Exploring Time-Varying Jump Intensities: Evidence from S&P500 Returns and Options

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Abstract

Standard empirical investigations of jump-dynamics in return and volatility are fairly complicated due to the presence of multiple latent continuous-time factors. We present a new discrete-time framework that combines GARCH processes with rich specifications of jumps in returns and volatility. Our models can be estimated with ease using standard maximum likelihood techniques. We provide a tractable risk neutralization framework for this class of models which allows for separate identification of risk premia for the jump and normal innovations. We anchor our models in the continuous time literature by providing continuous time limits of the models. The models are evaluated by return fitting on a long sample of S&P500 index returns as well as by option valuation on a large option data set. We find strong empirical support for time-varying jump intensities. A model with a jump intensity that is affine in the conditional variance performs particularly well both in return fitting and option valuation. Our implementation allows for multiple jumps per day and we find evidence of this most notably on Black Monday in October 1987. Our results also confirm the importance of jump risk premia for option valuation: jumps cannot significantly improve the performance of option pricing models unless sizeable jump risk premia are present.

JEL Classification: G12

Keywords: GARCH; compound Poisson process; option valuation; filtering; volatility jumps; jump risk premia; time-varying jump intensity.

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1 Introduction

This paper provides a modeling framework that allows for general specifications and easy maximum likelihood estimation of jump models. Our framework allows for correlated jumps in returns and volatility, as well as time-varying jump intensities. We provide the risk-neutral processes for use in option valuation and we develop continuous time limits of our discrete time models. We allow for multiple jumps each day and we suggest a filtering technique to identify these jumps. Our models allow for time-varying variance of the normal innovation as well as time-varying jump intensity of the jump innovation. The implementation of our framework is facilitated by the fact that we use a discrete-time approach, and by the way in which we model the time-variation in the jump intensity. As suggested in Fleming and Kirby (2003) we directly use GARCH processes as filters for the unobservable state variables. As a result, various specifications of complex jump models can be estimated on return data using a standard MLE procedure.

Table 1 provides an overview of some of the empirical studies using finite-activity jump processes. The bottom row of Table 1 indicates that our model ("J-GARCH") can accommodate various complex jump specifications currently used in the literature. Our approach is perhaps most closely related to the discrete-time approach in Maheu and McCurdy (2004). They find strong evidence of time-varying jump intensities in individual equity returns. We find similar evidence using equity index returns and importantly we provide theory and empirics on option valuation using our jump models as well as their continuous time limits. Our option valuation results also indicate that time-varying jump intensities are needed. Duan, Ritchken and Sun (2006) also provide risk neutralization of a GARCH model with jumps but they do not allow for time-varying jump intensities.

As the top part of Table 1 suggests, our work is related to the continuous-time approaches in Bates (2000, 2006), Andersen, Benzoni and Lund (ABL, 2002), and Eraker (2004), who estimate models that allow for state-dependent jump intensities. Huang and Wu (2004), and Pan (2002) also allow for state-dependent jump intensities but they extract the latent volatility process from options without filtering the underlying return. Recent studies such as Broadie, Chernov and Johannes (BCJ, 2007), Li, Wells and Yu (2007) assume constant jump intensities in their models.

Our discrete-time framework arguably has certain advantages. First, because implementing the filtering problem is relatively straightforward, we do not need to model the compound Poisson process as a Bernoulli process and we therefore allow for the possibility that there is more than one jump per time period.[1] We find strong evidence of this particularly on Black Monday in October 1987. Second, we provide the separate identification of the risk premia associated with the jump and the normal innovation, which follows from our risk-neutralization procedure which in this way differs from the existing approach in Duan, Ritchken and Sun (2006). Third, a potential advantage of our GARCH specification as compared to a (continuous-time or discrete-time) stochastic volatility specification is that all parameters needed for option valuation can be obtained from estimation on returns only.

We estimate four different but nested models. The simplest model, which we label J-GARCH(1), has a constant jump intensity. This model is closely related to the most popular

[1] In estimation, many papers make simplifying assumptions, such as approximating a compound Poisson process by a Bernoulli process, implying that there is maximum of one jump per time period. See for example Eraker (2004), Eraker, Johannes, and Polson (2003), Bakshi and Cao (2004), and Li, Wells and Yu (2007).

model in the continuous-time literature, the SVCJ model, which is studied among others by Eraker, Johannes, and Polson (EJP, 2003), Chernov, Gallant, Ghysels and Tauchen (CGGT, 2003) and Eraker (2004). The J-GARCH(2) has a time-varying jump intensity, but the normal innovation to the return process is assumed to be homoskedastic. The J-GARCH(4) model is the most general model we investigate: both the jump and the normal innovation are time-varying, and the dynamics are separately parameterized. In this case, the jump intensity carries its own GARCH dynamic. The J-GARCH(3) model is a special case of J-GARCH(4): both the jump and the normal innovation are time-varying, but parameterized identically. Our classification of models is inspired by Huang and Wu (2004) who investigate infinite-activity jump models using time-changed Lévy processes.

Our empirical investigation estimates these four models using a long time series of daily S&P500 returns going from 1962 through 2005. After estimating these processes using returns data, we risk-neutralize them and compare their option valuation performance using ten years of index call option data from 1996 through 2005.

The empirical results on returns and options allow us to address a number of important questions. (1) How should jump and normal (diffusive) innovations be jointly modeled in equity index returns? (2) Are jump intensities time-varying for the purpose of option valuation? (3) Do the data favor a specification that allows for more than one jump per day? How does this assumption impact on the evidence regarding time-varying jump intensity? (4) Do we need jump risk premia to model index option prices? (5) Are jumps needed for option valuation at all times or only in certain regimes? Finally, we also extract the time-series of conditional variance for jump and normal components. This allows us to address the following question (6) How large is the jump component and how much does it vary over time?

With regard to (1), we find that models without heteroskedasticity in the normal innovation are severely misspecified, which is entirely consistent with the continuous-time literature. Both the return-based and option-based evidence support the presence of jumps in returns as well as jumps in volatility. Our jump parameter estimates are roughly consistent with the results of Eraker (2004), EJP (2003), ABL (2002), and CGGT (2003).

With regard to (2), our MLE estimates show strong support for the J-GARCH(3) model, with time-varying jump intensities and linear dependence on the variance of the normal innovation. Under this specification, and assuming a dominant jump risk premium, we obtain up to 30% improvement in dollar root mean squared option pricing errors over the simple GARCH case. The J-GARCH(3) model is comparable to the continuous-time SVSCJ model. While to our knowledge time-series based estimation of the SVSCJ model is not available in the literature, ABL (2002) use a model without volatility jumps and find no time-series based evidence for a time-varying intensity, while Bates (2006) estimates the same model using same dataset and find evidence for time-varying jump intensity. Our results are therefore more supportive of time-varying intensities than the available literature, but it must be noted that due to the GARCH filter, jumps in return and volatility are perfectly correlated. Regarding (3), our estimates support the presence of multiple jumps per day in the October 1987 period. We speculate that not allowing for this possibility has biased existing studies against detecting time-varying jump intensities.

For (4), we find that in order to produce significant improvements in option valuation, jump models must allow for jump risk premia. We investigate if risk premia for the jump and the normal innovation can generate the various shapes and levels of implied volatility term structure, and we find that the implied volatility term structure is highly responsive to the level of the jump risk premium. On the other hand, unrealistically large magnitudes of risk premia for the normal innovation are required in order to generate levels and slopes of implied volatility comparable to the data. Therefore, we conclude that jump risk premia are necessary for realistic modeling of option prices. Regarding question (5), we conclude that jump models provide superior option pricing performance during high volatility periods, but jumps do not help when the VIX index is well below its average. Finally, regarding (6), we find that the contribution of the jump component to the total equity volatility is about 15%, which is consistent with the non-parametric time-series evidence of Huang and Tauchen (2005).

The remainder of the paper proceeds as follows. Section 2 presents the general J-GARCH modeling framework and discusses the four nested specifications. Section 3 provides the empirical results from estimating the models on daily returns. Section 4 develops the theoretical framework for risk-neutralization and option valuation. Sections 5 provides the empirical results on option valuation. Section 6 develops continuous time limits of our models and provides a filtration of the jump and normal components. Section 7 concludes.

2 Daily Returns with Jump Intensity Dynamics

In this section, we present the general return dynamic. This dynamic contains two components. The first is the jump component. The second is the component which corresponds to the diffusive term in continuous-time setups. Because we model this component using a normal innovation, we will henceforth refer to it as the normal component. This section discusses some aspects of the structure of the jump and normal components, but we postpone the specifications of the time-variation in their conditional variance until Section 3.

The return process is given by

$$R_{t+1} \equiv \log\left(\frac{S_{t+1}}{S_t}\right) = r + \left(\lambda_z - \tfrac{1}{2}\right) h_{z,t+1} + \left(\lambda_y - \xi\right) h_{y,t+1} + z_{t+1} + y_{t+1}, \tag{2.1}$$

where S_{t+1} denotes the underlying asset price at the close of day $t+1$, and r the risk free rate. Shocks to returns are generated by the normal component z_{t+1} and the pure jump component y_{t+1}, which are assumed to be contemporaneously independent. The normal component z_{t+1} is assumed to be distributed $N\left(0, h_{z,t+1}\right)$, where $h_{z,t+1}$ is the conditional variance. We model the jump component using the compound Poisson process, which is the standard jump process used in the continuous-time literature. See Merton (1976) for an early treatment of these processes in finance. We let y_{t+1} be conditionally distributed as compound Poisson $J\left(h_{y,t+1}, \theta, \delta^2\right)$, where $h_{y,t+1}$ denotes the jump intensity (or jump arrival rate), θ the mean jump size, and δ^2 the jump size variance. The convexity adjustment terms $\frac{h_{z,t+1}}{2}$ and $\xi h_{y,t+1} = \left(e^{\theta+\frac{\delta^2}{2}} - 1\right) h_{y,t+1}$ in (2.1) act as compensators to the normal and jump component respectively. Thus, when taking conditional expectations of the gross rate of return, we get

$$E_t\left[\frac{S_{t+1}}{S_t}\right] = e^{r+\lambda_z h_{z,t+1}+\lambda_y h_{y,t+1}}, \tag{2.2}$$

which shows that $\gamma_{t+1} = \lambda_z h_{z,t+1} + \lambda_y h_{y,t+1}$ is the conditional equity premium, with λ_z and λ_y the market prices of risk for the normal and jump components respectively. Our setup allows for the possibility of time-varying equity premia, and the dynamic will depend on the specification of $h_{z,t+1}$ and $h_{y,t+1}$.

Our specification in (2.1) is similar in spirit to the return process of Heston and Nandi (2000), with the addition of a jump component. Maheu and McCurdy (2004) and Duan, Ritchken, and Sun (2006, 2007) propose return dynamics similar to (2.1), but with different specifications of the equity premium. Maheu and McCurdy (2004) study similar return processes with jumps but do not provide the risk-neutral process which is necessary for option valuation. Duan, Ritchken and Sun (2006, 2007) provide risk-neutralization arguments but for processes without time-varying jump intensities. Our specification of the equity premium is affine in the state variables. The affine specification is not required but we use it in order to isolate the jump and diffusive risk premia and because it greatly facilitates the comparison with standard continuous time models that we undertake in Section 6. The structure also makes for straightforward identification of risk-neutral dynamic, which is important for option pricing. We develop the risk neutral process in Section 5.

2.1 The Structure of the Jump Innovation

The compound Poisson structure assumes that jump size is independently drawn from a normal distribution with mean θ and variance δ^2. The number of jumps n_{t+1} arriving between times t and $t+1$ is a Poisson counting process with intensity $h_{y,t+1}$. The jump component in the period $t+1$ returns is therefore given by

$$y_{t+1} = \sum_{j=1}^{n_t} x_{t+1}^j$$

where $x_{t+1}^j, j = 1, 2, ..$ is an i.i.d. sequence of normally distributed random variables with $x_{t+1}^j \sim N\left(\theta, \delta^2\right)$. The conditional expectation of the number of jumps arriving over the time interval $(t,\ t+1)$ equals the jump intensity $E_t\left[n_{t+1}\right] = h_{y,t+1}$. The mean and variance of the jump component are given by $\theta h_{y,t+1}$ and $\left(\delta^2 + \theta^2\right) h_{y,t+1}$ respectively. Intuitively, $h_{y,t+1}$ should be time-varying as the number of jumps occurring at any time period will depend on market conditions. Unfortunately, jump models with time-varying jump intensity are difficult to estimate and implement in latent-factor continuous-time models because the likelihood function typically is not available in closed form. The filtration procedure for the latent jump and stochastic volatility process is also far from straightforward in continuous time. Therefore, the literature contains limited evidence on equity returns and option pricing models with stochastic jump intensity. We will test several specifications for $h_{y,t+1}$, ranging from the simple case of a constant arrival rate to modeling it using a separate GARCH dynamic.

2.2 The Heston and Nandi GARCH(1,1) Benchmark Model

Heston and Nandi (2000) propose a class of GARCH models that allow for a closed-form solution for European options. The GARCH(1,1) version of this model is given by

$$\begin{aligned} R_{t+1} &= r + \left(\lambda - \tfrac{1}{2}\right) h_{z,t+1} + \sqrt{h_{z,t+1}}\varepsilon_{t+1} \\ h_{z,t+1} &= w_z + b_z h_{z,t} + a_z \left(\varepsilon_t - c_z \sqrt{h_{z,t}}\right)^2 \end{aligned} \tag{2.3}$$

where r is the risk free rate, and ε_{t+1} is the innovation term distributed i.i.d. $N(0,1)$. The variance of the return is $h_{z,t+1}$. The asymmetric variance response or the leverage effect is captured by the parameter c. This model is based on conditional normality and thus cannot generate one-period ahead conditional skewness and excess kurtosis. The GARCH dynamic in (2.3) is different from the more conventional NGARCH model used by Engle and Ng (1992) and Hentschel (1995), which is used by Duan (1995) to price options. We choose the GARCH dynamic (2.3) as our benchmark since it ensures the closest possible correspondence of our jump parameter estimates with the continuous-time literature. Our objective is to present a general J-GARCH model that can be applied to any dynamics, and we leave the determination of the preferred GARCH dynamics in the presence of joint normality and compound Poisson jumps for future research.

Before we extend the Heston-Nandi framework to the returns process in (2.1), we rewrite the GARCH(1,1) dynamic in (2.3). Letting $z_{t+1} = \sqrt{h_{z,t+1}}\varepsilon_{t+1}$, we get

$$
\begin{aligned}
R_{t+1} &= r + \left(\lambda - \tfrac{1}{2}\right) h_{z,t+1} + z_{t+1} \\
h_{z,t+1} &= w_z + b_z h_{z,t} + \frac{a_z}{h_{z,t}} \left(z_t - c_z h_{z,t}\right)^2 .
\end{aligned}
$$

The unconditional variance is given by $E\left[h_{z,t+1}\right] = \left(w_z + a_z\right) / \left(1 - b_z - a_z c_z^2\right)$, where $b_z + a_z c_z^2$ is the variance persistence. We will use the empirical performance of the Heston-Nandi model as a benchmark for the evaluation of our J-GARCH models. Because we only use the GARCH(1,1) implementation, we henceforth refer to it simply as the GARCH model.

2.3 Four Nested J-GARCH Models

We now apply the simple Heston-Nandi GARCH dynamic to our return dynamic in (2.1). We will refer to these two dynamics as the conditional variance (for the normal component) and the time-varying jump intensity (for the jump component).

For the most general J-GARCH specification, both the jump intensity and the variance of the normal innovation are governed by a Heston-Nandi type GARCH(1,1) dynamic.

$$
h_{z,t+1} = w_z + b_z h_{z,t} + \frac{a_z}{h_{z,t}} \left(z_t + y_t - c_z h_{z,t}\right)^2 \tag{2.4}
$$

$$
h_{y,t+1} = w_y + b_y h_{y,t} + \frac{a_y}{h_{y,t}} \left(z_t + y_t - c_y h_{y,t}\right)^2 \tag{2.5}
$$

The subscripts z and y are applied to distinguish the parameters governing the GARCH(1,1) dynamic of $h_{z,t+1}$ and $h_{y,t+1}$ respectively. In (2.4)-(2.5), the dynamics of $h_{z,t+1}$ and $h_{y,t+1}$ are predictable conditional on information available at time t, and it is the total return innovation, $z_t + y_t$, observable at time t that predicts the variance and jump intensity one period ahead. The specification therefore includes jumps in volatility, which are supported by the empirical findings of Eraker (2004) and EJP (2003).

In a continuous-time setting, adding jumps to stochastic volatility models involves an additional set of latent state variables. The study of option pricing in stochastic volatility models with jumps therefore relies heavily on econometric methods that can filter the unobserved state variables. CGGT (2003) use an EMM based method, Pan (2002) uses the implied-state GMM technique to fit her models to returns and option prices, while EJP

(2003), Eraker (2004) and Li, Wells, and Yu (2007) employ MCMC techniques. In comparison, our use of GARCH models in (2.4)-(2.5) is computationally convenient, because the GARCH model serves as a simple and convenient filter where the state variables $h_{z,t+1}$ and $h_{y,t+1}$ are directly computed from the observed shocks, and therefore all of our models can be estimated from returns data using standard Maximum Likelihood.

We investigate four nested models based on the general dynamic in (2.4)-(2.5). We maintain consistency with the continuous-time literature by borrowing the framework of Huang and Wu (2004). Similar to Huang and Wu (2004), our models generate time-varying higher moments of returns from the variance of the normal component $h_{z,t+1}$ and/or from the jump intensity $h_{y,t+1}$. We now present these four specifications.

The J-GARCH(1) Model

The first specification we explore is akin to the most common specification in the continuous-time affine jump diffusion literature. It is similar in spirit to the stochastic volatility with correlated jumps (SVCJ) model studied by EJP, CGGT and Eraker (2004). Compared to our most general specification (2.4)-(2.5), we turn off the time-varying jump intensity dynamic while maintaining the normal component's GARCH dynamic, which amounts to the restrictions

$$b_y = 0 \qquad a_y = 0 \qquad c_y = 0.$$

The J-GARCH(1) model contains nine parameters, three more than the Heston-Nandi GARCH model. In any given period, the J-GARCH(1) model implies that jumps arrive at a constant rate of w_y, regardless of the level of risk in the market. Although this may seem counter intuitive, it is assumed in most of the existing literature.

The J-GARCH(2) Model

The J-GARCH(2) allows for jump dynamics but dynamic eliminates a source of time-variation by turning off the dynamic in the conditional variance of the normal component. It is a special case of the general dynamic in (2.5), with the restrictions

$$b_z = 0 \qquad a_z = 0 \qquad c_z = 0.$$

In this specification, time-variation in returns is driven by the jump component. The normal component of returns is homoskedastic, with the variance equal to w_z. This is equivalent to applying stochastic time change only to the pure jump process in Carr, Geman, Madan and Yor (2003) and Carr and Wu (2004). Given that time variation is restricted to the jump intensity, we may see an increase in the relative importance of the jump component in returns for this specification.

The J-GARCH(3) Model

In the J-GARCH(3) specification $h_{z,t+1}$ and $h_{y,t+1}$ are both time-varying but driven by the same GARCH dynamic. We specify the jump intensity to be affine in the conditional variance of the normal component

$$h_{y,t+1} = kh_{z,t+1} \tag{2.6}$$

where k is a parameter to be estimated. This affine structure for (2.6) is studied in a continuous-time setting by Eraker (2004), ABL (2002), Bates (2000), and Pan (2002). Pan (2002) and Eraker (2004) are the only two papers that find statistically significant estimates for k. Moreover, these results are obtained by fitting their models to option data, and not by estimating using the time series of returns. Thus, the evidence on time-varying jump intensities is still inconclusive.[2]

The closest continuous-time counterpart of the J-GARCH(3) model is the SVSCJ model, which stands for stochastic volatility with state-dependent, correlated jumps.[3] The SVSCJ is the most general model considered by Eraker (2004), who finds it outperforms other models when simultaneously fitting option data and S&P500 index returns. The J-GARCH(3) specification can be written as a special case of the most general J-GARCH(4) specification, subject to the following restrictions on the GARCH parameters of $h_{y,t+1}$ in (2.5)

$$w_y = w_z k \qquad b_y = b_z \qquad a_y = a_z k^2 \qquad c_y = \frac{c_z}{k}$$

The J-GARCH(4) Model

We refer to the most general specification, where the conditional variance of the normal component and the jump intensity are governed by separate GARCH processes, as the J-GARCH(4) model. Since the two variance components vary separately over time, their relative contribution to returns will also be time-varying. Independent GARCH dynamics also allow the variance components to mean-revert at different rates. This is empirically relevant, because Eraker (2004) and Huang and Wu (2004) provide evidence that shocks from jump components are more persistent and decay at a slower rate than the shock associated with the normal innovation.

Huang and Wu (2004) estimate a continuous-time specification similar in spirit to (2.4)-(2.5) using European options on the S&P500 index, and find that it outperforms other models both in and out-of-sample. However, their result is based on minimizing option pricing errors over a short period, and the time series of underlying returns does not enter their objective function. To the best of our knowledge, the literature does not contain estimates for a continuous-time model akin to (2.4)-(2.5) based on returns data. This is perhaps due to large number of latent factors that renders the likelihood function unavailable which in turn makes the estimation much quite challenging. Our discrete-time setup and the convenient GARCH filtration allow us to estimate the J-GARCH(4) model using standard MLE techniques.

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[2] These different findings may be due to differences in model specification and/or estimation technique. Eraker (2004) estimates his model using MCMC and specifies correlated jumps in returns and volatility. ABL (2002) use EMM to estimate models with jumps in returns only. Bates (2000) and Pan (2002) do not estimate models with jumps in volatility.

[3] The J-GARCH(3) model is a special case of the most general SVSCJ, because jumps in returns and volatility are perfectly correlated due to the GARCH filter.

2.4 Conditional Moments of Returns

Conditional moments of returns can be readily derived using the moment generating function of the normal and compound Poisson processes. The first four conditional moments are

$$
\begin{aligned}
E_t(R_{t+1}) &\equiv \mu_{t+1} = r + \left(\lambda_z - \tfrac{1}{2}\right) h_{z,t+1} + (\lambda_y - \xi + \theta) h_{y,t+1} && (2.7)\\
Var_t(R_{t+1}) &= h_{z,t+1} + \left(\delta^2 + \theta^2\right) h_{y,t+1} && (2.8)\\
Skew_t(R_{t+1}) &= \frac{\theta\left(3\delta^2 + \theta^2\right) h_{y,t+1}}{\left(h_{z,t+1} + \left(\delta^2 + \theta^2\right) h_{y,t+1}\right)^{3/2}} && (2.9)\\
Kurt_t(R_{t+1}) &= 3 + \frac{\left(3\delta^4 + 6\delta^2\theta^2 + \theta^4\right) h_{y,t+1}}{\left(h_{z,t+1} + \left(\delta^2 + \theta^2\right) h_{y,t+1}\right)^2} && (2.10)
\end{aligned}
$$

where $Skew_t(R_{t+1})$ is the conditional skewness of returns, and $Kurt_t(R_{t+1})$ is the conditional kurtosis. The sign of the conditional skewness depends on the sign of the mean jump size θ. For positive $h_{y,t+1}$, which means in the presence of jumps, the dynamics of conditional skewness and kurtosis are driven by the conditional variance of the normal component as well as the jump intensity. In the empirical section we find that $h_{y,t+1}$ is orders of magnitude larger than $h_{y,t+1}$and that the higher-moment dynamics are trivial in models where $h_{y,t+1}$ is constant. Thus time-varying jump intensities are crucial for obtaining time-varying higher-moments. Harvey and Siddique (1999, 2000) document the importance of time-varying skewness in asset pricing.

The expressions for the persistence and long run variance of all four models are provided in Appendix A.

2.5 The Likelihood Function

The likelihood function for returns depends on the normal and Poisson distributions. First, notice that conditional on $n_{t+1} = j$ jumps occurring in a period, the conditional density of returns is normal

$$
f_t(R_{t+1} | n_{t+1} = j) = \frac{\exp\left(-\frac{\left(R_{t+1} - \mu_{t+1} - \theta_j\right)^2}{2\left(h_{z,t+1+j}\delta^2\right)}\right)}{\sqrt{2\pi\left(h_{z,t+1} + j\delta^2\right)}}, \tag{2.11}
$$

Because the number of jumps is finite in the compound Poisson process, the conditional probability density of returns can be derived by integrating out the number of jumps, distributed as a Poisson counting process

$$
\Pr_t(n_{t+1} = j) = \frac{\left(h_{y,t+1}\right)^j}{j!} \exp\left(-h_{y,t+1}\right). \tag{2.12}
$$

This yields the conditional density in terms of the observables

$$
f_t(R_{t+1}) = \sum_{j=1}^{\infty} f_t(R_{t+1} | n_{t+1} = j) \Pr_t(n_{t+1} = j). \tag{2.13}
$$

and likelihood function can now be constructed easily as the product of the conditional distributions across the sample.

When implementing the ML estimation the summation (2.13) must be truncated. Jumps of the Poisson type are large and rare, with on average one to two jumps per year.[4] However, we want to allow for the possibility of clustering of several jumps on a day and so we follow Maheu and McCurdy (2004) by truncating the summation at 25 jumps per day. In the empirical section we find evidence of up to 10 jumps on a single day. Through experimentation we have found that our estimation results are robust when the truncation is set to 25.

3 Daily Return Empirics

3.1 Data and Method

We estimate the models using the time series of S&P 500 returns from June 1, 1962 through December 31, 2005. The data are obtained from CRSP. The top panel of Figure 1 shows the daily logarithmic return of the S&P 500 for our sample. Several large or "jump-like" movements in returns are apparent. The largest price change is the crash of October 1987, when the index falls by almost 25 percent in a single day. We use a long sample of returns because it is well-known that it is difficult to estimate GARCH parameters precisely using relatively short samples. More importantly, jumps are rare events, and given an average occurrence of one or two jumps per year estimated in the existing literature, it is difficult to get sensible estimates on jump parameters with daily return data that spans less than ten to fifteen years. We estimate the models using standard Maximum Likelihood. For GARCH updating, we use starting values that correspond to the long-run jump intensity and the long-run variance. That is, we set $h_{z,1} = \sigma_z^2$ and $h_{y,1} = \sigma_y^2$. Our optimization converges quickly and our estimates are robust to the wide range of starting values that we try.

3.2 Maximum Likelihood Estimates

Table 2 presents Maximum Likelihood estimates (MLE) for the four J-GARCH models, obtained using returns data for 1962-2005. For reference, Table 3 presents MLE estimates for a number of benchmark models which include the Heston-Nandi GARCH, the Black and Scholes (1973) model, and the Merton (1976) Jump model. For each model, we divide parameter estimates into two columns, one for parameters representing the normal component, and the other for parameters representing the jump component. For example, the parameter λ in the "Jump" column refers to the λ_y parameter. Under each parameter estimate, we report its standard error computed using 100 bootstrapped MLE samples.

The log-likelihood values show that all jump models except the J-GARCH(2) significantly improve on the fit of the GARCH model. The relatively poor performance of the J-GARCH(2) specification is presumably due to the lack of time-variation in the variance of the normal component, which apparently cannot be compensated by allowing for a time-varying jump intensity. The jump intensity of the J-GARCH(2) process is highly persistent.

[4] Existing evidence regarding the jump frequency using S&P500 returns is quite consistent. See for example EJP (2003), Eraker (2004) and CGGT (2003). Please refer to Broadie, Chernov, and Johannes (2007) for a summary of these results.

Figure 2 shows the annualized conditional return variance for the four J-GARCH models. We plot the total conditional variance define above as

$$Var_t\left(R_{t+1}\right) = h_{z,t+1} + \left(\delta^2 + \theta^2\right) h_{y,t+1}$$

from January 1, 1986 through December 31, 2005. To aid the comparison with the conditional variance path of the GARCH model, we plot the difference between the conditional variance of each jump model and the Heston-Nandi GARCH in Figure 3. Note that J-GARCH(2) has its own scale in Figure 3. It is evident that the conditional variance in the J-GARCH(2) model does not exhibit much variation, which suggests an absence of time variability in the jump intensity. It is somewhat surprising, since one would expect a more volatile jump component given the homoskedastic normal innovation. Figure 4 shows the jump intensity over time in each model, that is $h_{y,t+1}$. It confirms that the J-GARCH(2) specification has a relatively constant jump intensity in comparison with the J-GARCH(3) and J-GARCH(4) models.

Next we look at the simplest jump structure, which is the J-GARCH(1) specification. Adding a simple constant jump component to a GARCH dynamic can significantly improve model fit, despite the fact that the GARCH parameters of the normal component in the J-GARCH(1) model are similar to those of the GARCH model. Table 3 reports a mean jump size of $\theta = -1.254\%$, and jump volatility of $\delta = 2.861\%$. As for the jump intensity $E\left[h_{y,t+1}\right] = w_y$, the model implies that jumps arrive at a frequency of $252 \times w_y \simeq 2.03$ jumps per year. These values are roughly consistent with estimates for the SVCJ model obtained by Eraker (2004), CGGT, and EJP. We find a slightly smaller jump mean size and slightly higher jump intensity rate.

The results for the J-GARCH(3) model indicate that allowing for state-dependent jump intensities can further improve model performance. The estimate of k is statistically significant, confirming that the arrival rate of jumps depends on the level of risk in the market. The mean jump size is smaller than for the J-GARCH(1) model. However, jumps arrive more frequently, with on average $E\left[h_{y,t+1}\right] = k\sigma_z^2 = 0.0384$ or 9.6782 jumps per year. Therefore, we conclude that by allowing for time-varying jump intensities, smaller jumps can occur at a frequency which depends on the level of risk in the market. The bottom-left panel in Figure 4 indeed shows considerable variation in the jump-intensity over time.

These findings regarding time-varying jump intensities in the J-GARCH(3) model stand in sharp contrast to our findings for the J-GARCH(2) model. We conjecture that this is due to the strong misspecification of the J-GARCH(2) model. Our evidence in support of time-varying jump intensities is in line with Eraker (2004) and Pan (2002), who estimate their models using joint information from returns and option prices. Bates (2006) also find support for time-varying jump intensity from returns when estimating the model using his approximate maximum likelihood method. However, ABL (2002), estimating on returns, and Bates (2000), estimating on option prices, do not find evidence which support time-varying intensities.

Note from Table 2 that in almost all models, the market price of risk parameters cannot be precisely estimated. This is not surprising, as we are estimating the models using the physical return process. Heston and Nandi (2000), EJP, and Eraker (2004) reach a similar conclusion. Nevertheless, these parameters critically impact on the volatility term structure of option prices. We discuss the implications of risk premia in more detail below.

Surprisingly, the likelihood for the most general specification, J-GARCH(4), offers little improvement over J-GARCH(3), despite the fact that it allows for the jump intensity to be driven by its own independent GARCH dynamic. The persistence of the jump intensity is higher than the variance of the normal component. The MLE estimates of the J-GARCH(4) model indicate that the average jump intensity is $E[h_{y,t+1}] = 0.0488$, which translates to more than 12 jumps per year. This jump arrival frequency is much higher than in the J-GARCH(1) and J-GARCH(3) models. Jumps are smaller, with an average jump size of -0.1622%. The bottom-right panel in Figure 4 shows that the general J-GARCH(4)model implies more variation in the jump-intensity over time than in the more restricted J-GARCH(3).

3.3 Variance Decomposition and Conditional Higher Moments

At the bottom of Table 2 we report on the decomposition of total return variance into the normal and jump components. With the exception of the J-GARCH(2) model, the contribution of return jumps to the overall equity variance is consistent with the non-parametric time-series evidence of Huang and Tauchen (2005). They find that jumps in prices account for about 10% to 15% of overall equity volatility. It is not surprising that the results differ in our J-GARCH(2) model because the variance dynamics are shut down in this model.

Figure 5 plots the conditional one-day-ahead skewness and excess kurtosis from the four models. We thus plot the expressions in (2.9) and (2.10) during the January 1, 1986 to December 31, 2005 period. The top row of panels contain the results from J-GARCH(1). Note that the scale for this model is different than of the other models. The J-GARCH(1) model which allows for dynamic variance and constant-intensity jumps implies conditional skewness down to -3 and conditional excess kurtosis as high as 60. The J-GARCH(2) model which has constant variance implies almost constant conditional skewness and kurtosis. The J-GARCH(3) and J-GARCH(4) models that allow for dynamic jump intensities imply conditional skewness down to -0.4 and conditional excess kurtosis up to 15 which seems more reasonable than the J-GARCH(1) moments.

Note that all the model with conditional variance dynamics imply that the conditional skewness and kurtosis are furthest from zero when conditional variance is low as it was during the mid-1990s.

4 Option Valuation Theory

We now derive results that allow us to value derivatives using the J-GARCH model. In our discrete-time framework, the stock price can jump to an infinite set of values in a single period, and therefore the uniqueness of the equivalent martingale measure cannot be guaranteed. Although we consequently cannot identify unique option values through the absence of arbitrage, we can proceed by establishing the existence of a risk-neutral probability density under which the returns on all assets are equal to the risk-free rate. We start by explaining our risk-neutralization procedure, with an emphasis on the importance of the assumption on the equity premium in (2.1). Given our assumptions, the functional form of the risk-neutral dynamic is identical to that of the physical dynamic. We further discuss the differences between our setup and the approaches used in existing studies in Appendix B.

4.1 Risk-Neutralization

We proceed along the lines of Gerber and Shiu (1994), who use the Esscher (1932) transform on the objective probability density $P(x)$

$$Q(x;\Lambda) = \frac{e^{\Lambda x} P(x)}{M(\Lambda)}, \tag{4.1}$$

where Λ is a real number such that the moment generating function $M(\Lambda)$ of the stochastic process x exists under P. The Esscher transform has previously been used in the derivatives literature, for instance by Carr and Wu (2004) for applications in continuous-time option pricing and by Ahn, Dai and Singleton (2006) for applications in discrete-time dynamic term structure modeling. When x is the index return, the assumption of the Esscher transform corresponds to the assumption of an economy with power utility, and Λ represents the coefficient of relative risk aversion (see Bakshi, Kapadia, and Madan (2003)). Buhlmann, Delbaen, Embrecths, and Shiryaev (1998) propose a discrete-time generalization of (4.1). For a discrete-time two-dimensional stochastic process, the Esscher transform corresponds to the following form on the conditional Radon-Nikodym derivative

$$\frac{\frac{dQ_{t+1}}{dP_{t+1}}}{\frac{dQ_t}{dP_t}} = \frac{\exp\left(\mathbf{\Lambda}' X_{t+1}\right)}{M\left(\mathbf{\Lambda};\ H_{t+1}\right)}, \tag{4.2}$$

where $X_{t+1} = (z_{t+1}, y_{t+1})'$ is a vector of shocks to returns and $H_{t+1} = (h_{z,t+1}, h_{y,t+1})'$ is a vector containing the variances of the normal component and the jump intensity. Because there are two types of shocks in this economy, $\mathbf{\Lambda} = (\Lambda_z, \Lambda_y)'$ is a two dimensional vector of equivalent martingale measure (EMM) coefficients that produce the wedge between the physical and the risk-neutral measure. The proof that (4.2) is a proper probability measure can be obtained by using the fact that the exponential term $\exp(\mathbf{\Lambda}' X_{t+1})$ is normalized by its joint moment generating function $M(\mathbf{\Lambda}; H_{t+1})$.[5]

Proposition 1 *If the dynamic of returns under the objective measure P is given by (2.1), the risk-neutral probability measure Q defined by the Radon-Nikodym derivative in (4.2) is an equivalent martingale measure (EMM) if and only if*

$$\log M\left(\mathbf{\Lambda}+\mathbf{1}; H_{t+1}\right) - \log M\left(\mathbf{\Lambda}; H_{t+1}\right) + \left(\lambda_z - \tfrac{1}{2}\right) h_{z,t+1} + \left(\lambda_y - \xi\right) h_{y,t+1} = 0 \tag{4.3}$$

where $\mathbf{1}$ is a two-dimensional vector of ones, and we recall that $\xi = \left(e^{\frac{\delta^2}{2}+\theta} - 1\right)$.

Proof. For an EMM to exist, the expected return of S_t from time t to $t+1$ must equal the risk-free rate

$$E_t\left[\left(\frac{\frac{dQ_{t+1}}{dP_{t+1}}}{\frac{dQ_t}{dP_t}}\right)\frac{S_{t+1}}{S_t}\right] = e^r.$$

[5] $M(\mathbf{\Lambda}; H_{t+1})$ is a joint moment generating function, as this function completely determines the distribution of (z_{t+1}, y_{t+1}). When $\Lambda_z = \Lambda_y$, $M(\mathbf{\Lambda}; H_{t+1})$ is the moment generating function of the process $z_{t+1} + y_{t+1}$.

Substituting the Radon-Nikodym derivative in (4.2) and the return dynamic in (2.1), and taking expectations we have

$$\frac{e^{r+\left(\lambda_z-\frac{1}{2}\right)h_{z,t+1}+(\lambda_y-\xi)h_{y,t+1}}E_t\left[\exp\left((\mathbf{\Lambda}+\mathbf{1})'X_{t+1}\right)\right]}{M\left(\mathbf{\Lambda};\ H_{t+1}\right)}=e^r.$$

Equating terms yields the required result. ■

Proposition 1 provides us with a simple relation (4.3) that can be used to solve for $\mathbf{\Lambda}$. However, this implies that we have to uniquely determine Λ_z and Λ_y from a single equation, which we can do using the affine structure of the model. First, we note that the logarithms of the joint moment generating function $\log M\left(\mathbf{\Lambda};H_{t+1}\right)$ is affine in H_{t+1}. Our assumption underlying the structure of the equity premium implies that the entire relation in (4.3) is also affine in H_{t+1}. This leads us to the following result.

Lemma 1 *For the return dynamic in (2.1), the solution to (4.3) for the J-GARCH model reduces to solving the following two equations*

$$\begin{aligned}\lambda_y-\left(e^{\frac{\delta^2}{2}+\theta}-1\right)-e^{\frac{\Lambda_y^2\delta^2}{2}+\Lambda_y\theta}\left(1-e^{\left(\frac{1}{2}+\Lambda_y\right)\delta^2+\theta}\right) &= 0 \qquad (4.4)\\ \Lambda_z+\lambda_z &= 0 \qquad (4.5)\end{aligned}$$

Proof. See Appendix B. ■

Equation (4.5) can be analytically solved for the EMM coefficient Λ_z, which gives $-\lambda_z$. This result is identical to the one obtained using Duan's (1995) LRNVR method which builds on Brennan (1979). It is not possible to solve for the second EMM coefficient Λ_y in (4.4) analytically. However, it is well behaved and can easily be solved for numerically. Note that due to the structure of the equity premium, the market prices of risk λ_z and λ_y enter separately into the above two equations (4.5) and (4.4). Given estimates of the physical parameters λ_z and λ_y, λ_z is sufficient to determine the EMM coefficient Λ_z, and hence the wedge that links the two measures for the normal innovation. Similarly, λ_y identifies the change of measure for the jump innovation. We can therefore identify the two sources of risk in the economy. An interesting special case is $\lambda_z=\lambda_y=0$, which means that the normal and jump risks are not priced in the market. In this case the solutions to Λ_z and Λ_y are zero and the distribution of returns under the physical and risk-neutral measures is identical. We will discuss the implications of the two different types of risk premia further below.

Finally, it is also important to note that our method for risk-neutralization of the jump component is different from several continuous-time studies including Pan (2002), Eraker (2004), and BCJ. We discuss their choice of risk-neutralization procedure in Section 6.

4.2 The Risk-Neutral J-GARCH Dynamic

We have now completely characterized the specification of the Radon-Nikodym derivative in (4.2). We can therefore derive the risk-neutral probability measure for the normal and jump components of returns using a simple change of measure.

Proposition 2 *Consider a stochastic process that is the sum of two contemporaneously independent random variables $z_{t+1}+y_{t+1}$, with each component distributed as*

$$z_{t+1}\sim N\left(0,h_{z,t+1}\right) \qquad\qquad y_{t+1}\sim J\left(h_{y,t+1},\theta,\delta^2\right)$$

under the objective measure P, $N()$ and $J()$ refer to the Normal and Compound Poisson distribution respectively. According to the Radon-Nikodym derivative in (4.2), under the risk-neutral measure Q the stochastic process can be written as $z^*_{t+1} + y^*_{t+1}$, where

$$z^*_{t+1} \sim N\left(\Lambda_z h_{z,t+1}, h_{z,t}\right) \qquad y^*_t \sim J\left(h^*_{y,t+1},\ \theta^*, \delta^2\right)$$

with

$$h^*_{y,t+1} = h_{y,t+1} \exp\left(\frac{\Lambda_y^2 \delta^2}{2} + \Lambda_y \theta\right) \qquad \theta^* = \theta + \Lambda_y \delta^2.$$

Proof. *See Appendix B.* ■

The change of measure shifts the mean of the normal component to the left by Λ_z, which amounts to $z^*_{t+1} = z_t - \lambda_z h_{z,t+1}$. This result is identical to the one of Duan (1995), who motivates the risk-neutralization using the power utility function. The compound Poisson process under the Q measure differs from its distribution under the physical measure in terms of the jump intensity $h^*_{y,t+1}$ and the mean jump size θ^*. This finding is consistent with available results on the change of measure in the continuous-time jump literature. For example, Naik and Lee (1990) assume power utility over consumption or wealth in a Lucas type model, and derive the difference in jump intensity and mean jump size between the two measures.

We have all the results needed to derive the risk-neutral J-GARCH dynamic. In order to avoid repetition, and recalling that the other three specifications are nested in the J-GARCH(4) specification, we show only the detailed result of the J-GARCH(4) model.

Proposition 3 ***Risk-Neutral J-GARCH(4) dynamic.*** *Under the risk-neutral measure, the stock return process can be written as*

$$\log\left(\frac{S_{t+1}}{S_t}\right) = r - \tfrac{1}{2} h_{z,t+1} - \xi^* h^*_{y,t+1} + z_{t+1} + y^*_{t+1}, \tag{4.6}$$

with the following GARCH Q-dynamic

$$\begin{aligned} h_{z,t+1} &= w_z + b_z h_{z,t} + \frac{a_z}{h_{z,t}} \left(z_t + y^*_t - c^*_z h_{z,t}\right)^2 \\ h^*_{y,t+1} &= w^*_y + b_y h^*_{y,t} + \frac{a^*_y}{h^*_{y,t}} \left(z_t + y^*_t + \Lambda_z h_{z,t} - c^*_y h^*_{y,t}\right)^2 . \end{aligned} \tag{4.7}$$

where we have the following transformation

$$\begin{aligned} h^*_{y,t+1} &= h_{y,t+1}\Pi \ , \qquad \xi^* = e^{\frac{\delta^2}{2}+\theta^*} - 1, \qquad w^*_y = w_y \Pi \\ a^*_y &= \Pi^2 a_y, \qquad c^*_z = \left(c_z - \Lambda_z\right), \qquad c^*_y = \frac{c_y}{\Pi} \end{aligned}$$

*for $\Pi = e^{\frac{\Lambda_y^2\delta^2}{2}+\Lambda_y\theta}$. Note that superscript * refers to a Q-parameter, and recall that $z_{t+1} \sim N\left(0, h_{z,t+1}\right)$ and $y^*_{t+1} \sim J\left(h^*_{y,t+1}, \theta^*, \delta^2\right)$.*

Proof. See Appendix B. ■

The discounted stock price process in (4.6) is a martingale where $-\frac{1}{2}h_{z,t+1}$ and $\xi^* h^*_{y,t+1}$ are the compensating terms for the normal and jump components respectively. The risk-neutral dynamic for the Heston-Nandi GARCH is a special case of (4.6) and (4.7), for $h^*_{y,t+1} = 0$.

Unlike in the normal GARCH case, closed-form option valuation results are not available. This is due to the jump innovation in the GARCH dynamic which does not yield an exponentially affine moment generating function. However, the discrete-time GARCH structure of the model renders option valuation straightforward via Monte-Carlo simulation.

5 Option Valuation Empirics

Estimating on returns data, the evidence in favor of jumps is overwhelming, and the results point to the importance of time-varying jump intensities. We now discuss the importance of jumps for the purpose of option valuation, and investigate how the jumps ought to be modeled from this perspective.

We evaluate the option pricing performance of our models using a rich sample of S&P 500 call option data for 1996-2005. We retrieve the European call option quotes from OptionMetrics and eliminate quotes that report zero trading volume. Subsequently, we apply the filters proposed by Bakshi, Cao, and Chen (1997) to the data. We only keep Wednesday options with maturity ranging from one week to a full calender year. We choose Wednesday because it is the least likely day to be a holiday and it is less likely to be affected by day-of-the-week effects. For further discussion of the advantages of Wednesday data, we refer to Dumas, Fleming and Whaley (1998).

Table 4 presents descriptive statistics for the option quotes by moneyness and maturity. The shape of the volatility smirk is evident from Panel C across all maturities, with short term options exhibiting the steepest volatility smirk. The middle panel of Figure 1 plots the Black-Scholes implied volatility using at-the-money options. The bottom panel of Figure 1 represents a time series for the CBOE-VIX index for the same period. Clearly the data in our sample are representative of the prevailing market conditions.

5.1 Calibrating The Risk Premia

We use the MLE estimates from Tables 2 and 3, risk-neutralize them, and price Wednesday call options for 1996-2005. Due to the GARCH structure of the models, all parameters needed to value options can in theory be identified from MLE estimates of the physical distribution.

We start by elaborating on the role of the equity premium in option pricing. Recall that we work with an affine equity premium specification in (2.2), where λ_z and λ_y are the market prices of risk associated with the normal and jump components respectively. The long run equity premium is found by taking unconditional expectations

$$E\left[\gamma_{t+1}\right] = \lambda_z \sigma_z^2 + \lambda_y \sigma_y^2$$

where σ_z^2 is the unconditional variance of the normal component and σ_y^2 is the unconditional jump intensity. Our general model allows for separate identification of the two market prices of risk, which is evident from (4.4) and (4.5). In models with jumps, the long-run total equity premium is therefore equal to the sum of two components, with $\lambda_z \sigma_z^2$ referring to the long-run risk premium associated with the normal innovation, and $\lambda_y \sigma_y^2$ referring to the long run jump risk premium.

Using the MLE estimates from Tables 2 the long run (unconditional) implied equity premium ranges from 2.67% (J-GARCH(2)) to 5.06% (J-GARCH(3)) per annum. However, even with more than forty years of daily S&P500 returns, we do not obtain statistically significant estimates of the market price of risk parameters λ_z and λ_y. This lack of reliable estimates creates problems for comparing models because these parameters are the key ingredients that drive a wedge between the physical and risk-neutral measures. BCJ (2007) estimate some of these parameters from option data. Although this approach is very helpful to identify risk premia, it complicates model comparison because it is likely to favour models with greater flexibility in the risk premium specification. Moreover, in their implementation different models are evaluated with different equity premia, and it is not clear if all risk-neutral equity premium estimates are economically justifiable. In our empirical implementation, we value options using historically observed equity premia, and we impose an identical long run equity premium across models.

The literature on estimates of the equity premium is too large to cite in full here.[6] Available estimates differ depending on the concepts, method, and data used in the calculations. Estimates of the historical equity return as reported in Ibbotson Associates (2006) range from 4.9 to 8.5%. Welch (2000) conducts two surveys on finance and economics academics' estimates of the expected equity premium over the next 30 years. He reports an average arithmetic equity premium of 7% over T-bills. Overall, most estimates in the literature are between 3% to 10%. For our sample period, our estimate on the realized expected return in excess of the 3-month Treasury bill is approximately 5.74%.

We first analyze option pricing performance using a long run equity premium of 6% for all models. This is close to the average available estimate, and consistent with our sample realized equity premium. Moreover, it is also economically reasonable. In our jump models, the total equity premium consists of a combination of jump and normal risk premia. Initially, we investigate two extreme cases, where the entire equity premium results either from the jump or the normal risk premium. We then proceed by assigning the jump risk to be the sole source of the equity risk premium. The effect of the risk-premia on the option pricing properties of the J-GARCH models will be studied further below.

5.2 Model Performance Based on Option Pricing Errors

Table 5 summarizes the models' option valuation performance. We use the MLE estimates to compute option prices for 1996-2005, assuming a 6% total equity premium. We report the pricing errors for two metrics, the dollar root mean squared error ($RMSE) and the implied volatility root mean squared error (IVRMSE). The computation of the $RMSE is straightforward. For the IVRMSE, we invert each computed call price C_j from the model using the Black-Scholes formula to get the implied volatilities $IV(C_j, K_j, \tau_j, S_j, r)$. The IVRMSE is

[6] See Cochrane (1997), Siegel and Thaler (1997), Mehra and Prescott (2003) and Fernandez (2006) for comprehensive surveys.

then computed as the square root of

$$\frac{1}{N}\sum_{j=1}^{N}\left(\sigma_j^{BS} - IV\left(C_j, K_j, \tau_j, S_j\right)\right)^2$$

where σ_j^{BS} is the Black-Scholes implied volatility of the j^{th} observed call price, and $N = 21,718$ is the total number of option contracts used in the analysis. The parameters K_j, τ_j, and S_j are the strike, maturity and the underlying index level associated with each option. We use \$RMSE and IVRMSE metrics because they are two commonly used loss functions in the literature. We report the raw \$RMSE and IVRMSE(%) for the benchmark GARCH model while for jump models, we report their RMSE ratios relative to the GARCH model for ease of comparison. In addition, we evaluate each jump model in two different ways. The two left columns contain RMSEs computed for $\lambda_y = 0$, where the equity premium is imputed to from risk premium on the normal innovation, while the two right columns refer to RMSEs computed for $\lambda_z = 0$, where the entire equity premium is imputed to the jump risk premium.

The most important finding is that without a jump risk premium, jump models provide little or no improvement for option pricing. However, when jumps are the source of the equity premium, the J-GARCH(1), J-GARCH(3) and J-GARCH(4) models lead to significant pricing improvements. These findings are consistent with Pan (2002) and BCJ (2007), who find significant and dominant jump risk premia in option data. Our findings are also consistent with those studies in a less obvious way. Pan (2002) and BCJ (2007) report statistically insignificant volatility risk premia. In our setup, this corresponds to insignificant risk premia associated with the normal innovation. The reason is that the GARCH filter constrains the return and volatility shocks associated with the normal component to be perfectly correlated. In the remaining part of this section, we focus on the case where the total equity premium is entirely due to jump risk. We will revisit the relative performance of different risk premium specifications in more detail below.

Table 5 indicates that the J-GARCH(3) model performs best. Its option pricing ability is quite remarkable, especially considering that we do not use option data to estimate the models. Using the assumption of a jump risk premium (and no risk premium associated with the normal component), it improves on the GARCH model by 30% and 18% using the \$RMSE and IVRMSE metrics respectively. Surprisingly, the most flexible model, J-GARCH(4), only modestly outperforms the constant jump intensity J-GARCH(1) model. Note that this results is actually consistent with the MLE estimation which found that the log likelihood value for the J-GARCH(4) model does not differ much from that of the J-GARCH(3) model. The J-GARCH(2) model performs poorly, with pricing errors similar to those of the Merton Jump model. This is not surprising, as the MLE estimates for the J-GARCH(2) model do not indicate much time variation in jump intensities, which means that the model essentially reduces to the Merton Jump model. The time-varying jump intensity is not sufficient to make up for the lack of variance dynamics in the J-GARCH(2) model: It performs poorly both in fitting returns and in pricing options.

The J-GARCH(1) model is a restricted form of the continuous-time SVCJ model. Thus we can compare our findings on its option pricing performance to the existing literature. Based on the IVRMSE metric, the J-GARCH(1) outperforms the GARCH by 9%. This is larger than the improvement of 2.3% in loss function found by Eraker (2004). BCJ (2007) conclude that the SVCJ model can improve on the IVRMSE over the Heston (1993) model by over 50%. However, their implementation is very different from ours. In their setup the spot volatility is estimated on options rather than filtered on returns and they also estimate the risk premia by minimizing the option pricing error.

5.3 Pricing Errors by Moneyness, Maturity and Volatility Level

Table 6 provides additional evidence on the option fit of J-GARCH models. We report option IVRMSE of the simple GARCH model and the IVRMSE ratio of the Jump models versus the simple GARCH model. In this and the next three subsections we will assume an equity risk premium of 6% that is all assigned to jump risk. We report the results by moneyness, maturity, and VIX index level. The performance of the J-GARCH(3) model is very robust across moneyness and maturity (see panels A-B). Overall, jump models, except for J-GARCH(2), perform very well for long maturity options. This may seem surprising, because jump models are well-known for their ability to fit short term options, see Pan (2002). However, recall that the J-GARCH models allow for jumps in volatility, which generate the higher persistence needed to price long maturity options.

Panel C of Table 6 reports IVRMSE and IVRMSE ratios sorted by the level of the VIX index. The J-GARCH(3) and J-GARCH(4) models outperform the GARCH model in medium and high volatility periods (when VIX $\geq$14). However, in low volatility periods, none of the jump models can improve on the simple GARCH model. This finding indicates that jumps provide little or no benefit for option pricing in low volatility periods. It may seem surprising that the J-GARCH(3) model, with a jump intensity specification that depends on the volatility level, performs so poorly in the low volatility period. However, the jump intensity in J-GARCH(3) is affine in the variance of the normal component, which is bounded below. Therefore, the jump intensity in the J-GARCH(3) model is also bounded below, and therefore jumps can occur even in very low volatility periods. The J-GARCH(4) model, on the other hand, performs comparably to the GARCH model in the low volatility period. The reason is that the J-GARCH(4) model has an independent GARCH dynamic for the jump intensity.

5.4 The Implied Volatility Bias over Time

The option RMSEs favour the J-GARCH(3) model. We now provide more insight behind this model's performance. We first look at the ability of the J-GARCH models to match the time path of average at-the-money implied volatilities. Figure 6 presents the average weekly at-the-money implied volatility bias (average observed market implied volatility less average model implied volatility) over the 1996-2005 option sample, using the MLE estimates from Tables 2 and 3.[7] No model can produce implied volatilities that are sufficiently high to match the data during high-volatility periods. However, the positive bias of the J-GARCH(3) model is much smaller than for the other models. Note that the time paths of implied volatility bias are nearly identical for the J-GARCH(1) and J-GARCH(4) models. The average biases in Figure 6 are as follows: J-GARCH(1): 0.0326, J-GARCH(2): 0.0516, J-GARCH(3): 0.0261, J-GARCH(4): 0.0363. These numbers can be compared with the simple

[7] We consider options to be at-the-money if their strike price lies within 2.5% of the underlying index.

GARCH bias of 0.0392. Thus the J-GARCH(3) has a clearly lower bias than the GARCH benchmark, whereas the J-GARCH(1) and J-GARCH(4) have only a slightly lower bias. The J-GARCH(2) performs poorly using this measure as well. The bias is very persistent in all models suggesting that perhaps richer conditional variance dynamics are needed.

5.5 The Implied Volatility Term Structure

The J-GARCH(3) performance could also be driven by its ability to better capture the implied volatility term structure. Thus, we now look at the models' ability to match average at-the-money implied volatility across maturity, for three different volatility periods. Figure 7 presents data and model implied volatility across maturities. In order to investigate the importance of different volatility regimes, we chose three periods each spanning four months. We chose four-month windows because they are small enough to capture different volatility regimes, while still containing sufficient data for robust analysis. The high volatility period (top panel) is from July 1st to October 31st of 1998. The average VIX level in this period is 31.50. The medium volatility period (middle panel) is from January 1st to April 30th of 2002. The average VIX level here is 21.01. Finally, the low volatility period (bottom panel) is from January 1st to April 30th of 2002, with an average VIX level of 13.22. To prevent clutter and due to its subpar performance we do not include the J-GARCH(2) model in this figure.

Note first that all models underestimate the level of volatility in the high volatility period. The GARCH is most biased followed by the J-GARCH(1) and J-GARCH(4) models. The J-GARCH(3) model performs relatively well in the high volatility period; its ability to price long-term options is presumably related to its higher persistence. For the medium volatility period, the three jump models outperform the GARCH model at medium to long maturities, but all are biased at short maturities. One possible explanation is that the 6% assumption on the equity premium is too low. This explanation is supported by the findings in Pan (2002), who reports an 18% jump risk premium when fitting jump models to near-the-money short-dated option prices. Another potential explanation is missing risk factors that are not attributable to volatility and jump risk premia, see for instance Jones (2006).

For the low volatility period, the GARCH model is clearly the best at matching the implied volatility term structure across maturities. Jump models produce volatility levels that are too high and the bias becomes increasingly negative as the maturity increases. The bias is especially severe for the J-GARCH(3) model at long maturities.

5.6 The Implied Volatility Smirks

Figure 8 presents the implied volatility smirks implied by the models. For each model, we compute the implied volatility smirk for the three different volatility periods identical to the ones used in Figure 7. In order to reduce the noise in the data, we pool the implied volatilities into moneyness and maturity bins. We plot moneyness smirks for four different maturity ranges (top to bottom): 15 to 30 days, 30 to 60 days, 90 to 120 days, and 250 to 300 days. The left column reports on the low volatility period, while the center and right columns report on the medium and high volatility periods. Again, to prevent clutter and due to its subpar performance we do not include the J-GARCH(2) model in this figure.

In the medium and high volatility periods, we notice that models cannot produce realistic volatility levels, especially for deep-in-the-money options. The slope of the smirk is fairly flat for all models, and they cannot capture the shape of the smirk implied by short maturity options. Nevertheless, J-GARCH(3) performs relatively well compared to GARCH, as it produces a steeper implied volatility slope across moneyness. As for the low volatility period, we observe the "hook" effect pointed out by Duffie, Pan, and Singleton (2000) and Pan (2002) in all models. Jump models once more do not perform well for long maturity options in the low volatility period, confirming our earlier findings.

5.7 Increasing the Level of the Equity Risk Premium

The option valuation results are based on parameter estimates obtained from physical returns, together with an economically plausible assumption on the equity premium. We now further explore the implications of different risk premia for option pricing performance. We first perform a robustness analysis by investigating alternatives to the assumption of the 6% equity premium. Subsequently we further investigate the different impact of the two risk premium components.

We start by relaxing the assumption of a 6% equity premium. Specifically, we use MLE estimates from Tables 2 and 3 and compute option RMSEs for other economically plausible values of the equity premium, while constraining the equity premium to be equal across models. For jump models, we allow both risk premia to jointly contribute to the total equity premium.

Table 7 reports option pricing performance based on the IVRMSE metric for different levels of the equity premium. Panel A shows the GARCH IVRMSE. Panels B, C, and D show the IVRMSE ratios for three J-GARCH models relative to the GARCH model. In each panel, the columns represents various levels of the equity premium ranging from 0% to 10%. The rows represents various levels of the risk premium associated with the normal component. Note that for the GARCH case, we only have entries on the diagonal since the only source of risk is associated with the normal innovation. For example, in Panel C, if the total equity premium is 6% and the risk premium for the normal innovation is 2%, which means that the jump risk premium is 4%, the IVRMSE ratio is 0.86. For this combination of risk premia, the J-GARCH(3) model improves over the GARCH model by 14%.

First, we investigate the case when the equity premium is entirely due to non-jump risk, which corresponds to the diagonal entries. Regardless of the equity premium, option pricing improvements are minimal in the absence of jump risk premia; the J-GARCH(3) leads to a 4% improvement, and the J-GARCH(1) actually underperforms the GARCH by 1%. On the other hand, when the equity premium is entirely due to jump risk, the J-GARCH models significantly improve on the GARCH model as the total equity premium increases. This corresponds to the bolded cells in the first rows of panels B-D. The strong dependence of option pricing performance on the presence of jump risk premia allows us to conclude with confidence that jump risk premia are a necessary element for option pricing. J-GARCH models improve option pricing performance through jump risk premia by reconciling the gap between the physical and the risk-neutral measures. To ensure that our finding is not due to a specific choice loss function, we repeat this analysis using \$RMSE instead of IVRMSE (not reported), and identical conclusions obtain. Table 7 also indicates that the superior performance of the J-GARCH(3) model extends to other equity premium levels.

Figure 9 presents volatility smirks implied by the J-GARCH(3) model at three different maturities, for different levels and sources of risk premia. The left column presents results for zero jump risk premia ($\lambda_y = 0$), with total equity premia of zero, five, ten, and fifteen percent. The right column presents results for $\lambda_z = 0$, and the middle column represents the mixed case where each component delivers half the equity risk premium. The conditional jump intensity and conditional variance of the normal component are set equal to their long run mean values.

The importance of the jump risk premia is clearly evident. The shape and particularly level of the implied volatility smirks are highly sensitive to the jump risk premium, but not to the risk premium associated with the normal innovation. We therefore conclude that models based on normal innovations alone cannot match the level of implied volatilities observed in the data when conventional estimates of the equity risk premium are imposed.

For options with 20 days to maturity, increasing jump risk premia result in steeper slopes. Interestingly, the location of the "hook" also changes as the jump risk premium increases. Our findings are consistent with Bakshi, Kapadia, and Madan (2003), who find that the more negative the risk-neutral skewness (and the higher the risk aversion), the steeper the implied volatility slopes. For longer maturities, we see that high implied volatilities can be generated with small jump risk premia. In the absence of jump risk premia this is not possible. Similar findings can be found in Bakshi, Cao, and Chen (1997) and Bates (2000), who show that the risk-neutral parameters required to fit stochastic volatility models to options prices are unrealistic. Our conclusions regarding the impact of jump risk premia are also in line with BCJ (2007).

6 Further Analysis of the Models

In this section we further investigate the GARCH jump models developed above.

First, recall that our models are designed so that the researcher does not need to separately identify the two unobserved shocks z_{t+1} and y_{t+1} neither to estimate the models nor to use them for option pricing. The likelihood is specified in terms of observed underlying returns and the variance and jump dynamics are updated using observed returns rather than using the two unobserved shocks individually. Nevertheless, in order to learn more about the models' performance we may want to separate the shocks and we do so below using the particle filter.

Second, our models are cast in discrete time. This is in line with a large body of work on equity return modeling using GARCH models. But the option valuation literature mainly proceeds using continuous time models. In order to anchor our models in the continuous time literature we therefore provide the continuous time limit of our models later on in this section.

6.1 Decomposition of Daily Returns by Particle Filtering

The two innovations z_{t+1} and y_{t+1} enter jointly into the dynamic (2.4), thus the GARCH updating procedure is straightforward. However, in order to appreciate the rich dynamics implied by the J-GARCH specification, we use particle filtering to separately identify both return components. The use of particle filtering was pioneered in finance by Pitt and Shephard (1999). Johannes, Polson, and Stroud (2007) discuss applications to jump-diffusion models. Using the MLE estimates in Table 2, we apply particle filtering to each J-GARCH model and back out the time series of three unobservables: the jump time n_{t+1}, the jump component of the return y_{t+1}, and the normal component of the return z_{t+1}.

The filtering density for the number of jumps at jump time $t+1$, n_{t+1}, is given by

$$\begin{aligned} \Pr_{t+1}(n_{t+1}=j) &= \frac{f_t(R_{t+1} \mid n_{t+1}=j)\Pr_t(n_{t+1}=j)}{f_t(R_{t+1})} \\ &\propto f_t(R_{t+1} \mid n_{t+1}=j)\Pr_t(n_{t+1}=j), \end{aligned} \tag{6.1}$$

where the expressions on the right hand side of (6.1) are given by (2.11), (2.12), and (2.13). See Maheu and McCurdy (2004) for a discussion. $\Pr_{t+1}(n_{t+1}=j)$ represents the ex-post inference on n_{t+1}, or the probability that j jumps have arrived between time t and $t+1$ conditional on the information available at time $t+1$.

The filter for y_{t+1} and z_{t+1} is given by

$$\Pr_{t+1}(z_{t+1},\ y_{t+1}) \propto \Pr_t(z_{t+1} \mid y_{t+1})\Pr_t(y_{t+1}) \tag{6.2}$$

This represents the ex-post joint inference on z_{t+1} and y_{t+1}, given time $t+1$ information. Note that the first term on the right hand side of (6.2) is conditionally normally distributed. It can also be written as $\Pr_t(R_{t+1} \mid y_{t+1})$. The second term on the right hand side of (6.2) is distributed as a Compound Poisson process.

Given the filtering densities in (6.1) and (6.2), we use the Sampling Importance Resampling (SIR) algorithm with 5,000 particles to integrate out the unobservables. We refer interested readers to the work of Pitt (2002) and Johannes, Polson, and Stroud (2007) for a more extensive discussion of the algorithm's implementation.

Figure 10 presents the results from applying particle filtering to the J-GARCH models. The expected ex-post number of jumps occurring in any given day is shown in the top panels. The filtered jump and normal components are presented in the middle and bottom panels respectively. We first look at the J-GARCH(1) model, which corresponds to the restricted case of the SVCJ in the continuous-time literature. The jump components for the J-GARCH(1) model are overall negative, thus the jumps induce negative skewness in the return distribution. At most one jump is observed on any given day, with the exception of the October 19th, 1987 crash, when the model indicates two jumps on the same day. This finding has important implications for many existing implementations of jump models. Several papers in the continuous-time literature approximate the jump component by $y_{t+1}=\varepsilon_{t+1}q_{t+1}$, with jump sizes retaining their distributional structure according to $\varepsilon_{t+1} \sim N(\theta, \delta^2)$, and jump times $q_{t+1} \in \{0,1\}$ are Bernoulli random variables with $\Pr(q_{t+1}=1)=h_{y,t+1}$ and $\Pr(q_{t+1}=0)=1-h_{y,t+1}$. Our findings indicate that this approximation may be inadequate for the 1987 crash.

When jump intensities are time-varying (in the J-GARCH(2), J-GARCH(3) and J-GARCH(4) models), jump times are more volatile with smaller jumps arriving at higher frequencies. The filtered state variables for the J-GARCH(3) and J-GARCH(4) models indicate the importance of time-varying jump intensities with clustering effects. The highest jump arrival frequencies are observed in 1987 and in the early period of the dot-com collapse. Interestingly, the crash of October 1987 is captured by the arrival of four or five jumps in one day for the J-GARCH(3) and J-GARCH(4) models respectively. Because more jumps can arrive when the level of risk rises, these models do not require a large negative jump means in order to produce the 1987 crash and the volatility of the late 1990s. While θ is slightly negative, we also observe jumps in the positive direction which represent the arrival of good news. When allowing only the jump intensity to be time-varying as in the J-GARCH(2) case, the jump component is very volatile, because jumps are the main shocks that determine changes in daily returns. It is also interesting to note that the expected number of jumps arriving in each day for the J-GARCH(2) model is always greater than zero and seems to be bounded below.

When jumps occur, they are usually the dominating shock to returns. Surprisingly, the crash of October 19th 1987 is an exception to this. On average (across models), the jump component only explains only 5% of the total 22.9% loss in the S&P500 index. The dominating shock for this crash is the normal component part of the return. This finding is somewhat different from EJP (2003), where the jump component accounts for at least half of the loss in the index on this date. At first, this may seem puzzling because a crash is usually associated with a jump in price. However, J-GARCH models contain jumps in volatility which are perfectly correlated with jumps in returns. Jumps in volatility therefore drive large shocks in the normally distributed return component, which then dominates in a period with a very large negative return, such as the 1987 crash.

6.2 Linking Results to the Continuous-Time Literature

Our discrete-time J-GARCH setup has considerable computational advantages. However, most related empirical results are obtained using a continuous-time setup. Thus, it is interesting to investigate the continuous-time limits of the J-GARCH model. Recall that when there is no jump in (2.1) and (2.4), hence $h_{y,t+1} = 0$, the model reduces to a simple Heston-Nandi (2000) GARCH(1,1). Heston and Nandi (2000, Appendix B) show that their model weakly converges to a diffusion limit which is the Heston (1993) square-root model. We now demonstrate that the continuous-time limit of our J-GARCH model falls into the category of non-affine quadratic jump-diffusion models. An alternative limit is part of the class of time-changed Lévy processes of Carr and Wu (2004).

First re-write the return dynamic in (2.1) using a new time-dependent parameterization, as follows

$$\log S_{t+\Delta} - \log S_t = r\Delta + \left(\lambda_z - \tfrac{1}{2}\right) h_z\left(t+\Delta\right) + \left(\lambda_y - \xi\right) h_y\left(t+\Delta\right) \quad + \mathbf{1}' X_{t+\Delta}$$

where the shocks to the return process are part of a two-dimensional vector $X_{t+\Delta}$

$$X_{t+\Delta} = \begin{pmatrix} z\left(t+\Delta\right) & y\left(t+\Delta\right) \end{pmatrix}'.$$

Using vector notation, we can also write (2.4)-(2.5) using time-dependent parameterization. This gives

$$\begin{aligned} \begin{pmatrix} h_z\left(t+\Delta\right) \\ h_y\left(t+\Delta\right) \end{pmatrix} \quad &= \quad \begin{pmatrix} w_z\left(\Delta\right) \\ w_y\left(\Delta\right) \end{pmatrix} + diag\begin{pmatrix} b_z\left(\Delta\right) + a_z\left(\Delta\right) c_z^2\left(\Delta\right) \\ b_y\left(\Delta\right) + a_y\left(\Delta\right) c_y^2\left(\Delta\right) \end{pmatrix} \cdot \begin{pmatrix} h_z\left(t\right) \\ h_y\left(t\right) \end{pmatrix} \\ &\quad + diag\begin{pmatrix} -2a_z\left(\Delta\right) c_z\left(\Delta\right) \\ -2a_y\left(\Delta\right) c_y\left(\Delta\right) \end{pmatrix} \cdot X_t + diag\begin{pmatrix} a_z\left(\Delta\right) h_z^{-1}\left(t\right) \\ a_y\left(\Delta\right) h_y^{-1}\left(t\right) \end{pmatrix} \cdot X_t' \boldsymbol{\Sigma} X_t \end{aligned}$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and $diag\binom{\xi_1}{\xi_2}$ represents a 2×2 diagonal matrix with first and second element given by ξ_1 and ξ_2 respectively. Define $\upsilon_z(t) = h_z(t)/\Delta$ and $\upsilon_y(t) = h_y(t)/\Delta$ as the diffusive variance and jump intensity per unit time. Now, consider letting the time interval shrink with parameter specifications

$$\begin{aligned} w_j(\Delta) &= \omega_j \Delta^2 & a_j(\Delta) = \alpha_j \Delta^2 \\ c_j(\Delta) &= \gamma_j \Delta^{-1} & b_j(\Delta) = 1 - \left(\alpha_j \gamma_j^2 - \beta_j\right) \Delta \end{aligned}$$

for $j = y,\ z$. In the limit as $\Delta t \to dt$, we get the process

$$d \log S_t = (r + \lambda_z \upsilon_z(t) + \lambda_y \upsilon_y(t))\, dt + \mathbf{1}' dX_t - \frac{\upsilon_z(t)}{2} dt - \xi \upsilon_y(t)\, dt \tag{6.3}$$

$$\begin{pmatrix} d\upsilon_z(t) \\ d\upsilon_y(t) \end{pmatrix} = \left[\mathbf{W} + \mathbf{B} \begin{pmatrix} \upsilon_z(t) \\ \upsilon_y(t) \end{pmatrix}\right] dt + \mathbf{C} dX_t + \mathbf{A} \begin{pmatrix} \upsilon_z^{-1}(t) \\ \upsilon_y^{-1}(t) \end{pmatrix} dX_t' \mathbf{\Sigma} dX_t \tag{6.4}$$

where

$$\begin{aligned} \mathbf{W} &= \begin{pmatrix} \omega_z \\ \omega_y \end{pmatrix} & \mathbf{C} = \begin{pmatrix} -2\alpha_z \gamma_z & 0 \\ 0 & -2\alpha_y \gamma_y \end{pmatrix} \\ \mathbf{B} &= \begin{pmatrix} \beta_z & 0 \\ 0 & \beta_y \end{pmatrix} & \mathbf{A} = \begin{pmatrix} \alpha_z & 0 \\ 0 & \alpha_y \end{pmatrix}. \end{aligned}$$

In the limit $\Delta t \to dt$, the normal and jump shocks to the return converge to

$$dX(t) = \begin{pmatrix} dZ(\upsilon_z(t)) \\ dY(\upsilon_y(t), k, \theta) \end{pmatrix} = \begin{pmatrix} \sqrt{\upsilon_z(t)}\, dW(t) \\ Q(t)\, dN(t) \end{pmatrix}.$$

The diffusion term can be written as the product of the Brownian motion $W(t)$ and the square root of the stochastic variance $\upsilon_z(t)$. The jump term has a jump-size component $Q(t)$ and a component given by a Poisson counting process $N(t)$ with instantaneous arrival rate of $\upsilon_y(t)$. The continuous-time limit of the return in (6.3) is nested within the jump diffusive specification of Duffie, Pan, and Singleton (2000). However, the stochastic variance and jump intensity in (6.4) are clearly non-affine, with quadratic dependence on dX_t.

Next we show how the above continuous time limit can be interpreted as a time-changed Lévy process. Consider the vector of potential stochastic time changes $T_t = \left[T_t^d, T_t^j\right]$ applied to two Lévy components Z_t and Y_t. By definition, the time change T_t is an increasing, right continuous process with left limits satisfying the usual conditions, for all $t \geq 0$ and $T_t \to \infty$ as $t \to \infty$. At any time t, the dynamic of the random time is characterized by

$$T_t^d = \int_0^t \upsilon_z(s)\, ds \qquad \text{and} \qquad T_t^j = \int_0^t \upsilon_y(s)\, ds,$$

where $\upsilon_z(s)$ and $\upsilon_y(s)$ are usually referred to as the instantaneous activity rates, with $\upsilon_z(0) = \upsilon_y(0) = 0$. Following Carr and Wu (2004), we can intuitively think of t as the calender time and of T_t as business activity on calender day t. Days with higher volatility represent active business days with higher instantaneous activity rates. Using (6.3), we can integrate and write the price process at any given calender time t as

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t d\left(\log S_s\right) ds\right) \\ &= S_0 \exp\left(rt + \left(\lambda_z - \tfrac{1}{2}\right) T_t^d + \left(\lambda_y - \xi\right) T_t^j + Z_t + Y_t\right). \end{aligned} \tag{6.5}$$

The first of the two Lévy shocks to the stock price is a Brownian process Z_t evaluated at a stochastic time change T_t^d. Thus, it is normally distributed with with mean zero and variance T_t^d. The second Lévy shock is a pure jump process, which we model as compound Poisson. Jumps in Y_t arrive over the interval $[t, t+dt]$ at a stochastic instantaneous rate of $\upsilon_y(t)$. Therefore, the expected number of jumps occurring between times 0 and t is T_t^j, with each jump normally distributed with mean θ and variance δ^2. The stochastic time change T_t that determines the distribution of Z_t and Y_t is governed by the dynamics in (6.4). Note that the stock price process in (6.5) is very similar to the MJDSV4 process in Huang and Wu (2004), with a somewhat different specification of the stochastic time change process.

The special case of the J-GARCH(3) model which performed so well empirically has the continuous-time limit

$$\begin{aligned} dS_t &= \left(r + \gamma_t\right) S_t dt + \sqrt{\upsilon_z(t)} S_t dW_t + dY_t - \xi S_t k \upsilon_z(t)\, dt \\ d\upsilon_z(t) &= \left(\omega_z + \beta_z \upsilon_z(t)\right) dt - 2\alpha_z \gamma_z \sqrt{\upsilon_z(t)} dW_t + \frac{\alpha_z}{\upsilon_z(t)}\left(\sqrt{\upsilon_z(t)} dW_t + dY_t\right)^2 \end{aligned}$$

where $\upsilon_z(t) = h_z(t)/\Delta$ can be thought as the stochastic variance process in the conventional continuous-time models. The pure jump component in the model has the limit of $dY_t = Q(t)\, dN(t)$ where jumps arrive according to a Poisson counting process $dN(t)$ at an instantaneous rate of $\upsilon_y(t)$ and the size of each jump $Q(t)$ is log-normally distributed with mean θ and variance δ^2. A quick look at the continuous-time limits of J-GARCH(3) show that it has great resemblance with the SVSCJ model in the continuous-time jump-diffusion literature but with a non-affine stochastic variance process.

6.3 Risk-Neutralization in Continuous-Time Jump Models

Bates (1988, 2000) and Naik and Lee (1990) provide a general equilibrium treatment of jumps in a continuous-time setup. They assume that the jump in the Compound Poisson process is normally distributed. Under the risk-neutral measure, the Compound Poisson process undergoes changes in the jump intensity $h_{y,t+1}$ and mean jump size θ. That is, the change of measure will result in

$$\left\{h_{y,t+1},\ \theta\right\} \implies \left\{h_{y,t+1}^*,\ \theta^*\right\}.$$

Most empirical implementations of jump models are inspired by this general equilibrium setup, in the sense that they adopt a stochastic discount factor (SDF) which allows for jump intensity and jump mean size to differ across the two measures. However, these implementations do not impose the full structure in Bates (1988, 2000) and Naik and Lee (1990). The most popular choice of SDF in the continuous-time jump-diffusion literature is $L_t = L_t^D L_t^J$, where the SDF for the diffusive part L_t^D is specified as

$$L_t^D = \exp\left(\int_0^t \Theta(s)\, dW(s) - \frac{1}{2}\int_0^t \|\Theta(s)\|^2\, ds\right).$$

and the SDF component associated with the jump is given by

$$L_t^J = \prod_{n=1}^{N_t} \left(\frac{\lambda_{\tau_n}^Q\ f^Q(\tau_n, X_n)}{\lambda_{\tau_n}\ f(\tau_n, X_n)}\right) \exp\left(\int_0^t \left\{\int_Z \left[\lambda_s f(s,X) - \lambda_s^Q f^Q(s,X)\right] dX\right\} ds\right) \tag{6.6}$$

where λ_s is the jump intensity and $f(s,X)$ is the distribution of each jump X. This choice of SDF has both advantages and disadvantages. The main advantage is that it is very flexible. The risk neutral process is also Compound Poisson with jump intensity $h_{y,t+1}^*$ and each jump size distributed according to $f^Q(s,X)$, but the risk-neutral jump size distribution can be different from the physical one. The same flexibility applies to the jump intensity.

This choice of SDF therefore allows for a highly flexible Compound Poisson jump structure in the risk-neutral measure. The disadvantage of this SDF specification is that it is not possible to identify the risk neutral measure $h_{y,t+1}^*$ and $f^Q(s,X)$ from a given level of jump risk premium. To see this, we will set the diffusion component equal to zero, so that only jump risk premium matters, and focus on a simple geometric compound Poisson process.

$$dS(t) = S(t)\left(r_t + \gamma_t - \overline{\mu}\lambda\right) dt + S(t-)\, d\left(\sum_{n=1}^{N_t}\left[e^{X_n} - 1\right]\right) \tag{6.7}$$

where $X_n \sim N\left(\theta, \delta^2\right)$ is the distribution of each jump and $\overline{\mu} = \exp\left(\theta + \frac{\delta^2}{2}\right) - 1$ is the jump compensator. Using the change of measure according to L_t^J above, the stock price under the Q measure will be

$$dS^Q(t) = S(t)\left(r_t - \overline{\mu}^Q\lambda^Q\right) dt + S(t-)\, d\left(\sum_{n=1}^{N_t(Q)}\left[e^{X_n(Q)} - 1\right]\right). \tag{6.8}$$

The above process is a martingale under the Q measure. Comparing (6.7) and (6.8), we see that the instantaneous total equity premium γ_t is given by

$$\gamma_t = \overline{\mu}\lambda - \overline{\mu}^Q\lambda^Q,$$

This illustrates the weakness of this SDF specification. Given an equity premium γ_t and known P measure jump parameters $\overline{\mu}\lambda$, one can only identify the Q measure compensator $\overline{\mu}^Q\lambda^Q$. Therefore, it is not clear how a given level of equity premium is split up between $\overline{\mu} - \overline{\mu}^Q$ and $\lambda - \lambda^Q$. The solution in many empirical implementations is to assume $\lambda^Q = \lambda$, which means that all of the jump risk premium is absorbed through the $\overline{\mu} - \overline{\mu}^Q$ factor. See Pan (2002), EJP (2003), and Eraker (2004) for examples of this approach. Broadie, Chernov and Johannes (2007) use a different approach, and allow for additional flexibility by assuming $\delta \neq \delta^Q$. They note that prior studies constrain $\delta = \delta^Q$ because of an underlying equilibrium model that assumes power utility over consumption or wealth, as in Bates (1988) and Naik and Lee (1990), and argue that when valuing options based on the absence of arbitrage, there is therefore no need to restrict $\delta = \delta^Q$. However, it is worth noting that the pricing kernel used to risk-neutralize the diffusion component in most jump-diffusion models, including in Broadie, Chernov and Johannes (2007), is based on power utility. Therefore, there seems to be an internal inconsistency in this approach, in the sense that the pricing kernels for the diffusion and jump components are possibly supported by two different utility functions. In the approach we use, this is not the case. Moreover, this is not due to the use of a discrete- versus a continuous-time framework. Our risk-neutralization can be implemented in continuous-time, and the conventional continuous-time approach described above can be implemented in discrete time.

7 Conclusion and Directions for Future Work

This paper presents a new framework for modeling and estimating jumps in returns and volatility. The specification of the jump models is inspired by a popular class of jump-diffusion models in the continuous-time literature. However, we specify the models using a discrete-time GARCH setting, and as a result models with time-varying jump intensities and jumps in volatility can be easily estimated from long time series of return data using a standard MLE procedure. This enables us to analyze complex jump models that in a continuous-time setup are difficult to study because of their computational complexity. Our general model, which we refer to as J-GARCH, builds on the framework of Huang and Wu (2004) and studies four nested specifications which exhaust all possible sources of heteroskedasticity in the general model. The time-varying properties of the J-GARCH model are driven by the dynamics of the variance of the normal shock and the dynamics of the jump intensity. Our model shares some similarities with the models of Maheu and McCurdy (2004) who study individual equity returns but not options, and Duan, Ritchken, and Sun (2006) who study option pricing. However, our model has different implications for option valuation, because our assumptions enable us to characterize the risk-neutral dynamic systematically with separate identification of the jump and diffusion (normal) risk premia. We also provide continuous time limits of our models which allow us the anchor our framework in the continuous time literature.

Our empirical analysis on S&P 500 index return data and option prices leads to important conclusions regarding the implication of jumps for asset pricing and option valuation. We conclude that jump models should allow for heteroskedasticity both in the conditional variance of the normal innovation and the jump intensity. Although jumps can complement a heteroskedastic normal innovation by improving the modeling of the tails of the distribution, they cannot replace the normal innovations. In fact, models without time-varying conditional variances for the conditional normal innovation are severely misspecified, as evident from the evidence on the J-GARCH(2) model.

Our option valuation results also demonstrate that without jump risk premia, jump models cannot improve model fit. For a reasonable range of equity premia, we find that jump models lead to smaller errors in fitting options when the jump risk premium is the dominant factor in the equity premium. We also find that the risk premium associated with the conditional normal innovation has little impact on the implied volatility term structure, and that realistic shapes of the implied volatility term structure can only be generated with sizeable jump risk premia. This finding is in line with Pan (2002) and Broadie, Chernov, and Johannes (2007), but contradicts the results in Eraker (2004).

We also include that the frequency of jump arrivals should be time-varying and dependent on the level of risk in the market. This finding contributes to the current debate on the specification of jump intensities. Contrary to the findings of ABL (2002), we find evidence for time-varying jump intensities when estimating the model using a long time series of returns. Our option pricing results favor a model with jump intensities that are affine in the variance of the normal component. We also find that jumps are not useful in low volatility regimes. Therefore, future specifications should allow for frequent jump arrivals in high volatility regimes, and little or no possibility of jump arrivals in the low volatility regimes.

Our results can be extended in a number of ways. First, the models in this paper can be estimated using a long time series of cross-sections of options data, and the results can be compared with the parameter estimates and the option valuation results in this paper. Joint estimation on returns and options as in Chernov and Ghysels (2000) would be interesting as well. Second, we analyze jumps of finite activity and finite variation, but the framework can be extended to incorporate the infinite activity Lévy processes of Huang and Wu (2004). Third, it would be interesting to investigate the marginal contribution of jumps to the return process when the variance of the normal innovation follows a non-affine dynamic as in Christoffersen and Jacobs (2004), or a richer GARCH dynamic such as the component GARCH model of Christoffersen, Jacobs, Ornthanalai and Wang (2008).

8 Appendix A: Long run properties and persistence

J-GARCH(1)

Without time-varying jump intensity, the long run variance of the normal innovation is

$$\sigma_z^2 = \frac{a_z + w_z - 2a_z c_z w_y \theta + \sqrt{C}}{2\left(1 - b_z - a_z c_z^2\right)}$$

where

$$C = \left(a_z + w_z - 2a_z c_z w_y \theta\right)^2 - 4a_z\left(-1 + b_z + a_z c_z^2\right) w_y \left(\delta^2 + \left(1 + w_y\right)\theta^2\right)$$

and the persistence is $\rho_z = b_z + a_z c_z^2$.

J-GARCH(2)

Without time-varying variance in the normal component, we only need to solve for the expression for the long run jump intensity. It is given by

$$\sigma_y^2 = \frac{1}{2}\frac{w_y + a_y\left(\theta^2 + \delta^2\right) + \sqrt{C}}{\left(1 - b_y - a_y\left(c_y - \theta\right)^2\right)}$$

where C is given by

$$C = \left(w_y + a_y\left(\theta^2 + \delta^2\right)\right)^2 + 4a_y w_z\left(1 - b_y - a_y\left(c_y - \theta\right)^2\right)$$

We can then easily see that the persistence is given by $\rho_y = b_y + a_y\left(c_y - \theta\right)^2$.

J-GARCH(3)

The long run variance of the normal innovation is

$$\sigma_z^2 = \frac{1}{2} \frac{w_z - a_z \left(-1 - \left(\delta^2 + \theta^2\right) k\right) + \sqrt{C}}{\left(1 - b_z - a_z c_z^2 - a_z \theta k \left(-2c_z + \theta k\right)\right)}$$

where C is given by

$$C = \left(a_z + w_z + a_z \left(\delta^2 + \theta^2\right) k\right)^2.$$

It can then be seen that the persistence $h_{z,t+1}$ is given by $\rho_z = b_z + a_z c_z^2 + a_z \theta k \left(-2c_z + \theta k\right)$. The long run jump intensity can be derived using the affine structure of the variance of the normal component as $\sigma_y^2 = k\sigma_z^2$.

J-GARCH(4)

The derivation of the unconditional variance and persistence implied by the J-GARCH(4) dynamic requires some elaboration. We derive the long run mean of $h_{z,t+1}$ and $h_{y,t+1}$ by taking the unconditional expectation of (2.5) and applying the law of iterated expectations,

$$\sigma_z^2 = E\left[\ E_t\left[h_{z,t+2}\right]\ \right] \qquad \text{and} \qquad \sigma_y^2 = E\left[\ E_t\left[h_{z,t+2}\right]\ \right]. \tag{8.1}$$

The expressions on the right hand side are nonlinear functions of σ_z^2 and σ_y^2. Solving the system of two equations, we get

$$\begin{aligned} 0 \;=\;& a_z + w_z - w_y + \left(1 - b_y + a_y c_y \left(2\theta - c_y\right) - 2\theta a_z c_z\right) \sigma_y^2 + \\ & \left(b_z + a_z c_z^2 - 1 - a_y c_y d_y - a_y d_y \left(c_y - 2\theta\right) - a_y \sigma_y^{-2}\right) \sigma_z^2. \end{aligned} \tag{8.2}$$

Notice that (8.2) does not imply a unique solution for σ_z^2 and σ_y^2. We therefore define

$$\sigma^2 \equiv Var\left(R_t\right) = \sigma_z^2 + \left(\delta^2 + \theta^2\right) \sigma_y^2 \tag{8.3}$$

where σ^2 is the unconditional variance of the returns which can be estimated outside the model by simply using the sample second moment. This relation amounts to variance targeting, where the model's implied second moment is matched to the variance of the estimation sample. Applying (8.3) to (8.2), we can solve for σ_z^2 and σ_y^2 analytically. The long run variance and persistence for the other three models are derived in a similar fashion.

For the long rung jump intensity, the solution is

$$E\left[h_{y,t+1}\right] \equiv \sigma_y^2 = \frac{2\sigma^2 a_y}{1 - \rho_y},$$

where ρ_y is the persistence of the jump intensity and is given by

$$\rho_y = 1 - \left(\delta^2 + \theta^2\right) a_y - a_z - \sigma^2 \left(-1 + b_z + a_z c_z^2\right) + w_y - w_z - \sqrt{C}$$

where C is given by

$$\begin{aligned} C \;=\;& 4\sigma^2 a_y \left(1 + \delta^2 + \theta^2 - b_y - \left(\delta^2 + \theta^2\right) b_z - a_y c_y \left(-2\theta + c_y\right) - a_z c_z \left(2\theta + \left(\delta^2 + \theta^2\right) c_z\right)\right) + \\ & \left(-\sigma^2 + \left(\delta^2 + \theta^2\right) a_y + \sigma^2 b_z + a_z \left(1 + \sigma^2 c_z^2\right) - w_y + w_z\right)^2. \end{aligned}$$

The long run variance of the normal innovation is

$$E\left[h_{z,t+1}\right] \equiv \sigma_z^2 = \frac{D_0 + \sqrt{-\left(\delta^2+\theta^2\right)^2 D_1}}{2\left(1-\rho_z\right)}$$

where

$$\begin{aligned}
D_0 &= -2\theta\sigma^2 a_z \left(\theta + \left(\delta^2+\theta^2\right)^2\right) + \left(\delta^2+\theta^2\right)^2 w_z \\
D_1 &= 4\sigma^2 a_z^2 \left(\theta + \left(\delta^2+\theta^2\right) c_z\right)^2 - \left(\delta^2+\theta^2\right)^2 w_z^2 + \\
&\quad 4\sigma^2 a_z \left(\left(\left(\delta^2+\theta^2\right)^2 + \theta^2\sigma^2\right)\left(b_z - 1\right) + \theta\left(\theta + \left(\delta^2+\theta^2\right) c_z\right) w_z\right)
\end{aligned}$$

The persistence of the variance of the normal component is given by

$$\rho_z = -\delta^2 - \theta^2 + b_y + \left(\delta^2+\theta^2\right) b_z + a_y c_y \left(c_y - 2\theta\right) + a_z c_z \left(2\theta + \left(\delta^2+\theta^2\right) c_z\right).$$

9 Appendix B: risk-neutralization

Proof of Lemma 1

First, we need to find the joint moment generating function (MGF) of the return innovation. Because the normal and jump components are contemporaneously independent, the conditional joint MGF of $z_{t+1} + y_{t+1}$ can be written as a simple product of their moment generating functions

$$\begin{aligned}
E_t\left[\exp\left(\phi_z z_{t+1} + \phi_y y_{t+1}\right)\right] &\equiv \exp\left(\ \Psi\left(\boldsymbol{\phi}; H_{t+1}\right)\ \right) \\
&= \exp\left(\Psi_z\left(\phi_z; h_{z,t+1}\right) + \Psi_y\left(\phi_y; h_{y,t+1}\right)\right)
\end{aligned} \tag{9.1}$$

The expression $\Psi_z\left(\phi_z; h_{z,t+1}\right) = \frac{1}{2}\phi_z^2 h_{z,t+1}$ is the exponent of the Normal MGF with mean zero and variance $h_{z,t+1}$. For the compound Poisson process with jump intensity $h_{y,t+1}$, jump mean size θ, and jump variance δ^2, the exponent of its MGF is given by

$$\Psi_y\left(\phi_y; h_{y,t+1}\right) = h_{y,t+1}\left(\exp\left(\phi_y\theta + \tfrac{1}{2}\phi_y^2\delta^2\right) - 1\right)$$

Substituting (9.1) for $\log M\left(\boldsymbol{\Lambda}; H_{t+1}\right) = \Psi\left(\boldsymbol{\Lambda}; H_{t+1}\right)$ in the EMM restriction (4.3) gives, after simplification and collecting terms

$$h_{y,t+1}\left(\lambda_y - \left(e^{\frac{\delta^2}{2}+\theta} - 1\right) - e^{\frac{\Lambda_y^2\delta^2}{2}+\Lambda_y\theta}\left(1 - e^{\left(\frac{1}{2}+\Lambda_y\right)\delta^2+\theta}\right)\right) + h_{z,t+1}\left(\lambda_z + \Lambda_z\right) = 0.$$

This is solved by equating the coefficients of $h_{y,t+1}$ and $h_{z,t+1}$ to zero, which gives (4.5) and (4.4).

Proof of Proposition 2

We will prove this through the moment generating function. Our procedure is as follows. First, we find the MGF of $z_t + y_t$ under the risk-neutral measure Q,

$$E_t^Q\left[\exp\left(\phi\left(z_{t+1} + y_{t+1}\right)\right)\right] = \exp\left(\ \Psi^Q\left(\phi; H_{t+1}\right)\ \right) \tag{9.2}$$

where we let $\Psi^{Q}(\phi; H_{t+1})$ be the exponent of the Q measure MGF . Next, we apply the change of measure through the Radon-Nikodym derivative in (4.2). Subsequently, we retrieve the stochastic process that will yield such form of MGF. Using the notation in (9.1), the change of measure to (9.2) gives

$$\begin{aligned} E_t^{Q}\left[\exp\left(\phi\left(z_{t+1}+y_{t+1}\right)\right)\right] &= E_t\left[\frac{\frac{dQ^{\Lambda}}{dP}\mid\mathcal{F}_{t+1}}{\frac{dQ^{\Lambda}}{dP}\mid\mathcal{F}_{t}}\exp\left(\phi\left(z_{t+1}+y_{t+1}\right)\right)\right] \\ &= E_t\left[\frac{\exp\left(\left(\phi+\Lambda_z\right)z_{t+1}+\left(\phi+\Lambda_y\right)y_{t+1}\right)}{M\left(\mathbf{\Lambda};\ H_{t+1}\right)}\right], \end{aligned}$$

where the conditional expectation on the right hand side is now under the objective measure. Noting that $M(\mathbf{\Lambda};\ H_{t+1}) = \exp\left(\Psi_z\left(\Lambda_z; h_{z,t+1}\right)+\Psi_y\left(\Lambda_y; h_{y,t+1}\right)\right)$ is predictable at time t, and that z_t and y_t are conditionally independent, this gives

$$E_t^{Q}\left[\exp\left(\phi\left(z_{t+1}+y_{t+1}\right)\right)\right] = \exp\left(A_z+A_y\right)$$

where

$$\begin{aligned} A_z &= \Psi_z\left(\phi+\Lambda_z; h_{z,t+1}\right)-\Psi_z\left(\Lambda_z; h_{z,t+1}\right) \\ A_y &= \Psi_y\left(\phi+\Lambda_y; h_{y,t+1}\right)-\Psi_y\left(\Lambda_y; h_{y,t+1}\right) \end{aligned}$$

It turns out that A_z is the exponent of the Normal MGF with mean $-\Lambda_z$ and variance $h_{z,t+1}$

$$A_z = -\Lambda_z\phi+\tfrac{1}{2}\phi^2 h_{z,t+1}.$$

We denote this risk-neutral measure normal component $z_{t+1}^{*} \sim N\left(-\Lambda_z, h_{z,t+1}\right)$. Similarly, rearranging the expression in A_y yields

$$A_y = h_{y,t+1}e^{\left(\frac{\Lambda_y^2\delta^2}{2}+\Lambda_y\theta\right)}\left(e^{\phi\left(\theta+\Lambda_y\delta^2\right)+\frac{\delta^2\phi^2}{2}}-1\right),$$

which is the exponent of the Compound Poisson MGF with jump mean size $\theta^{*}=\theta+\Lambda_y\delta^2$, and jump intensity $h_{y,t+1}^{*} = h_{y,t+1}\exp\left(\frac{\Lambda_y^2\delta^2}{2}+\Lambda_y\theta\right)$. Again, we denote this risk-neutral measure jump component as $y_t^{*} \sim J\left(h_{y,t+1}^{*},\ \theta^{*},\delta^2\right)$. The proof is complete as we have shown that $E_t^{Q}\left[\exp\left(\phi\left(z_{t+1}+y_{t+1}\right)\right)\right]$ is the moment generating function of the stochastic process $z_{t+1}^{*}+y_{t+1}^{*}$.

Proof of Proposition 3

Using the result from Proposition 2, we see that under the risk-neutral Q measure, the returns process in (2.1) can be written as

$$\log\left(\frac{S_{t+1}}{S_t}\right) = r+\left(\lambda_z-\frac{1}{2}\right)h_{z,t+1}+\left(\lambda_y-\xi\right)h_{y,t+1}-\theta h_{y,t+1}+z_{t+1}^{*}+y_{t+1}^{*}. \tag{9.3}$$

The change of measure affects only the return innovations. Therefore, other parameters will remain the same after the measure change. The risk-neutral GARCH dynamics in (2.4) can be written as

$$
\begin{aligned}
h_{z,t+1} &= w_z + b_z h_{z,t} + \frac{a_z}{h_{z,t}} \left(z_t^* + y_t^* - c_z h_{z,t}\right)^2 \\
h_{y,t+1} &= w_y + b_y h_{y,t} + \frac{a_y}{h_{y,t}} \left(z_t^* + y_t^* - c_y h_{y,t}\right)^2 .
\end{aligned}
\tag{9.4}
$$

Note that the risk-neutral distributions of the two shocks are $z_{t+1}^* \sim N\left(\Lambda_z, h_{z,t+1}\right)$ and $y_{t+1}^* \sim J\left(h_{y,t+1}^*, \theta^*, \delta^2\right)$. The convention in the GARCH literature is to express the normal shock as a mean zero innovation. We therefore use the simple transformation $z_{t+1}^* = z_t - \Lambda_z h_{z,t+1}$. Recalling that the analytical solution to Λ_z is $-\lambda_z$, we can write (9.4)) as

$$
\log\left(\frac{S_{t+1}}{S_t}\right) = r - \tfrac{1}{2} h_{z,t+1} + \left(\lambda_y - \xi\right) h_{y,t+1} + z_{t+1} + y_{t+1}^* .
$$

Because the Q measure is constructed such that discounted price process of S_t is martingale, and we already know that $E_t\left[\exp\left(z_{t+1} - \frac{1}{2} h_{z,t+1}\right)\right]$ is martingale, we must have

$$
\begin{aligned}
\exp\left(\left(\lambda_y - \xi\right) h_{y,t+1}\right) &= E_t\left[\exp\left(-y_{t+1}^*\right)\right] = \Psi_y^Q\left(\Lambda_y; h_{y,t+1}^*\right) \\
&= \exp\left(-\left(e^{\frac{\delta^2}{2}+\theta^*} - 1\right)\right) h_{y,t+1}^* = \exp\left(-\xi^* h_{y,t+1}^*\right)
\end{aligned}
$$

and thus $\xi^* = \left(e^{\frac{\delta^2}{2}+\theta^*} - 1\right)$. Additionally, we apply the following GARCH parametrization

$$
\begin{aligned}
w_y^* &= w_y \Pi, \qquad a_y^* = \Pi^2 a_y, \qquad c_z^* = \left(c_z - \Lambda_z\right), \\
c_y^* &= \frac{c_y}{\Pi}, \qquad h_{y,t}^* = h_{y,t} \Pi
\end{aligned}
$$

for $\Pi = \exp\left(\frac{\Lambda_y^2 \delta^2}{2} + \Lambda_y \theta\right)$. Substituting them into (4.7) and (4.6) will yield the expressions in (9.3) and (9.4).

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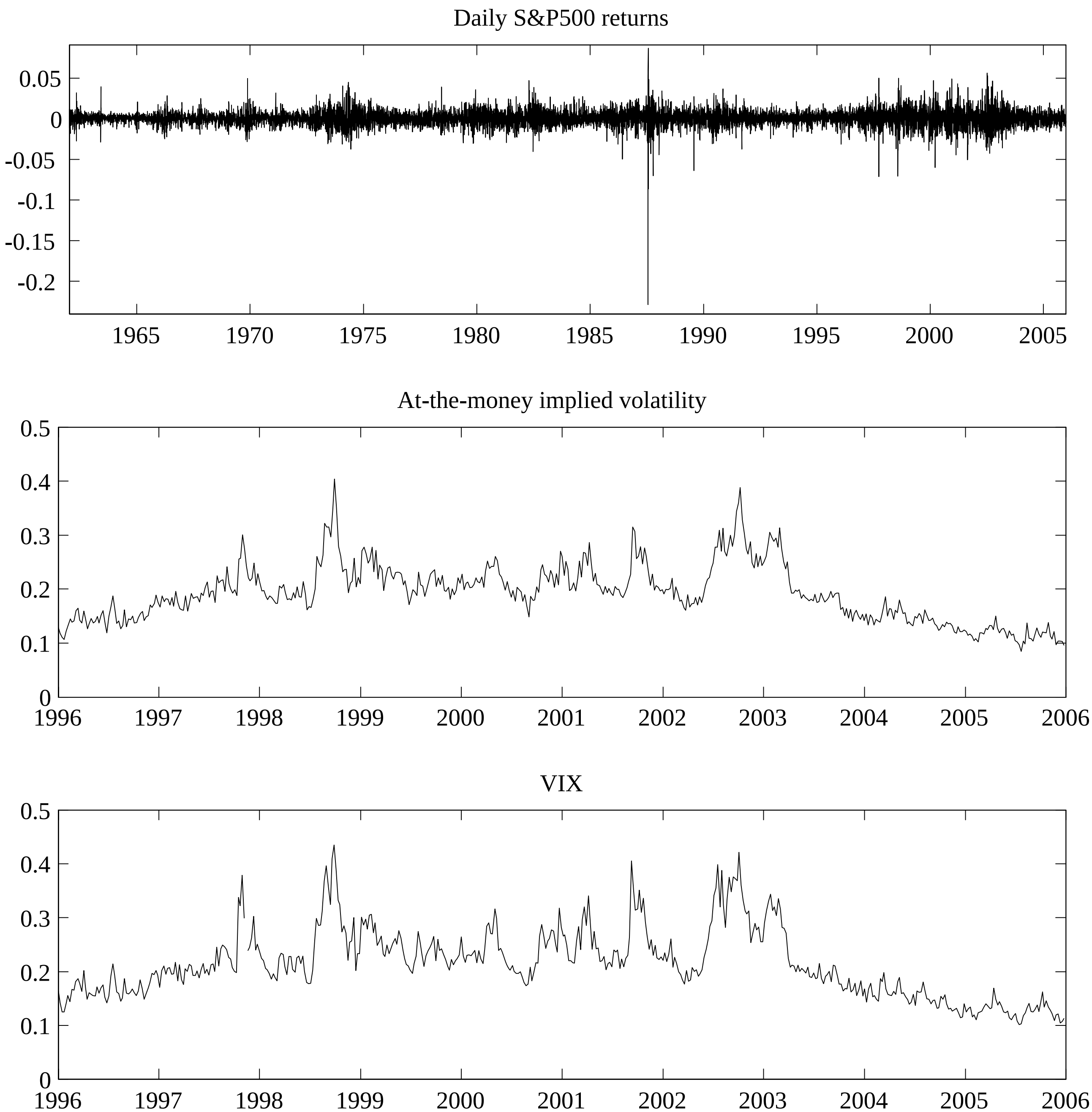
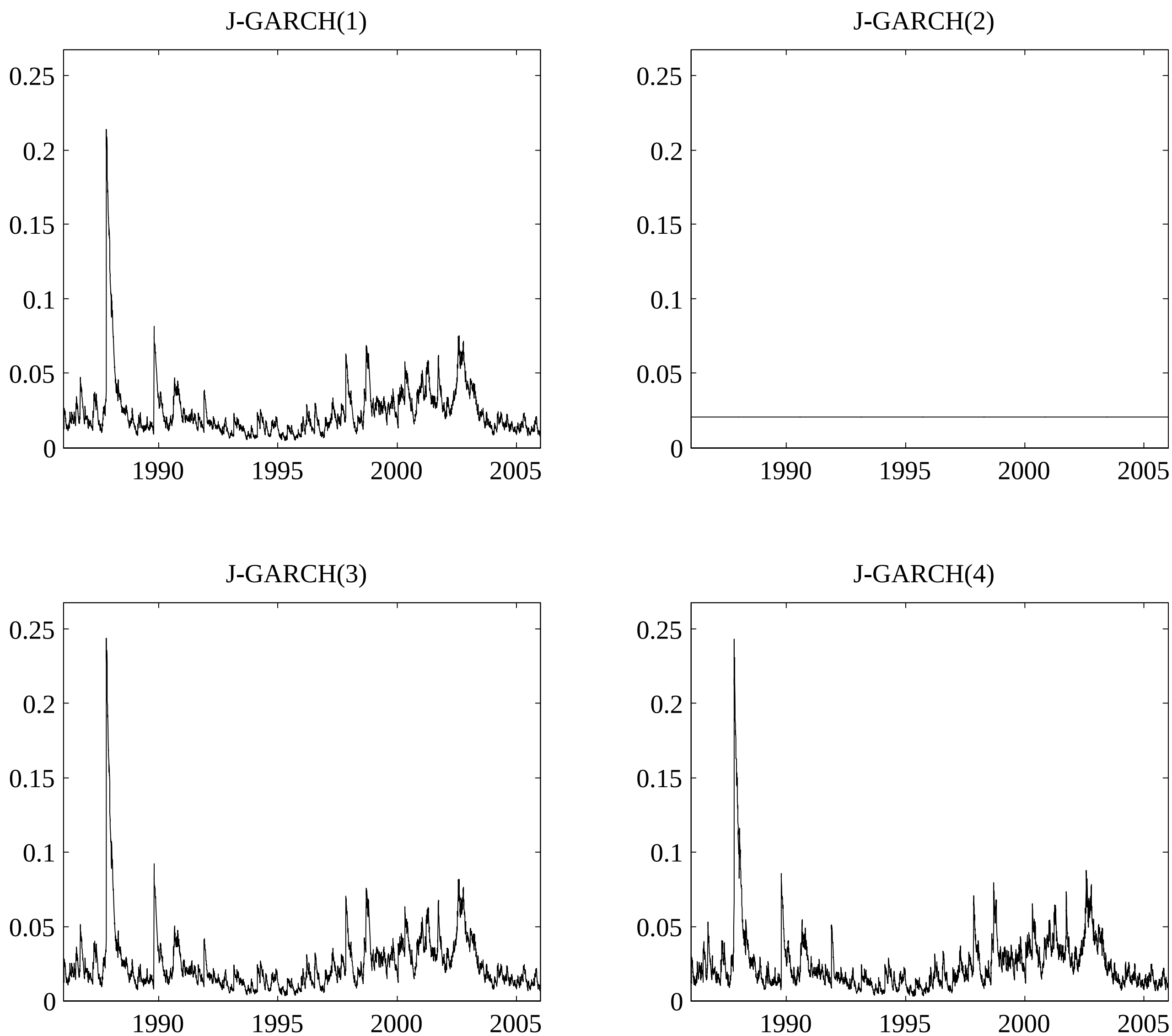
Figure 1: Daily Return and Implied Volatility on the S&P500.



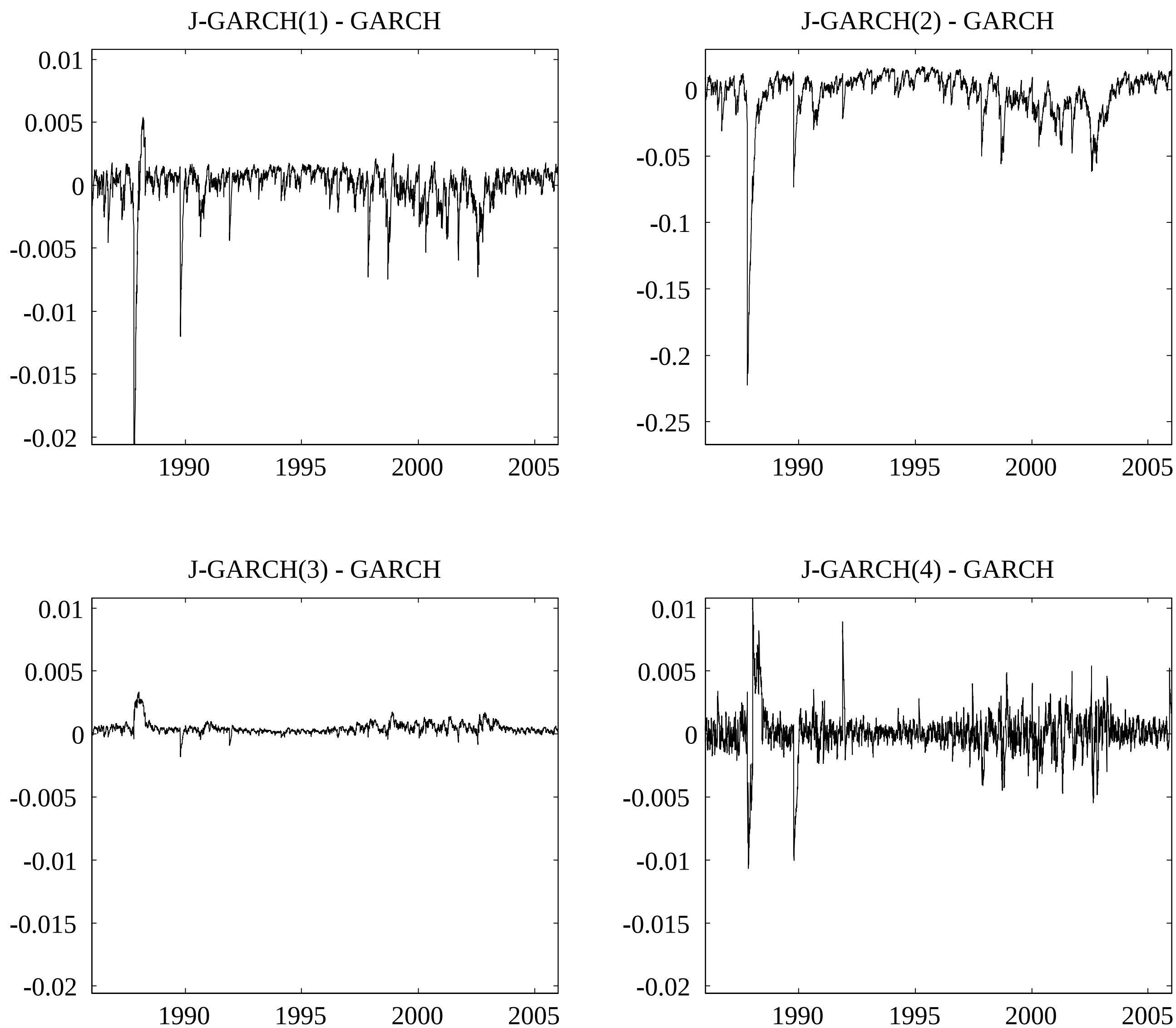
Notes to Figures: The top panel plots the daily S&P500 return from 1962 through 2005. The middle panel plots the average weekly implied Black-Scholes volatility for the at-the-money S&P500 call options in our sample which goes from 1996 through 2005. The bottom panel plots the VIX index from the CBOE for comparison.

Figure 2: Conditional Variance Paths (Annualized)



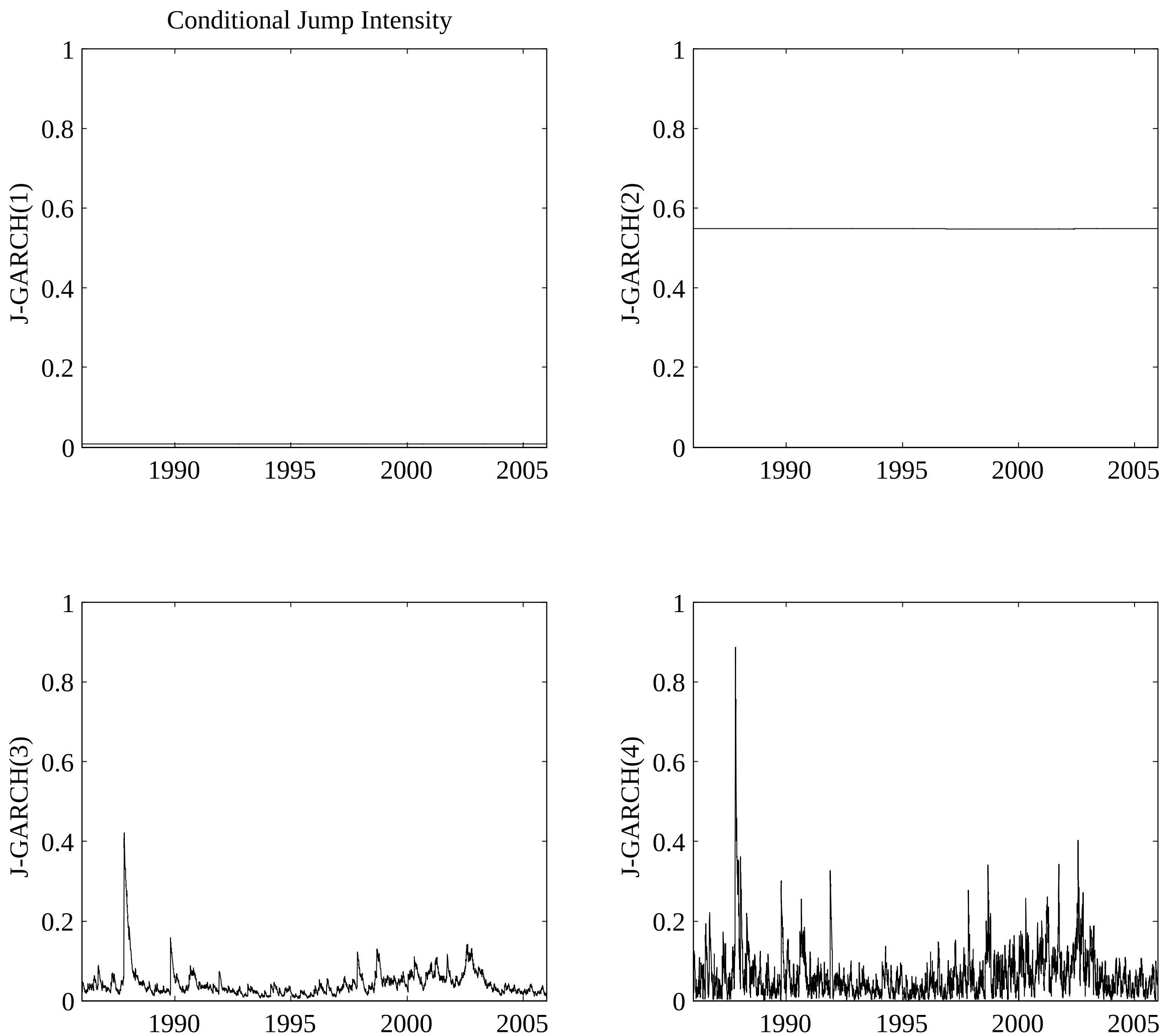
Notes to Figure: We plot the conditional variance from each of the four J-GARCH models. The values are expressed in annualized terms. The parameter values are obtained from the MLE estimates on returns in Table 2.

Figure 3: Conditional Variance: J-GARCH less Benchmark GARCH (Annualized)



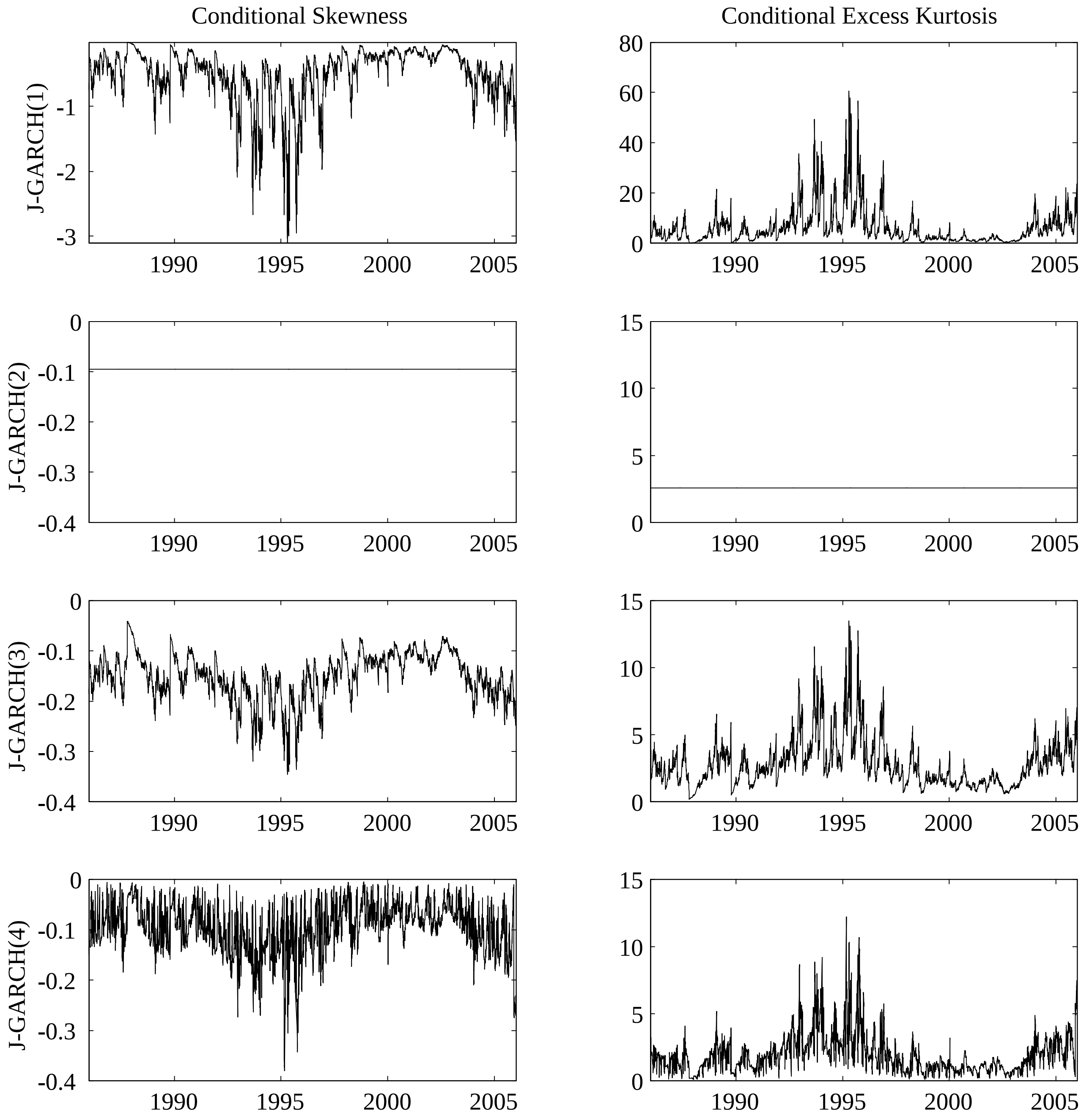
Notes to Figure: We plot the difference between the conditional variance of the J-GARCH models and that of the benchmark conditionally normal GARCH model from. All values are expressed in annualized term. The underlying parameter estimates are from Tables 2 and 3. Note that the scale is differerent for the J-GARCH(2) model in the top-right panel.

Figure 4: Conditional Jump Intensities



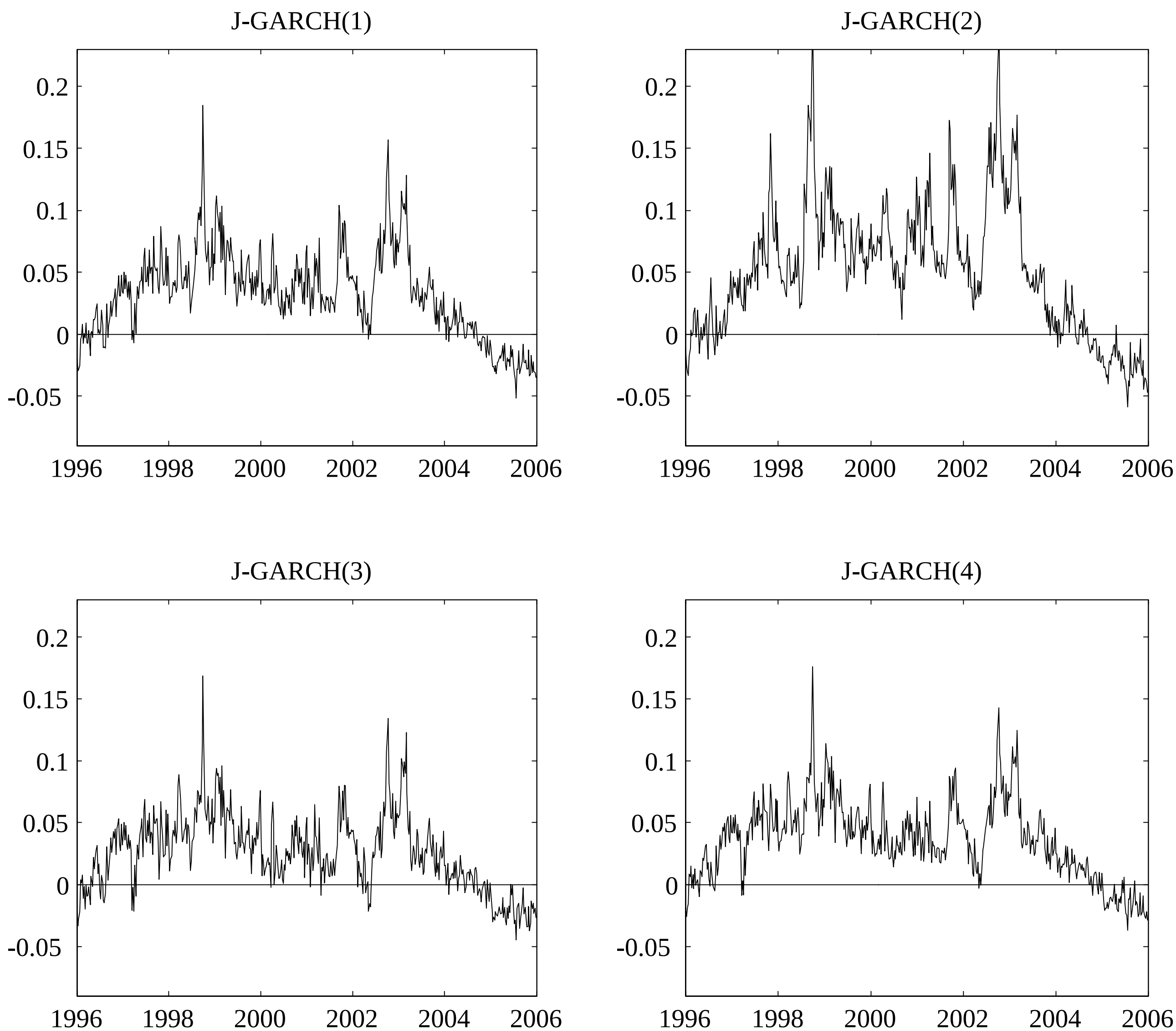
Notes to Figure: Using the parameter estimates in Table 2 we plot the daily conditional jump intensity, $h_{y,t+1}$ for each of the J-GARCH models. The plot covers the January 1, 1986 to December 31, 2005 period.

Figure 5: Conditional Skewness and Kurtosis



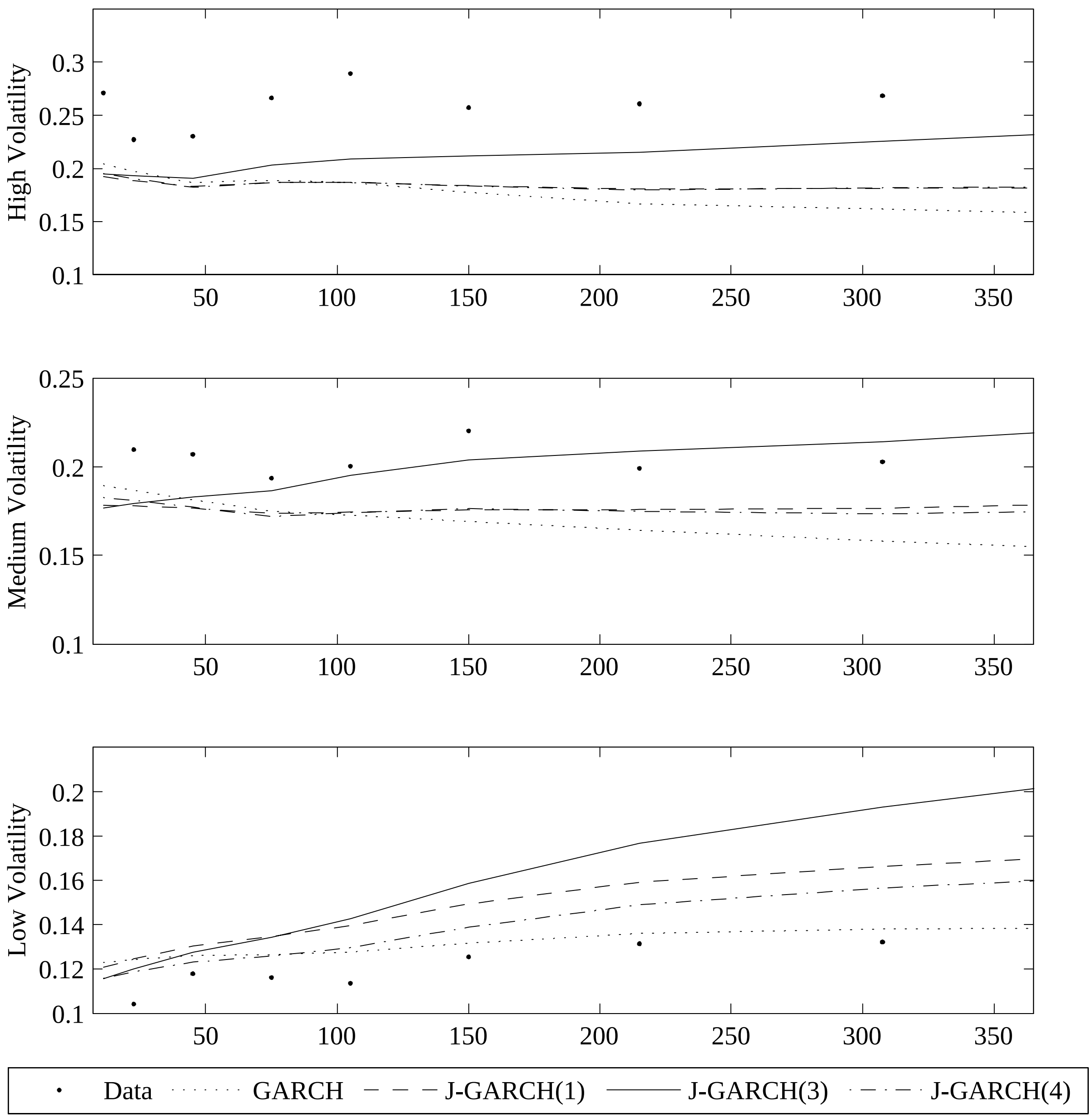
Notes to Figure: Using the parameter estimates in Table 2 we plot the daily conditional skewness and excess kurtosis from the four J-GARCH models. The plot starts on January 1, 1986 and ends on December 31, 2005. Note that the scale is different for the J-GARCH(1) model in the top row of panels.

Figure 6: Weekly Implied Volatility Bias for At-the-Money Options



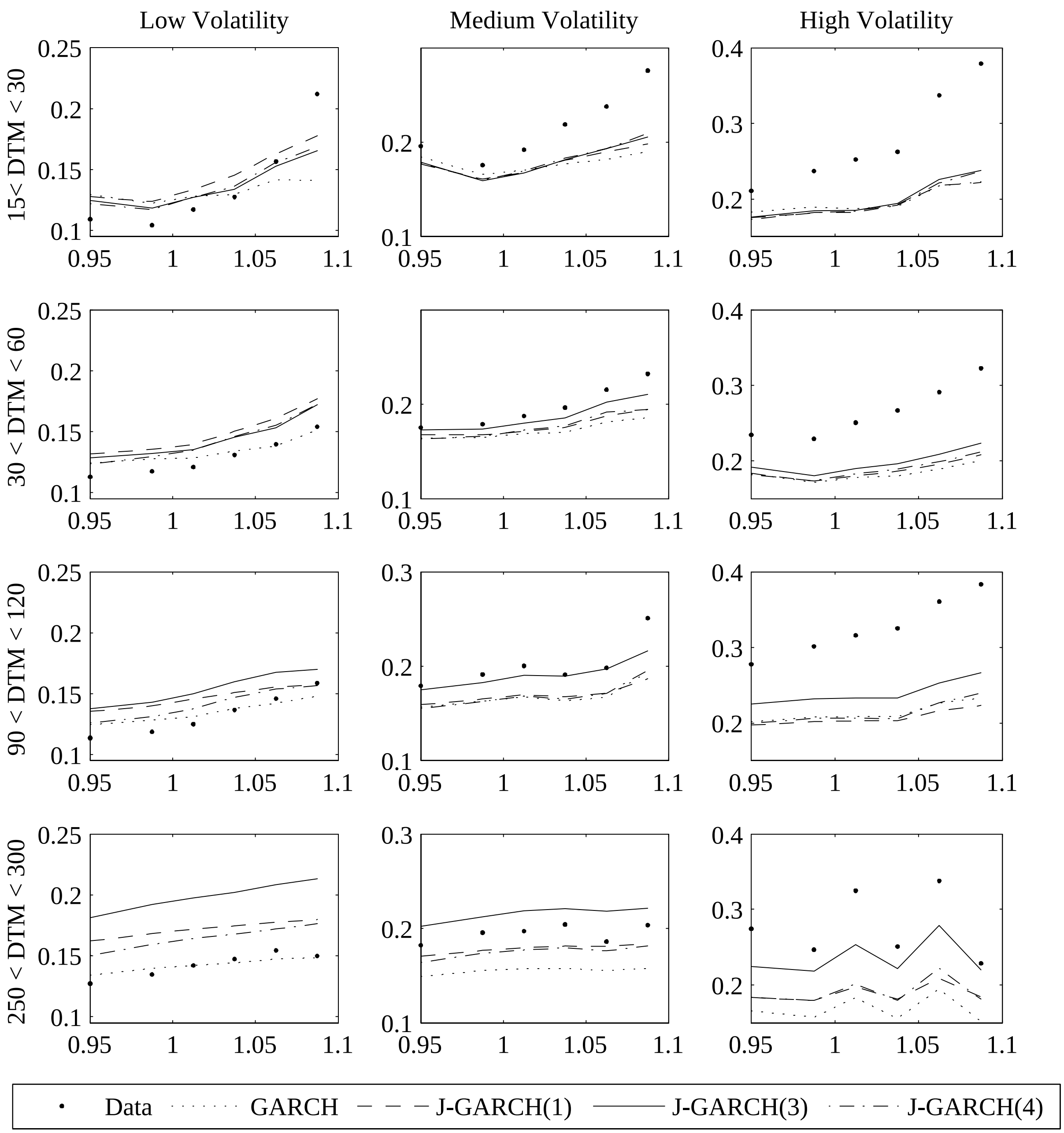
Notes to Figure: We plot the weekly average difference between the market and model implied volatility for options with moneyness (index value over strike price) between 0.975 and 1.025. We assume the long run total equity premium of 6% across all models. For models with jumps, we assume that the total equity premium comes from jump risk.

Figure 7: Implied Volatility Term Structures



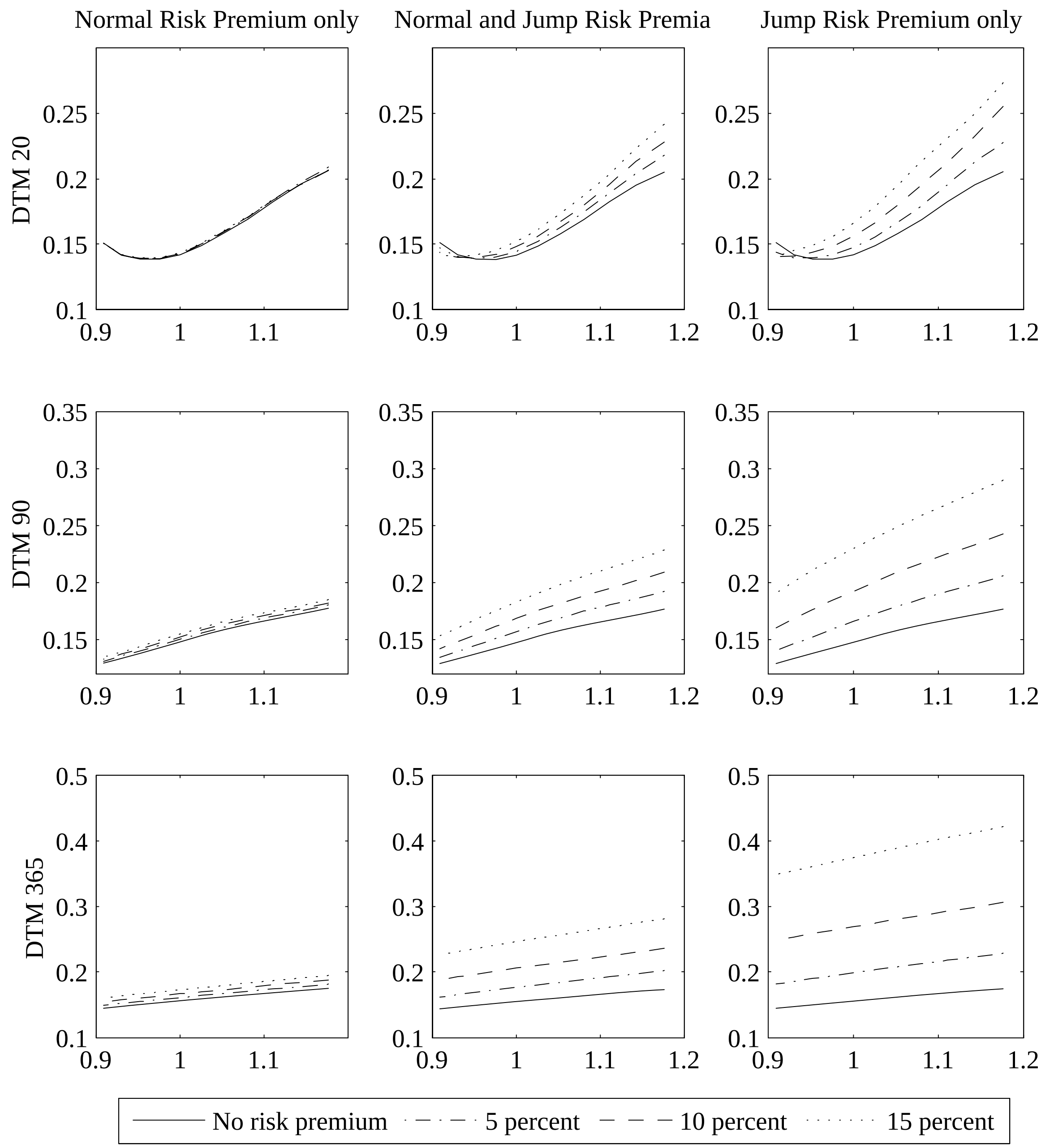
Notes to figure: We compare the average implied Black-Scholes volatility term structure from the GARCH and selected J-GARCH models in three volatility regimes. The dots mark the average implied volatility from the data. The high volatility period is 1998/07/01 - 1998/10/31, where the average VIX level is 31.50. The medium volatility period is 2002/01/01 - 2002/04/31, where the average VIX level is 21.01. The low volatility period is 2005/01/01 - 2005/04/31, where the average VIX level is 13.22.

Figure 8: Implied Volatility Smirks for Various Maturities



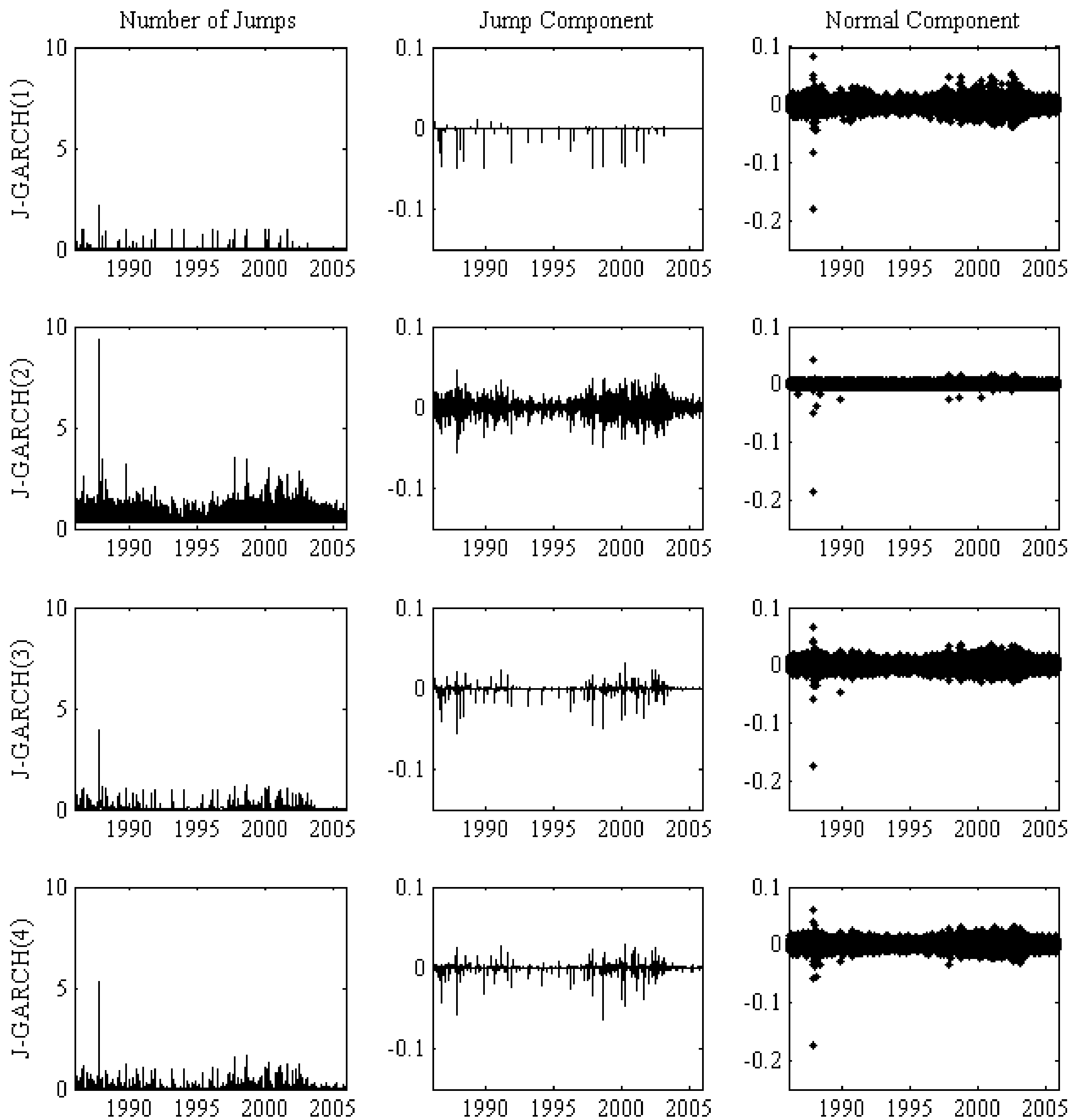
Notes to Figure: We compare the average implied Black-Scholes volatility smirks from the GARCH and selected J-GARCH models in three volatility regimes. The dots mark the average implied volatility from the data. The high volatility period is 1998/07/01 - 1998/10/31, where the average VIX level is 31.50. The medium volatility period is 2002/01/01 - 2002/04/31, where the average VIX level is 21.01. The low volatility period is 2005/01/01 - 2005/04/31, where the average VIX level is 13.22.

Figure 9: The Impact of Risk Premia on the IV Smirk in the J-GARCH(3) Model



Notes to Figure: We plot the volatility smirks at different maturities and according to different levels and sources of risk premia. In the left column the equity premium is entirely from the normal risk. In the middle column jump and normal risk premia provide equal contribution. In the right column the equity premium is entirely from jump risk. The conditional variance of the normal component and the conditional jump intensity are set to their model's implied long run mean.

Figure 10: Decomposition of Daily Returns using the Particle Filter



Notes to figure: We filter the ex-post expected number of jumps (left), the jump variable (middle), and the normal variable (right) using 5,000 particles. We use the MLE estimates in Table 2. The top to bottom panels show the results for J-GARCH (1), J-GARCH (2), J-GARCH(3), and J-GARCH(4) respectively.

Table 1: Selected Summary of the Literature on Finite-Activity Jump-Diffusion Estimation

	Jump Specification				Estimated from		
Continuous Time Models	Jumps in Volatility	Correlated Return and Volatility Jumps	State Dependent Jump Intensity	Stochastic Jump Intensity	Returns only	Options only	Joint Returns and Options
Merton (1976)					✓		
Bakshi, Cao, Chen (1997)						✓	
Bates (1996)					✓	✓	✓
Bates (2000)			✓			✓	
Pan (2002)			✓				✓
Andersen, Benzoni, Lund (2002)			✓		✓		
Chernov, Gallant, Ghysels, Tauchen (2003)	✓	✓			✓		
Eraker, Johannes, Polson (2003)	✓	✓			✓		
Eraker (2004)	✓	✓	✓		✓		✓
Huang, Wu (2004)			✓	✓		✓	
Bates (2006)			✓		✓		
Li, Wells, Yu (2007)	✓	✓			✓		
Broadie, Chernov, Johannes (2007)	✓					✓	
Discrete Time Models							
Maheu, McCurdy (2004)	✓	✓	✓	✓	✓		
Duan, Ritchken, Sun (2006)	✓	✓			✓		
J-GARCH	✓	✓	✓	✓	✓		

Notes to Table: For each study we choose its most flexible specification and categorized it according to: (1) The presence of jumps in volatility (2) Are jumps in returns and volatility correlated? (3) Is the jump intensity a function of a state variable such as volatility? "State Dependent Jump intensity", (4) Is the jump intensity modeled as a separate stochastic process (not as a function of other state variable)? "Stochastic Jump intensity". We also include information on how the model is estimated in each paper in the three right hand columns. Note that all models considered here have jumps in returns. We include our J-GARCH in the final row for comparison.

Table 2: MLE Estimates of J-GARCH models on S&P500 Returns, 1962-2005

	J-GARCH (1)		J-GARCH (2)		J-GARCH (3)		J-GARCH (4)	
Parameters	**Normal**	**Jump**	**Normal**	**Jump**	**Normal**	**Jump**	**Normal**	**Jump**
λ	1.968E+00	-4.369E-03	8.638E+01	-3.930E-03	2.774E+00	-8.788E-05	2.211E+00	-1.686E-04
	(1.312E+00)	(5.495E-03)	(6.668E+01)	(3.390E-03)	(2.643E+00)	(1.385E-04)	(1.155E+00)	(2.385E-05)
w	-1.210E-06	8.053E-03	2.612E-05	5.472E-06	-1.073E-06		-1.005E-06	7.549E-09
	(1.458E-07)	(2.002E-03)	(1.180E-06)	(2.457E-06)	(1.647E-07)		(1.252E-07)	(4.905E-09)
b	9.549E-01			9.355E-01	9.539E-01		9.652E-01	4.625E-01
	(5.023E-03)			(2.991E-06)	(5.751E-03)		(4.467E-03)	(8.376E-02)
a	2.144E-06			1.987E-06	1.976E-06		1.654E-06	1.317E+00
	(1.536E-07)			(6.886E-10)	(1.714E-07)		(1.405E-07)	(1.591E-01)
c	1.154E+02			1.802E+02	1.190E+02		1.085E+02	6.119E-01
	(1.048E+01)			(2.880E-02)	(1.424E+01)		(1.327E+01)	(4.820E-02)
θ		-1.254E-02		-4.211E-04		-2.628E-03		-1.662E-03
		(4.511E-03)		(1.501E-08)		(1.411E-03)		(8.355E-04)
δ		2.861E-02		1.012E-02		1.924E-02		1.613E-02
		(3.260E-03)		(3.318E-04)		(1.583E-03)		(8.355E-04)
k						5.209E+02		
						(1.21E+02)		
Properties								
Persistence	0.98345			0.99999	0.98168		0.95826	0.99524
Percent of Annual Variance	90.24	9.76	31.75	68.25	83.57	16.43	81.28	18.72
Avg Annual Vol	0.1362		0.1440		0.1354		0.1352	
LogLikelihood	37554		36581		37573		37576	

Notes to Table: We apply MLE to the daily return series of the S&P500 index from June 1962 to December 2005. Values under “Normal” columns refer to estimates of parameters governing the normal component. Similarly, the estimates of parameters governing the jump component are reported in the “Jump” columns. Reported in the bracket are the bootstrap standard error computed using 100 bootstrapped samples. Under Properties we report “Persistence” which refers to variance and jump intensity persistence respectively, “Percent of Annual Variance” which refers to the contribution to overall return variation arising from the normal and jump components respectively, as well as the average annualized volatility (standard deviation). The last row contains the log-likelihood values.

Table 3: MLE Estimates of benchmark models on S&P 500 Returns, 1962-2005

	BSM		Merton		GARCH	
Parameters	**Normal**	**Jump**	**Normal**	**Jump**	**Normal**	**Jump**
λ	1.16E+00		1.79E+01	-6.06E-04	1.336E+00	
	(8.71E-01)		(5.03E+00)	(1.46E-04)	(1.17E+00)	
w	8.83E-05		2.61E-05	5.49E-01	-1.296E-06	
	(1.46E-06)		(1.50E-06)	(4.88E-02)	(1.93E-07)	
b					9.495E-01	
					(5.33E-03)	
a					2.792E-06	
					(2.32E-07)	
c					1.065E+02	
					(9.48E+00)	
θ				-3.67E-04		
				(1.95E-04)		
δ				1.01E-02		
				(3.50E-04)		
Properties						
Persistence					0.9812	
Percent of Annual Variance	100.00	-	31.69	68.31	100.00	-
Avg Annual Vol	0.14913		0.14397		0.1340	
LogLikelihood	35573		36580		37342	

Notes to Table: We apply MLE to the daily return series of the S&P500 index from June 1962 to December 2005. Values under "Normal" columns refer to estimates of parameters governing the normal component. Similarly, estimates of parameters governing the jump component are reported under the "Jump" columns. Reported in the bracket are the bootstrap standard error computed using 100 bootstrapped samples. "BSM" refers to the standard Black-Scholes model, while "Merton" refers to the pure jump model of Merton (1976). "GARCH" is the Heston-Nandi (2000) GARCH(1,1) model which serves as our benchmark. Under Properties we report "Persistence" which refers to variance persistence, "Percent of Annual Variance" which refers to the contribution to overall return variation arising from the normal and jump components respectively, as well as the average annualized volatility (standard deviation). The last row contains the log-likelihood values.

Table 4: S&P 500 Index Call Option Data (1996-2005)

Panel A. Number of Call Option Contracts

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	123	1,841	2,078	2,293	6,416
0.975<S/X<1.00	554	2,557	1,076	645	4,851
1.00<S/X<1.025	867	2,282	717	366	4,236
1.025<S/X<1.05	571	1,337	413	191	2,516
1.05<S/X<1.075	257	839	263	139	1,501
1.075<S/X	298	1,190	466	237	2,198
All	2,670	10,046	5,013	3,871	21,718

Panel B. Average Call Option Price

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	5.39	13.96	26.29	43.56	28.95
0.975<S/X<1.00	11.82	24.12	44.31	77.13	34.58
1.00<S/X<1.025	23.86	36.25	60.76	92.19	42.77
1.025<S/X<1.05	43.30	55.37	79.43	110.79	60.90
1.05<S/X<1.075	66.65	76.40	99.07	127.03	83.53
1.075<S/X	111.08	120.90	135.19	169.19	127.98
All	38.52	45.00	53.41	67.76	50.40

Panel C. Average Implied Volatility from Call Options

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	0.2075	0.1876	0.1875	0.1831	0.1863
0.975<S/X<1.00	0.1768	0.1768	0.1831	0.1865	0.1796
1.00<S/X<1.025	0.1785	0.1813	0.1948	0.1955	0.1842
1.025<S/X<1.05	0.2034	0.1983	0.2040	0.2041	0.2009
1.05<S/X<1.075	0.2554	0.2187	0.2122	0.2056	0.2227
1.075<S/X	0.3561	0.2691	0.2379	0.2266	0.2695
All	0.2120	0.1971	0.1950	0.1893	0.1970

Notes to Table: We use European call options on the S&P500 index. The data are obtained from OptionMetrics. The prices are taken from non-zero trading volume quotes on each Wednesday during the January 1, 1996 to December 31, 2005 period. We apply the moneyness and maturity filters used by Bakshi, Cao and Chen (1997) to the data. The implied volatilities are calculated using the Black-Scholes formula.

Table 5: Option Pricing Performance for Jump Models relative to GARCH

Model Specification	All Normal Risk		All Jump Risk	
	$RMSE ratio	IVRMSE ratio	$RMSE ratio	IVRMSE ratio
BSM	1.1912	1.2974	1.1912	1.2974
Merton	1.25	1.32	1.24	1.32
J-GARCH(1)	1.00	1.03	0.87	0.91
J-GARCH(2)	1.24	1.33	1.23	1.32
J-GARCH(3)	0.91	0.98	**0.70**	**0.82**
J-GARCH(4)	0.96	0.99	0.88	0.92

Notes to Figure: We use the MLE estimates from Table 2 amd 3 to compute the Black-Scholes Implied Volatility RMSE (IVRMSE) (%) and dollar pricing RMSE ($RMSE) on the S&P500 Wednesday call options from 1996-2005. For comparisons across models, we set the long run equity premium of each model equal to six percent. For the jump models, we assume two extreme cases: first is when the long run equity premium is purely from normal risk (two left columns), and second is when the long run equity premium is purely from jump risk (two right columns). We only report $RMSE/IVRMSE ratios of the selected model relative to the benchmark GARCH model. The raw $RMSE of the GARCH model is 11.16, and the raw IVRMSE (%) of the GARCH model is 6.03.

Table 6. IVRMSE and Ratio IVRMSE by Moneyness, Maturity, and VIX level

Panel A: Sorting by Moneyness

	GARCH	J-GARCH(1)	J-GARCH(2)	J-GARCH(3)	J-GARCH(4)
Moneyness	IVRMSE(%)	IVRMSE Ratio	IVRMSE Ratio	IVRMSE Ratio	IVRMSE Ratio
S/X<0.975	4.934	0.898	1.311	0.742	0.946
0.975<S/X<1.00	4.595	0.950	1.419	0.850	0.968
1.00<S/X<1.025	5.012	0.951	1.423	0.866	0.956
1.025<S/X<1.05	6.170	0.905	1.347	0.848	0.919
1.05<S/X<1.075	7.591	0.877	1.295	0.826	0.891
1.075<S/X	10.493	0.909	1.216	0.809	0.880
All	6.028	0.914	1.316	0.816	0.921

Panel B: Sorting by Maturity

	GARCH	J-GARCH(1)	J-GARCH(2)	J-GARCH(3)	J-GARCH(4)
Maturity	IVRMSE(%)	IVRMSE Ratio	IVRMSE Ratio	IVRMSE Ratio	IVRMSE Ratio
DTM<20	7.719	0.943	1.270	0.988	0.981
20<DTM<80	5.894	0.934	1.379	0.841	0.925
80<DTM<180	5.559	0.915	1.329	0.683	0.928
DTM>180	5.624	0.813	1.172	0.638	0.820
All	6.028	0.914	1.316	0.816	0.921

Panel C: Sorting by VIX level

	GARCH	J-GARCH(1)	J-GARCH(2)	J-GARCH(3)	J-GARCH(4)
Maturity	IVRMSE(%)	IVRMSE Ratio	IVRMSE Ratio	IVRMSE Ratio	IVRMSE Ratio
VIX<14	2.610	1.241	1.341	1.450	1.064
14<VIX<18	3.838	0.727	0.869	0.929	0.860
18<VIX<22	5.412	0.798	1.059	0.762	0.875
22<VIX<26	6.519	0.875	1.248	0.728	0.910
26<VIX<30	7.748	0.955	1.418	0.787	0.940
30<VIX	10.013	1.011	1.566	0.831	0.956
All	6.028	0.914	1.316	0.816	0.921

Notes to Table: We use the MLE estimates from Tables 2 and 3 to compute the implied volatility root mean squared error (IVRMSE) for various moneyness, maturity, and VIX level bins. The IVRMSE is reported in levels for the GARCH model and for the J-GARCH models we report the IVRMSE ratio with the GARCH model. The equity premia is assumed to consist only of the jump risk premium.

Table 7: The Risk Premia Effects on IVRMSE Option Pricing Performance

Panel A: GARCH IVRMSE (%)

Normal Risk Premium (%)	**Total Equity Premium (%)** 0.0	2.0	4.0	6.0	8.0	10.0
0.0	**6.42**					
2.0		**6.26**				
4.0			**6.15**			
6.0				**6.03**		
8.0					**5.90**	
10.0						**5.78**

Panel B: J-GARCH (1) over GARCH IVRMSE ratio

Normal Risk Premium (%)	**Total Equity Premium (%)** 0.0	2.0	4.0	6.0	8.0	10.0
0.0	**1.02**	**0.99**	**0.96**	**0.91**	**0.89**	**0.87**
2.0		**1.02**	0.99	0.96	0.92	0.89
4.0			**1.03**	0.99	0.97	0.93
6.0				**1.03**	1.00	0.97
8.0					**1.03**	1.00
10.0						**1.04**

Panel C: J-GARCH (3) over GARCH IVRMSE ratio

Normal Risk Premium (%)	**Total Equity Premium (%)** 0.0	2.0	4.0	6.0	8.0	10.0
0.0	**0.97**	**0.92**	**0.86**	**0.82**	**0.81**	**0.88**
2.0		**0.97**	0.92	0.86	0.82	0.83
4.0			**0.98**	0.92	0.87	0.83
6.0				**0.98**	0.92	0.87
8.0					**0.98**	0.93
10.0						**0.98**

Panel D: J-GARCH (4) over GARCH IVRMSE ratio

Normal Risk Premium (%)	**Total Equity Premium (%)** 0.0	2.0	4.0	6.0	8.0	10.0
0.0	**0.98**	**0.97**	**0.94**	**0.92**	**0.89**	**0.86**
2.0		**0.98**	0.97	0.95	0.92	0.90
4.0			**0.99**	0.97	0.95	0.93
6.0				**0.99**	0.98	0.96
8.0					**0.99**	0.98
10.0						**1.00**

Notes to Table: We compute S&P500 Wednesday call option prices for 1996-2005 using MLE estimates from Table 2 and 3 together with various assumption for the long-run equity risk premium. Reported here are the IVRMSEs of the benchmark GARCH model of Heston-Nandi (2000) and IVRMSE ratios of selected jump models relative to the simple GARCH model. The columns represent pricing errors as the total equity premium increases, and the rows represent the pricing errors as the normal risk premium increases. For example, when the total equity premium is 6% and the normal risk premium is 2%, this implies a jump risk premium of 4%, etc. The top and bottom cells in each non-shaded region (also bolded) represent the case where the total equity premium consist purely of either the jump or normal risk premium.