the roots of this equation. With some simplifications we can write the above equation as

[15] As a quick consistency test, we note that using L'Hôpital's rule we have that as $\upsilon \to 0$ we have $c_{z1s} \to I * \alpha / (\alpha + (1-\lambda_z)(1-\alpha))$, just as it should.

$$I + \frac{2\left(1-\lambda_z\right)\left(\left(p_z - q_z^M\right) - 2\upsilon\bar{k}_z\right)^2}{16\upsilon} - \left(1-\lambda_z\right)\upsilon\bar{k}_z^2 = \left(1 + \frac{\left(1-\lambda_z\right)\left(1-\alpha\right)}{2\alpha}\right) c_{z1s}$$
$$+ \frac{2\left(1-\lambda_z\right)\left(\left(p_z - q_z^M\right) - 2\upsilon\bar{k}_z\right)\sqrt{\left(\left(p_z - q_z^M\right) - 2\upsilon\bar{k}_z\right)^2 + 8\upsilon\left(1-\alpha\right)c_{z1s}/\alpha}}{16\upsilon} \quad (18)$$

More compactly, we can write

$$T = Uc_{z1s} + V\sqrt{W + Xc_{z1s}} \quad (19)$$

where

$$\begin{aligned} T &= I + \tfrac{2(1-\lambda_z)\left(\left(p_z - q_z^M\right) - 2\upsilon\bar{k}_z\right)^2}{16\upsilon} - \left(1-\lambda_z\right)\upsilon\bar{k}_z^2 \\ U &= \left(1 + \tfrac{(1-\lambda_z)(1-\alpha)}{2\alpha}\right) \\ V &= \tfrac{(1-\lambda)\left(\left(p_z - q_z^M\right) - 2\upsilon\bar{k}_z\right)}{8\upsilon} \\ W &= \left(\left(p_z - q_z^M\right) - 2\upsilon\bar{k}_z\right)^2 \\ X &= \tfrac{8\upsilon(1-\alpha)}{\alpha} \end{aligned} \quad (20)$$

The quadratic in c_{z1s} is

$$\left[U^2\right]c_{z1s}^2 - \left[2TU + V^2X\right]c_{z1s} + \left[T^2 - WV^2\right] = 0 \quad (21)$$

The above quadratic has either one or two positive real roots (the discriminant is positive). If there are two, then since the agent prefers more to less, the higher root is optimal. In either case, the larger root is the optimal choice and

$$c_{z1s} = \frac{2TU + V^2X + \sqrt{\left(2TU + V^2X\right)^2 - 4U^2\left(T^2 - WV^2\right)}}{2U^2} \quad (22)$$

This closed-form solution is important since it reduces the amount of numerical computations needed to solve the model, which makes the solution more accurate. Since two choices of c_{z1s} may satisfy the first-order conditions, it is important that we determine the correct root before turning to the numerical solution that is only based on first-order conditions.

If the old own $\bar{k}_z$ units of durable good, then, since the young are endowed with $\delta\omega_k$, we know by market clearing that the estates must own $\left(\omega_k - \left(\delta\omega_k + \bar{k}_z\right)\right)$ units of the durable good. In equilibrium, the old must begin the period with a positive position in the reverse mortgages worth exactly the value of the property held by the estates. We therefore conclude that the quantity of reverse mortgages issued is $\left((1-\delta)\,\omega_k - \bar{k}_z\right) / (1-\delta)$. In addition, the old own the entire fruit tree (in unit net supply). This argument demonstrates that the durable holdings of the old are enough to pin down the state of the economy. That is, once we know the old begin the period with $\bar{k}_z$ units of durable good, we are able to determine the initial holdings of all types of assets for all agents in the economy. In equilibrium financial wealth for the old will be

$$A_z = (1 + r_z) * 0 + \left(d_z + q_z^F\right) * 1 + p_z\,(1-\delta) * \left[\left((1-\delta)\,\omega_k - \bar{k}_z\right) / (1-\delta)\right] \quad (23)$$

Her wealth is the sum bond holdings (she has none), the cum-dividend value of shares in the fruit tree (she has the entire unit), and the value of her stake in the reverse mortgage contracts.

In Economy $\mathcal{E}_1$ there were (potentially) two types of agents in the older generation, those who planned to hold when old and those who planned to sell their durable good when old. In Economy $\mathcal{E}_2$ there is only one type of old agent since all agents in the young generation will make the same choices. Therefore, all agents must begin the older period with the same amount of durable good, regardless of the realization of the exogenous state of nature. For fixed $\bar{k}_z$ the older agent's objective function is concave in λ_z for each potential realization of z. It does not matter what value of $\bar{k}_z$ the young agent chooses, it is still fixed when old. We show in Figure 9 that the old agent's objective function, concentrated in λ_z, is concave for each z. The old agent has two wealth accounts that are not perfect substitutes, one represents financial wealth and the other represents durable wealth, which is subject to transaction costs. The young agent can directly hedge against fluctuations in the value her financial account, but she can only indirectly hedge against changes in the value of her durable account through the purchase of financial assets. A direct hedge for durable wealth account would allow the agent to wake up in a smaller home (without paying transaction costs) in the event her wealth declines. The distinction between durable and financial wealth is the fundamental reason why the agents prefer to randomize consumption when old.

The last order of business relating to the older generation's problem is to determine the initial conditions for our system consistent with recursive equilibrium. Since we have a dynamic economy, we need to initialize the system with an initial generation that begins life as an older person. Their younger period of life is not modeled, and we just assume the endowment for this initial old generation includes both financial and durable wealth. An equivalent assumption is that the initial old are endowed only with durable goods, but only part of her durable stock is subject to transaction costs. For a given value of durable wealth, the initial old are endowed with the financial wealth A_z, described by Equation 23.

Program for the Young. The young agent's value function only depends on the current state, as we assumed that she is not subject to transaction costs when young. In state z her endowment is worth $\omega_c + \delta p_z \omega_k$ and she solves:

$$
\begin{aligned}
& V^0(z) = \max_{c_{z0}, k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}} \log\left(c^{\alpha}_{z0} k^{1-\alpha}_{z0}\right) + \beta \textstyle\sum_{\hat{z}=1}^{J} \pi_{z\hat{z}} V^1\left(\bar{k}_{\hat{z}}, A_{\hat{z}}\right) \\
& \hfill \textit{(objective function)} \\
& \text{SUBJECT TO:} \\
& c_{z0} + p_z k_{z0} + \theta^B_{z0} + q^F_z \theta^F_{z0} + q^M_z \theta^M_{z0} \leq \omega_c + \delta p_z \omega_k \\
& \hfill \textit{(budget feasibility when young)} \\
& A_{\hat{z}} = (1 + r_z)\,\theta^B_{z0} + \left(d_{\hat{z}} + q^F_{\hat{z}}\right)\theta^F_{z0} + (1 - \delta)\, p_{\hat{z}} \theta^M_{z0} \\
& \hfill \textit{(evolution of financial wealth)} \\
& \bar{k}_{\hat{z}} = (1 - \delta)\, k_{z0} \\
& \hfill \textit{(evolution of durable wealth)}
\end{aligned}
\tag{24}
$$

Using the budget constraints and the calculation of the initial old from the previous section we can write this program as:

$$
\begin{aligned}
V^0(z) = & \max_{k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}} \log\left(\left(\omega_c + \delta p_z \omega_k - \left(p_z k_{z0} + \theta^B_{z0} + q^F_z \theta^F_{z0} + q^M_z \theta^M_{z0}\right)\right)^{\alpha} k^{1-\alpha}_{z0}\right) \\
& + \beta \sum_{\hat{z}=1}^{J} \pi_{z\hat{z}} V^1\left((1 - \delta)\, k_{z0}\,,\ (1 + r_z)\,\theta^B_{z0} + \left(d_{\hat{z}} + q^F_{\hat{z}}\right)\theta^F_{z0} + (1 - \delta)\, p_{\hat{z}} \theta^M_{z0}\right)
\end{aligned}
\tag{25}
$$

Additional closed-form simplifications are not available for the young agent's problem. This is not a serious problem as we already have enough results to move forward with a highly accurate numerical solution.

3.3 Stationary Markov Equilibrium.

So far, our analysis has basically been static. We asked, for a given state, what are the agent-optimal choices? The OLG structure allows us to make the preceding static analysis into a recursive analysis. Time matters only through its infinite horizon. We do not need to specify time subscripts, we just need to keep track of which variables represent future values. The entire past history of shocks is conveniently summarized by the state of the economy. Conditional on the state, a pricing function posts market clearing prices and the policy function describes the evolution of our system. We stress that the pricing and policy functions described below hold at all time periods and although our analysis appears somewhat static, it is actually recursive.

Definition 1 *The **state** of the economy is a vector $z \in Z = [0, (1-\delta) * \omega_k] \times \{1, 2, \ldots J\}$. The first element of this vector is the amount of durable good owned by the older generation and the second is the exogenous state of the dividend.*

Definition 2 *The **pricing function** is a mapping from the current state of the economy to the price of the durable good, the interest rate, and the prices of the fruit tree and mortgage contract.*

$$g = (g^p, g^r, g^{q^F}, g^{q^M}) : Z \to \mathbb{R}_{++} \times (-1, \infty) \times \mathbb{R}_{++} \times \mathbb{R}_{++} \tag{26}$$

Definition 3 *The **policy function** is a map from the current endogenous state of the economy to the endogenous state of the economy next period.*[16]

$$f^k : Z \to [0, (1-\delta)\,\omega_k] \tag{27}$$

Definition 4 *A **stationary Markov equilibrium** for Economy $\mathcal{E}_2$ is a pricing function and a policy function such that, for any realization of the exogenous dividend process, agent optimality, market clearing, and correct expectations are satisfied.*

[16] We could expand the definition of the policy function to include all of the choices variables for both agents, but we leave it as just f^k to emphasize that $\bar{k}_z$ summarizes all of the relevant information about the endogenous state of the system.

Market Clearing

$$\begin{aligned}
\omega_c + d_z &= c_{z0} + \lambda c_{z1h} + (1-\lambda)\, c_{z1s} + (1-\lambda)\left(\tau \bar{k}_z + \upsilon \left(\bar{k}_z - k_{z1s}\right)^2\right) \\
&\qquad \textit{(market clearing nondurable goods)} \\
\omega_k &= k_{z0} + \lambda k_{z1h} + (1-\lambda)\, k_{z1s} \\
&\qquad \textit{(market clearing durable goods)} \\
0 &= \theta^M_{z0} + \theta^M_{z1} \\
&\qquad \textit{(market clearing reverse mortgages)} \\
0 &= \theta^B_{z0} \\
&\qquad \textit{(market clearing risk-free bonds)} \\
1 &= \theta^F_{z0} \\
&\qquad \textit{(market clearing fruit tree)}
\end{aligned} \tag{28}$$

Optimality of Choices *The policy function f^k is consistent with optimal choices of all agents.*

Correct Expectations *Although implicit in the above definitions, we note here that the expected prices and interest rates actually do materialize and clear the market.*

Existence of Equilibrium. Existence of stationary Markov equilibria is not guaranteed and Kubler and Polemarchakis (2004) prove the potential for nonexistence by counter-example. Although a competitive equilibrium will exist, it is not necessary for the competitive prices to be described by (relatively) simple recursive functions. Those authors go on to describe an equilibrium concept that always exists and relates directly to current computational routines. The Markov ε-equilibria described by Kubler and Polemarchakis (2004) are ε close to a stationary Markov equilibrium in the sense that market clearing and correct expectations are satisfied, but agents need only be within ε of their optimum level of utility. That is, agents are permitted to make small errors in their decision rules.

Routines for computing Markov ε-equilibria follow the same general steps. First, one makes a make a guess at the solution and determines the error associated with that guess. Errors are usually defined via deviations from equality in the agent's Euler equation. If the errors associated with the current guess are small the routine stops, otherwise the routine makes a new guess at the solution and the process repeats. Since we only have finite time to make the computations, our final guess for the pricing and policy function does not zero out the errors. Kubler and Polemarchakis (2004) point out that due to this fact applied recursive equilibrium papers are indeed computing Markov ε-equilibria.

3.4 Numerical Example and Asset Pricing.

In order to contrast a fixed costs model with a convex costs model we numerically solve for two examples of Economy $\mathcal{E}_2$.[17] Many of the values for the exogenous parameters used in the simulation of Economy $\mathcal{E}_2$ are the same as those used in the simulation of Economy $\mathcal{E}_1$. The unchanged parameter values are: $\alpha = 0.85$ (weight of nondurables in the utility function), $\beta = 0.95$ (subjective discount factor), and $\omega_k = 19.2$ (aggregate stock of durable goods). We will assume three equiprobable future states of the dividend process, so $\pi_{z\hat{z}} = 1/3\,\forall z, \hat{z}$, and we assume the corresponding dividends are 1, 5, or 9 units of nondurable good. We set $\omega_c = 5$, so the expected aggregate amount of nondurable goods is ten units. The depreciation rate is set to $\delta = 0.0383$. In the model with convex costs we set $\upsilon = 0.025$ and $\tau = 0$ and for the fixed costs case we set $\upsilon = 0$ and $\tau = 0.025$. Our transaction cost is a technological assumption, so $\tau = 0.025$ means the agent must "burn" 2.5 units of the nondurable good in order to sell 100 units of the durable good. Since these relationships do not depend on prices, one cannot calibrate this model to a world where transaction costs are based on a fraction of value (e.g. commissions). When calculated as a percent of equilibrium value, the parameters we use for the transaction costs in the fixed costs case are too high. The recent survey of property by The Economist[18] puts the cost of a real estate transaction in the US at about 10% percent of the value of the home. In our model, when the price of the durable good is at its lowest point, transaction costs may reach 50% of value. Although this may seem like an unappealing feature of the model, it is exactly what fixed costs imply. When the price of the durable good drops, the associated fixed transaction costs, as measured in value, rise. Since we conclude that fixed transaction costs have relatively little impact on macroeconomic variables, using transaction costs that may be too high actually supports this finding. On the other hand, the implications for individual behavior are probably overstated.

[17] Appendix A contains all the details. The code is available from the author upon request.
[18] "Close to Bursting" May 31 2003. See Economist.com/surveys.

Equilibrium Choices of the Old Agents Figure 5 shows the optimal choice of durable good for the old agent as a function of the current state for both the fixed and convex costs case. Since transaction costs are paid by the old agents, this where we should expect to find the largest differences in behavior. In this graph and in the four subsequent graphs we want to make clear the interpretation of the x-axis. When the old agent begins the period with a low level of durable good, this does not mean she is poor. If the old agent owns a small amount of durable good, then, in equilibrium, the estates must have a relatively large position in durable goods. The old must have a claim on the value of the durable goods owned by the estates via the reverse mortgage contract. Low values of durable good holdings for the old agent therefore imply she is house poor and cash rich. Likewise, we should associate states when the old own a large amount of durable good as implying the old are house rich and cash poor.

In the convex costs case the optimal policy is just what we might expect. In the high dividend state ($d = 9$) the old agent is relatively more wealthy and she consumes more of the durable good than she does in the low dividend state. Convex costs make a complete adjustment too costly and the more housing she owns, the more housing she will consume.

Since our model is calibrated to twenty-year intervals, the effects of depreciation are substantial. Once the old agent decides to make a transaction, she will purchase additional durable good to replenish her stock. If our computational routine could accommodate more trading periods, then we suspect agents might also choose to vary their durable stock, up or down, with aggregate fluctuations.

With fixed costs, the optimal policy involves the use of proper randomization in some states of the world. Figure 6 shows the optimal choice of λ as a function of the current state. When the old are cash rich–this occurs when either their holdings of durable goods are low or when the dividend realization is high, or both–they always make a transaction. Conversely, when the old are cash poor–this occurs when their holdings of durable goods are high and the dividend realization is medium to low–they are less likely to make a transaction.

Returning to Figure 5 we can clearly see the effects of fixed costs and randomization on the optimal policies. The easiest case to describe is the line showing the optimal policy in the high dividend ($d = 9$) state. When the dividend is high, the old agent is cash rich and she makes a transaction with probability one. Next, we consider the optimal policy in the low dividend ($d = 1$) state. We represent this policy as a correspondence of state of the economy. Areas where the correspondence is single-valued refer to situations where the agent is either deterministically holding or selling. Where there are two values for a given level of durable good, the higher value represents the bundle conditional on selling while the lower value represents the bundle conditional on holding. For simplicity, we have not shown the associated probability weights. The optimal policies conditional on the hold or sell decision are dramatically different, and she clearly prefers the outcome of the lottery in which she consumes more. Purchasing a random bundle is a way to possibly get the benefits of adjustment while avoiding some of the transaction costs.

Prices Figure 7 shows the equilibrium price functions and annualized interest rate function in the fixed cost model. Since the lines describing these functions are not monotonic, they beg for an explanation. In the fixed costs model when the agent is making a transaction with probability one, prices are relatively insensitive to the endogenous state. The reason for this is that fixed costs no longer affect marginal decisions (λ is at a corner) and the agent's location within the endogenous state space only matters through a small impact on wealth (more wealth held as durable goods implies more transaction costs). Next consider the regions where the old agent chooses a proper lottery. In these cases, fixed costs matter for marginal decisions. The agent responds to increased holdings of durable goods by increasing the probability of not making a transaction, thus avoiding some of the increased transaction costs. This action reveals itself as a decline in the price of durable goods, as demand by the old declines (the agent consumes less when holding than when selling) as the probability of holding, λ, ranges from zero to one. Once the agent is holding with probability one, fixed costs no longer affect marginal decisions, but the location of the old agent within the endogenous state space determines her consumption. The more durable good she has, the more she must consume (this is the definition of holding). This implies that demand for durable goods by the old is increasing over this range of the state space, causing an increase in the price of durable goods.

The interest rate and the prices of the fruit tree and reverse mortgage contract also respond to the choices of the old agent. The intuition is a mirror of the intuition provided above for the price of the durable good. When the old purchase less durable good, the young must, in equilibrium, purchase more. The more the young spend on durable goods, the less they have to spend on financial assets. Thus our intuition suggests that declines in the price of the durable good (caused by reduced demand from the old) should coincide with declines in the prices of other assets. Since the quantity of the reverse mortgage contract traded varies with the state of the economy, these interpretations may not hold for this asset.

In Figure 8 we observe that convex costs always impact marginal decisions and we see a noticeable dependence over the entire range of the endogenous state. One curious feature of equilibrium in the convex costs case is that the equilibrium price of the fruit tree is actually higher in the recession than in the boom. We expect asset prices should fall in the recession, and when we solve the model with a much lower level of convex transaction costs, this is indeed what occurs. We are willing to accept this behavior as accurate for two reasons. First, the numerical errors of our solution are smaller than generally accepted practice, and, second, the quantity of reverse mortgages issued is smaller in a recessions and thus makes a price rise plausible.

Asset Pricing Implications of Fixed vs. Convex Transaction Costs. To conclude this section we consider the impact of transaction costs on the risk-reward profile of financial assets. Before we contrast the results, we need a framework within which we can make the comparison. As stressed by Cochrane (2001)[19] all asset pricing models are of the form

$$q = \mathbb{E}[mx] \tag{29}$$

where q is the price of the asset, x is the numeraire payoff of the asset, and $m > 0$ is a random variable[20] known as the stochastic discount factor that prices all assets. If we want to compare asset pricing models, then we have to make a comparison of the implied stochastic discount factors. How should we compare random variables (the m's) generated from theoretical models to the data? In the paper of Hansen and Jagannathan (1991) the authors show that the ratio of the standard deviation of the stochastic discount factor to its mean must be greater than the Sharpe ratio of the market return (the market we have in mind the US equity market). The Sharpe ratio of the market is defined as the market return

[19] See also Harrison and Kreps (1979).

[20] The notation $m > 0$ means $m(\omega) > 0$ for almost all $\omega \in \Omega$, where the random variable m lives on some probability space $(\Omega, F, \mathbb{P})$.

less the risk-free rate of return (denoted by R^e below) divided by the standard deviation of the market return.

$$\frac{\sigma\ (m)}{\mathbb{E}\,[m]} \geq \frac{|\mathbb{E}\left[R^e\right]|}{\sigma\ (R^e)} \tag{30}$$

According to Cochrane (2001) the Sharpe ratio for the U.S. market is roughly 0.5 and the gross risk-free rate is 1.01. In short, $\frac{\sigma(m)}{\mathbb{E}(m)} \geq 0.5$ and $R_f \approx 1.01$ are two quick checks on the plausibility of an asset pricing model. A quick way to compare stochastic discount factors across models is to check their performance in these dimensions.

A short argument reveals the form of m in our model. The young agent is free so save ε units if numeraire today for $\varepsilon * (1+r)$ units of numeraire tomorrow. In equilibrium, this deviation must yield no extra utility, therefore

$$\frac{\alpha}{c_{z0}} = \beta\mathbb{E}_0\left[\frac{\alpha\,(1+r_z)}{c_{\hat{z}1s}}\right] \tag{31}$$

Since the payoff $(1+r)$ is not random, the stochastic discount factor must satisfy $\frac{1}{1+r} = \mathbb{E}_0\,[m]$, so

$$m = \beta\frac{c_{z0}}{c_{\hat{z}1s}} \tag{32}$$

The quantity m is random since future consumption $c_{\hat{z}1s}$ will depend on the realization of the exogenous dividend from the fruit tree. Consider below the key moments of the stochastic discount factor implied by our model[21].

	$\frac{\sigma(m)}{\mathbb{E}[m]}$	$(1+r)$
Convex Costs Model	0.0296	1.065
Fixed Costs Model	0.0294	1.061

(33)

Our model suffers from the same calibration failure as most other general equilibrium models. Fixed costs in the housing market have a large impact on the optimal policies of individual agents, but they explain little of the observed asset market returns. To be sure, the logarithmic utility function

[21]This model is roughly calibrated to twenty-year intervals and if we assume that m is generated by an *i.i.d.* process, then the annualized stochastic discount factor implied by the model is $m^{1/20}$. The moments reported in the paper are unconditional averages of the annualized calculations. As an aside, we note that the annalized stochastic discount factor need not price the annualized returns generated by our equilibrium model.

4 Conclusions.

Before we conclude, we need to step back and consider the broader trajectory in recursive equilibrium research. The trend is, and should be, to construct ever more realistic descriptions of the economy. Nobody knows how useful a detailed recursive equilibrium model of the economy would be to policy makers because none has been solved. Since the curse of dimensionality stands in the way of progress, papers that develop enhanced numerical techniques for high-dimensional functional approximation or circumvent the problem with closed-form solutions would dramatically extend the research frontier. For now, researchers must formulate their models with a small number of state variables and focus on some particular aspect of the economy.

We have examined fixed transaction costs in the housing market within the context of an OLG model. Our methodological contribution is to show that the lottery framework of Prescott and Townsend (1984a) fits naturally into an OLG model since the heterogeneity generated by the lotteries is eventually eliminated from the model through the finite lifetimes of the agents. Controlling heterogeneity is one way to control the number of state variables, and thus keep the model numerically tractable. Lotteries themselves are useful since randomization at the level of individual agents is one way to overcome noncovexities such as those generated by fixed costs.

Fixed costs in the housing market have a large impact on the optimal policies of individual agents, but they have little impact on the characteristics of financial assets. This result is somewhat driven by our assumption of a utility function that is separable in nondurable and durable goods. Our goal, however, was not to construct solutions to asset pricing puzzles with imaginative utility functions. Rather, we seek to identify fundamental constraints at the level of individual agents that could have macroeconomic consequences. It is in this spirit that we chose to examine fixed costs in the housing market. The importance of the housing market is determined by its size relative to the economy, its large contribution to the net worth of many older Americans, and the demographic forces that will almost double the number of older Americans in the coming generation. Reverse mortgages are not widely used in the US, but we have seen how important they are to a life-cycle model of housing, suggesting this market may continue to develop in the coming years.

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A Numerical Methodology

Economy $\mathcal{E}_1$: No Aggregate Uncertainty. In the nonstochastic we solve the program by suggesting values for γ (fraction who are assigned to hold by the auctioneer) and then computing the market clearing prices and choices assuming agents take this value of γ as given. We repeat this step, adjusting the value of γ each time, until Equation 10 is satisfied. Conditional on γ, we have closed-form solutions for the optimal choices, therefore the numerical errors are negligible and are not reported.

Economy $\mathcal{E}_2$: Aggregate Uncertainty. We solve one program where $\upsilon > 0$ and $\tau = 0$ and a separate program where $\upsilon = 0$ and $\tau > 0$. In an slight abuse of notation, let $V^0\left(z|k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}\right)$ be the objective function of the young agent, so

$$V^0(z) = \max_{k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}} V^0\left(z|k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}\right) \tag{34}$$

and let $V^1\left(\bar{k}_z, A_z|\lambda_z\right)$ be the objective function of the old agent concentrated in λ (the choices other than λ are known in closed form). We have

$$V^1\left(\bar{k}_z, A_z\right) = \max_{\lambda_z} V^1\left(\bar{k}_z, A_z|\lambda_z\right) \tag{35}$$

In the $\upsilon = 0$ and $\tau > 0$ case we search for a solution to the following system of equations

$$\begin{aligned}
\partial V^0\left(z|k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}\right)/\partial k_0 &= 0\\
\partial V^0\left(z|k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}\right)/\partial \theta^B_0 &= 0\\
\partial V^0\left(z|k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}\right)/\partial \theta^F_0 &= 0\\
\partial V^0\left(z|k_{z0}, \theta^B_{z0}, \theta^F_{z0}, \theta^M_{z0}\right)/\partial \theta^M_0 &= 0\\
\partial V^1\left(\bar{k}_z, A_z|\lambda_z\right)/\partial \lambda_z + \left(\max\left[0, \mu_1\right]\right)^2 - \left(\max\left[0, \mu_2\right]\right)^2 &= 0\\
\lambda_z - \left(\max\left[0, -\mu_1\right]\right)^2 &= 0\\
1 - \lambda_z - \left(\max\left[0, -\mu_2\right]\right)^2 &= 0
\end{aligned} \tag{36}$$

The first four equations are the first-order conditions of the young agent. The fifth condition is the first order condition for the older agent with respect to λ_z and the final two restrictions come from the complementary slackness conditions for $0 \leq \lambda_z \leq 1$. As is described in Judd, Kubler, and Schmedders (2003) we have already transformed to the system of Kuhn-Tucker inequalities into a system of differentiable equations. This transformation makes for a slightly convoluted presentation, but we solve the system using a set of routines that can only solve a system of equalities. For a particular state these are seven equations in seven unknowns, $p_z, r_z, q_z^F, q_z^M, \lambda_z, \mu_1, \mu_2$. For given prices and an optimal choice of λ_z, the optimal choices for the older agent are known in closed form. Market clearing then implies all of the choices for the younger agent. Therefore, we will not have a numerical approximation to f^k. Instead, we have an exact solution for f^k given the approximations to the functions describing $p_z, r_z, q_z^F, q_z^M, \lambda_z, \mu_1, \mu_2$. In the case that $\tau = 0$ we know that, optimally, $\lambda = 0$ (everyone sells) and we need not account for the last three equations.

We use the time iteration method as described in Judd, Kubler, and Schmedders (2003). We approximate the functions using linear interpolation. Numerical errors in the stochastic case are also negligible. This is not too surprising given that we have closed-form solutions for the equilibrium values conditional on prices (this is part of the payoff from choosing logarithmic utility). In contrast, a typical system of equations defining a recursive equilibrium are the Euler equations that only implicitly define the equilibrium values. In addition, our model had only a single continuous state variable, which further simplifies the numerical routines. Suppose we define the numerical errors as relative error in the young agent's first order condition with respect to bond holdings.

$$ErrorVec_z = \frac{\frac{1}{c_{z0}} - \mathbb{E}\left[\beta\,(1+r)\,\frac{1}{c_{\hat{z}1s}}\right]}{\frac{1}{c_{z0}}} \tag{37}$$

We could look at other dimensions, but if we consider the table below we see the numerical errors are extremely small.

Numerical Errors	Fixed Costs Model	Convex Costs Model
$\max_{z \in Z}(ErrorVec_z)$	6×10^{-16}	2×10^{-15}

(38)

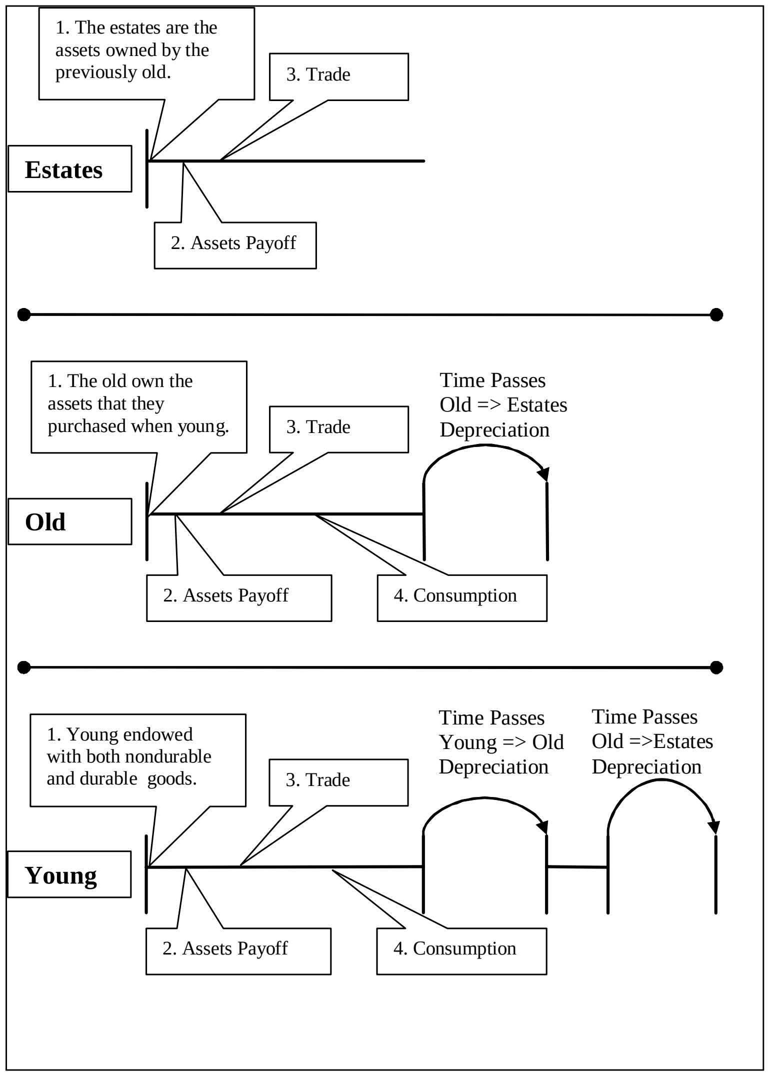
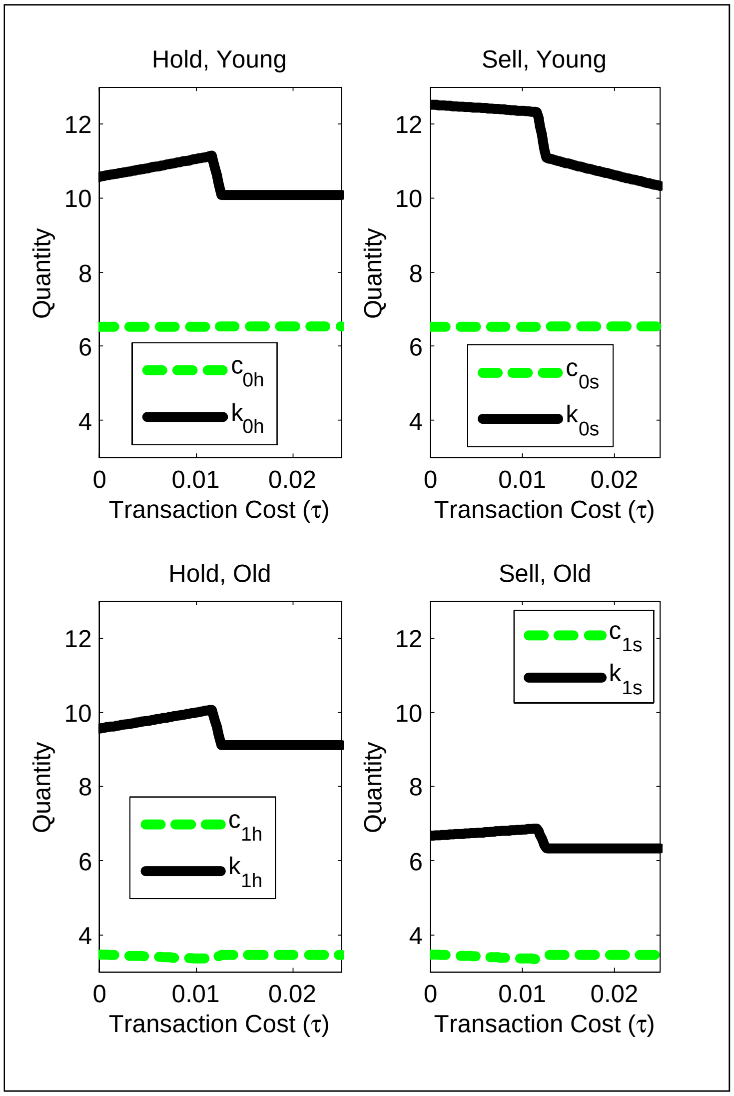
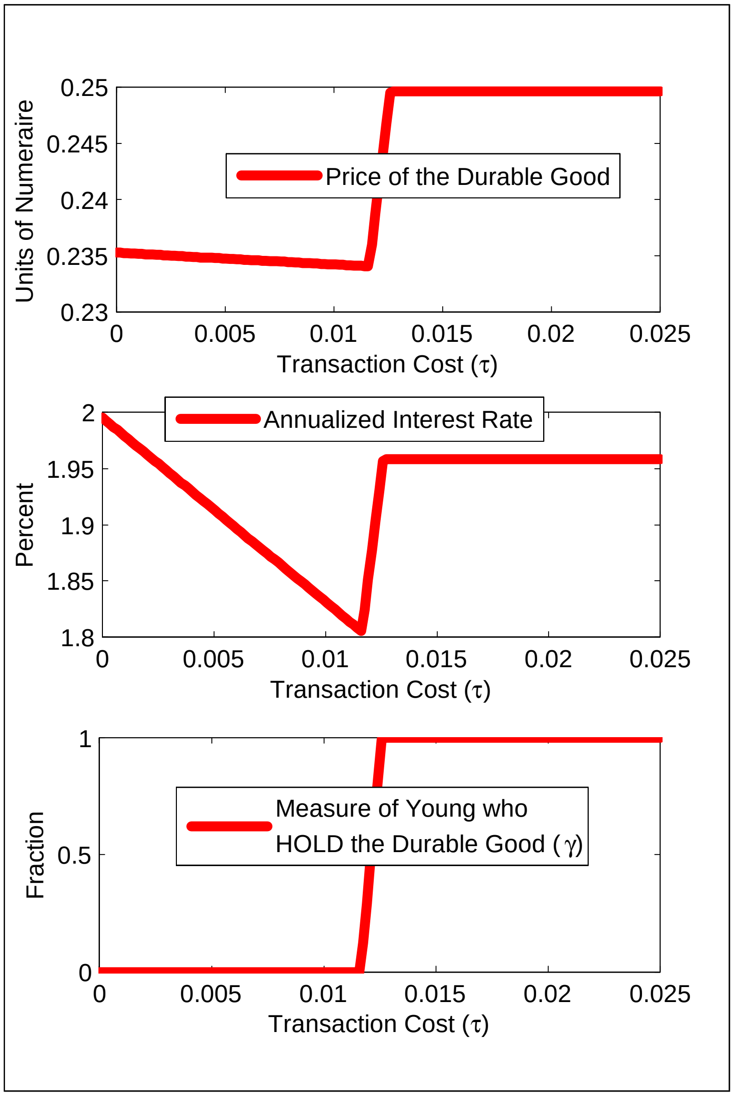

1. The estates are the assets owned by the previously old.
3. Trade
Estates
2. Assets Payoff
1. The old own the assets that they purchased when young.
3. Trade
Time Passes
Old => Estates
Depreciation
Old
2. Assets Payoff
4. Consumption
1. Young endowed with both nondurable and durable goods.
3. Trade
Time Passes
Young => Old
Depreciation
Time Passes
Old =>Estates
Depreciation
Young
2. Assets Payoff
4. Consumption

Figure 2:



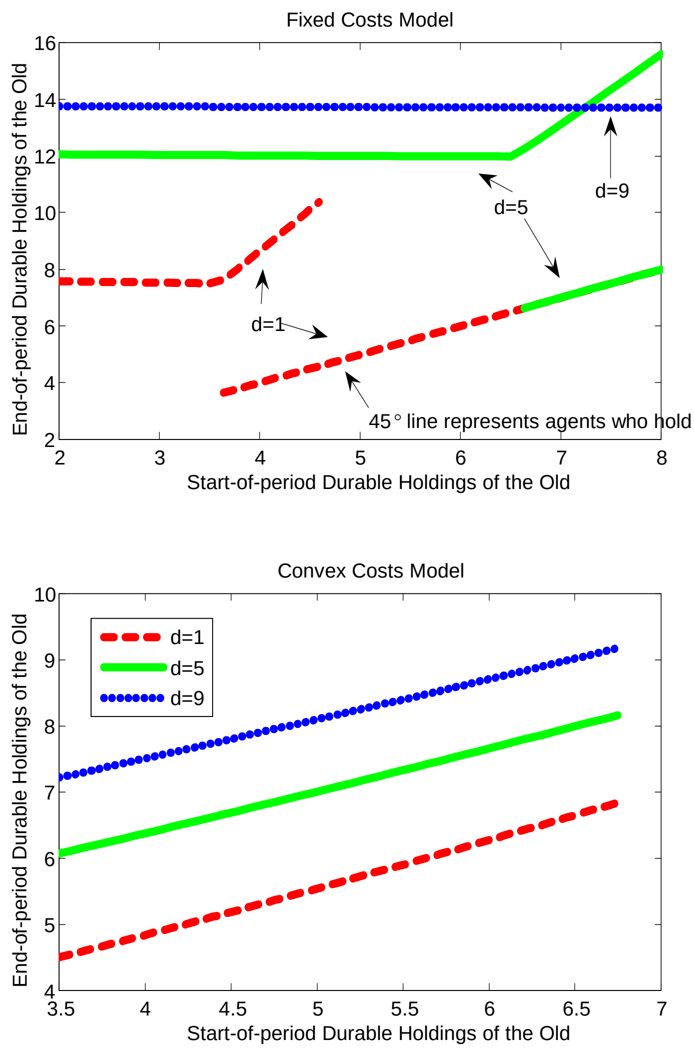
Hold, Young
Sell, Young
Hold, Old
Sell, Old
Quantity
Transaction Cost (τ)
c_{0h}
k_{0h}
c_{0s}
k_{0s}
c_{1h}
k_{1h}
c_{1s}
k_{1s}

Figure 3:



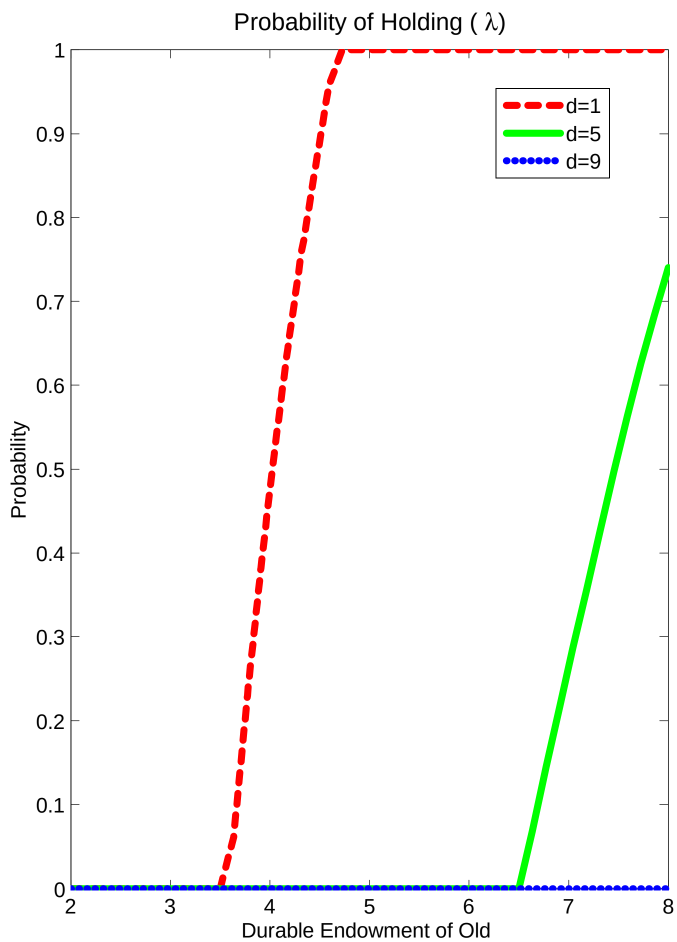
Units of Numeraire
0.25
0.245
0.24
0.235
0.23
Price of the Durable Good
0
0.005
0.01
0.015
0.02
0.025
Transaction Cost (τ)
Percent
2
1.95
1.9
1.85
1.8
Annualized Interest Rate
0
0.005
0.01
0.015
0.02
0.025
Transaction Cost (τ)
Fraction
1
0.5
0
Measure of Young who
HOLD the Durable Good (γ)
0
0.005
0.01
0.015
0.02
0.025
Transaction Cost (τ)

Figure 4:



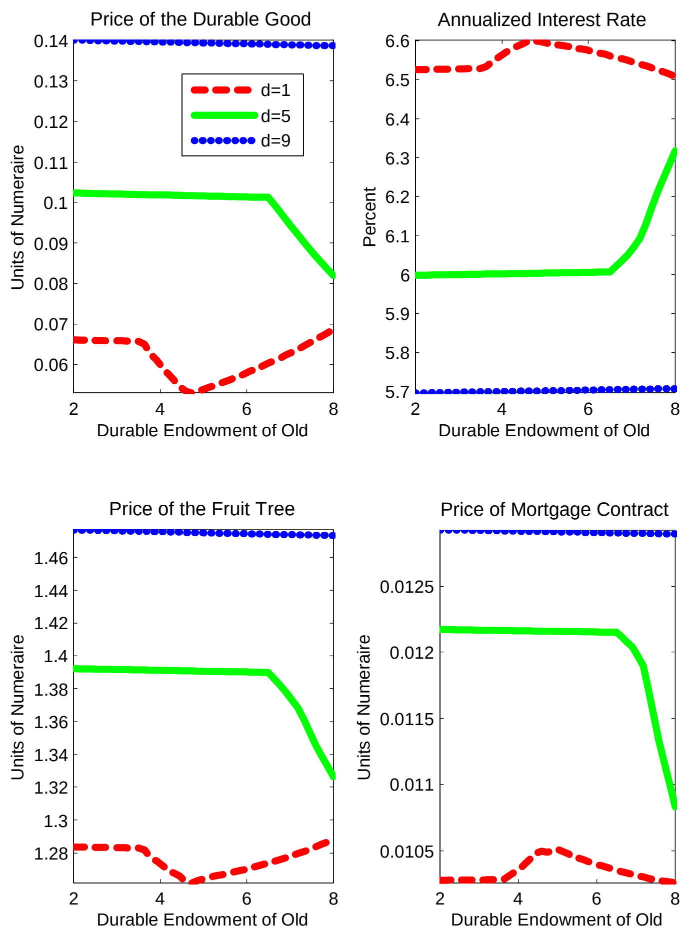
Fixed Costs Model
End-of-period Durable Holdings of the Old
Start-of-period Durable Holdings of the Old
d=1
d=5
d=9
45° line represents agents who hold
Convex Costs Model
End-of-period Durable Holdings of the Old
Start-of-period Durable Holdings of the Old
d=1
d=5
d=9

Figure 5:



Probability of Holding (λ)
Probability
Durable Endowment of Old
d=1
d=5
d=9
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
3
4
5
6
7
8

Figure 6:



Price of the Durable Good
Units of Numeraire
Durable Endowment of Old
d=1
d=5
d=9
Annualized Interest Rate
Percent
Durable Endowment of Old
Price of the Fruit Tree
Units of Numeraire
Durable Endowment of Old
Price of Mortgage Contract
Units of Numeraire
Durable Endowment of Old

Figure 7:

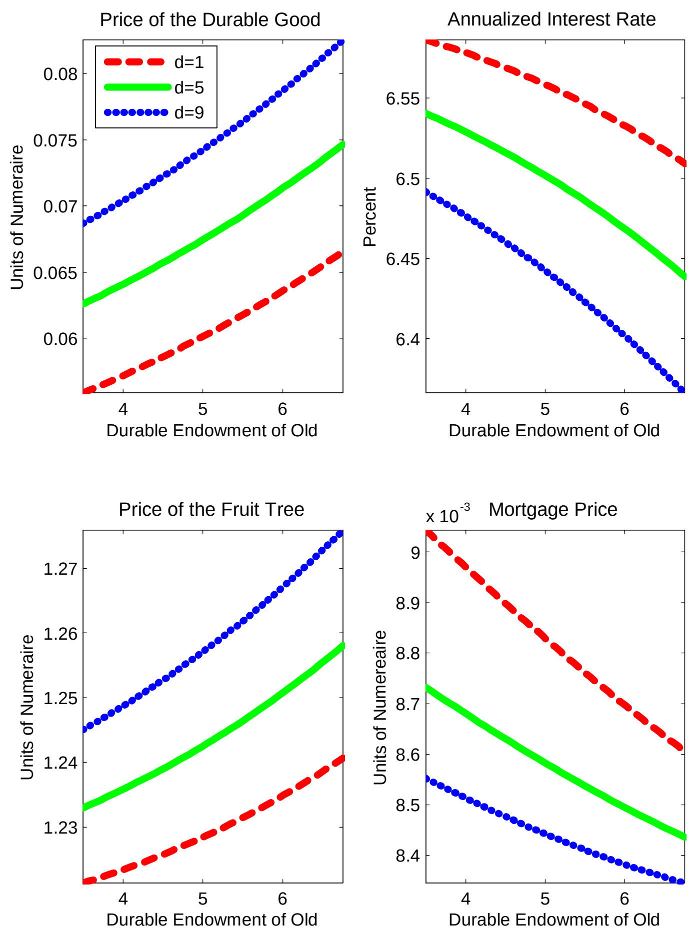

Price of the Durable Good
Units of Numeraire
d=1
d=5
d=9
Durable Endowment of Old
Annualized Interest Rate
Percent
Durable Endowment of Old
Price of the Fruit Tree
Units of Numeraire
Durable Endowment of Old
x 10^-3
Mortgage Price
Units of Numereaire
Durable Endowment of Old

Figure 8:

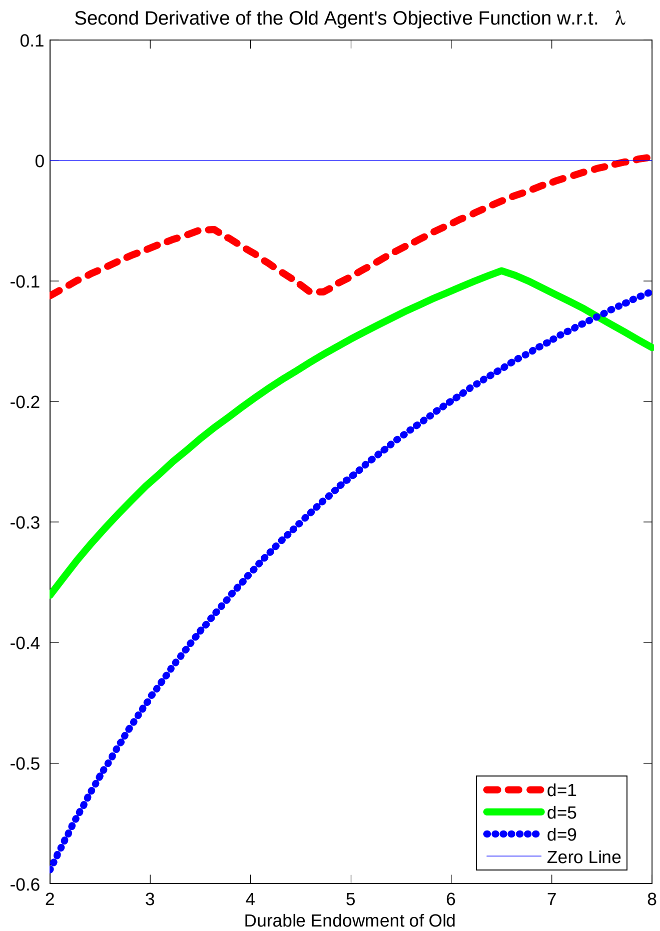


Figure 9: This is a graph of the second derivative of the old agent's objective function with respect to λ. The value is negative over the entire state space (at equilibrium prices). Therefore agent's objective function, concentrated in λ, is concave and the agents may optimally choose nondegenerate lotteries.