

# The Price of Protection: Derivatives, Default Risk, and Margining\*

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## ABSTRACT

Like many financial contracts, derivatives are subject to default risk. A very popular mechanism in derivatives markets to mitigate the risk of non-performance on contracts is margining. By attaching collateral to a contract, margining supposedly reduces default risk. The broader impacts of the different types of margins are more ambiguous, however. In this paper we develop both, a theoretical model and a simulated market model to investigate the effects of margining on trading volume, wealth, default risk, and welfare. Capturing some of the main characteristics of derivatives markets, we identify situations where margining may increase default risk while reducing welfare. This is the case, in particular, when collateral is scarce. Our results suggest that margining might have a negative effect on default risk and on welfare at times when market participants expect it to be most valuable, such as during periods of market stress.

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# The Price of Protection: Derivatives, Default Risk, and Margining

Several recent credit events prompted public attention on the vulnerability of the financial system to the default of one of its participant institutions.<sup>1</sup> Of particular concern are situations where a shock to one institution propagates through the financial system leading to a failure of other institutions and thereby impairing the system's functioning. One might argue that the increasing importance of derivatives for the business of financial institutions have made those contracts both, a likely source as well as a potential propagation channel of such a systemic shock.<sup>2</sup> Exposure to a shock through derivatives contracts may be direct, as in the case of counterparty default; or indirect, for example, through asset markets. Financial institutions, banks in particular, have of course recognized the risks in derivatives contracts, especially counterparty default risk, and they have taken steps to reduce these risks.<sup>3</sup>

A central means to mitigate default risk in derivatives contracts is margining. It is employed to attach collateral either to a single contract or to a portfolio of contracts.<sup>4</sup> While margin rates in derivatives markets have been increasing significantly over the last few years, the effects of margin on default risk, trading volume, wealth, and welfare are not yet well understood. We therefore address the following research question: How do the various margining mechanisms observed in derivatives markets affect trading volume, capital, default risk, and welfare in the banking sector?

We follow two approaches to answer this question. First, we develop a theoretical model to investigate the effects of margin requirements on a single bank. The bank, exposed to unilateral, bilateral, or multilateral default risk, solves a standard utility maximization problem while being subject to different types of margin requirements. We find that margin typically serves its purpose of reducing default exposure per contract. However, we also find that it may increase default rates and overall losses given default while reducing trading volume and welfare. This is particularly the case when margin rates are high and collateral is scarce as might be the case during a market crisis. Subsequently, we further investigate these results by developing

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<sup>1</sup>Examples include the cases of Metallgesellschaft, LTCM and Enron.

<sup>2</sup>This view is advocated, e.g., in Schinasi (2006).

<sup>3</sup>Among others, both, the number of collateral agreements and margin rates have been increasing. International Swaps and Derivatives Association (2006) reported that the number of collateral agreements in over-the-counter markets increased by a factor of 9.1 between 2000 and 2006. In addition, the amount of margin posted increased by a factor of 6.7 to USD 922 billion in the same period (and 1.9 between 2003 and 2006). Average coverage of credit exposure increased by 97% to 59% between 2003 and 2006.

<sup>4</sup>In the following, we will refer to the assets used as security deposit as *collateral*. We will refer to the collateral attached to a contract or to a position of several contracts as *margin*. This means that we will abstract from certain legal details of margining that we consider irrelevant for our economic analysis.

a simulation model of a derivatives market under severe stress. In order to capture the features of actual derivatives markets, we calibrate the model's parameters with recent market data. The simulation confirms the results of our theoretical analysis.

Default risk, the risk of non-performance of a counterparty, is inherent in derivatives contracts as much as in any other contract. However, default risk in a derivatives contract appears considerably more complex and less predictable than the default risk in, for example, a simple loan. Payoff and credit exposure of derivatives contracts are time-varying and depend on the prices of the underlying assets. Therefore, liabilities in relation to derivatives contracts are correlated with the underlying price changes being hedged.<sup>5</sup> In addition, contracts are often reciprocal meaning that each counterparty to a derivatives contract is potentially both, a creditor and a debtor. Since the early days of (modern) derivatives markets, traders have tried to manage default risk in derivatives contracts through contractual innovations. These innovations include provisions for margin requirements as well as central counterparties. They can be viewed as mechanisms to mitigate default risk. Today, the contracts underlying derivatives transactions vary widely with regards to the mitigation mechanisms they employ.<sup>6</sup>

Margining is supposed to increase the lower bound of the delivery rate in case a default occurs. One gains the impression that margining is sometimes regarded as a panacea to cure default risk, as the following quote from International Swaps and Derivatives Association (2005) illustrates:

“The mechanism by which collateral provides benefit is through improvement of the recovery rate. Collateral does not make it more or less likely that a counterparty will default and does not change the value of a defaulted transaction.”

The increases in margin rates over the last few years are, supposedly, reactions to several severe credit events experienced by OTC markets including the cases mentioned above. A common argument in support of margining refers to the fact that derivatives clearinghouses, which have been employing margining extensively, have rarely defaulted. As we will discuss in Section 6, though, clearinghouses rely on several other risk management tools in addition to margining. It is therefore questionable whether margining alone

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<sup>5</sup>Cf. e.g. Duffie and Singleton (2003).

<sup>6</sup>We describe the most important mechanisms employed in derivatives markets to mitigate default risk in Appendix A.

ensures the mitigation of default risk. In some of the crises mentioned earlier, it is not entirely clear to what extent margin requirements mitigated default risk and to what extent they also contributed to those crises.

It is worth mentioning that margining does not only yield benefits but that it may also engender costs. First, collateral posted as margin may have opportunity cost in the form of foregone interest. Secondly, margin requirements might limit the number of contracts traded by a market participant and thus might prevent him from implementing his optimal position. Thirdly, a trader might be indirectly constrained if other market participants reduce the number of contracts traded as a consequence of increasing margin requirements. Thus, margin requirements might constrain risk sharing among market participants in several ways. As a result, while they do reduce credit exposure, their effects on banks' capital and on welfare are ambiguous.

Various theoretical and empirical studies, discussed in Section 1 and including Brunnermeier and Petersen (2005), Cuoco and Liu (2000), and Johannes and Sundaesan (2006), analyzed the implications of margin requirements on agents' investment decisions and on asset prices. Most of these studies take the perspective of a single agent. What is less well understood is the effect of margin requirements on "aggregate phenomena" such as market liquidity. Moreover, no study has, to the best of our knowledge, formally analyzed the effects of margin requirements on default risk.

Our results show that, in certain situations, default risk mitigation mechanisms, in various combinations, might not only have a negative effect on a single institution but also exacerbate systemic risk by amplifying a shock. Given the size of OTC derivatives markets and the fact that most of the contracts are being held by a small number of large banks<sup>7</sup>, we believe that our results warrant a closer examination of default risk mitigation mechanisms, and of margining in particular, with regards to their impact on the stability of the financial system.

We proceed as follows. In Section 1 we briefly review the literature relevant to our study. In Section 2, we discuss the economic setting of our theoretical analysis. Subsequently, in Section 3, we describe the types of margins we consider. These margins are analyzed in detail in Section 4 for contracts with unilateral default risk, and in Section 5 for contracts with bilateral default risk. We discuss multilateral trading and

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<sup>7</sup>OCC (2006) reports that 96.6% of notional held by U.S. insured commercial banks were held by the five largest institutions as of 30 June 2006.

default risk in Section 6. In Section 7, we investigate margins in a dynamic market model with heterogeneous agents. Finally, Section 8 concludes.

## 1. Literature Review

The measurement of default risk in derivatives contracts is inherently more complex than in most other financial contracts. Standard measurement approaches to credit risk often fail in case of derivatives. Duffee (1996) suggests two main reasons for this complexity. First, credit exposure fluctuates with the price of the underlying security. And secondly, exposures on derivatives contracts are correlated with the probabilities of default. Duffie and Singleton (2003, Chapter 9) provide a framework for the analysis of default risk in derivatives. In particular, they suggest to differentiate between current and potential future credit exposure. As we will see later, this perspective helps to understand the different types of margin found in derivatives markets. The incorporation of default risk into the valuation of derivatives contracts was first considered by Hull and White (1995) and has since been addressed by Collin-Dufresne and Hugonnier (2002) and others for a rather broad class of instruments.

Equally challenging as the measurement of default risk is its mitigation. As Swan (2000) reports, market participants have been preoccupied with the development of mechanisms to mitigate default risk in derivatives contracts ever since the inception of modern derivatives markets. Among the first means employed by market participants were appraisals of counterparties and collateral. Over time, rather sophisticated mechanisms evolved including derivatives clearinghouses and central counterparties. A historical account of these mechanisms is provided by Loman (1931) and Moser (1994, 1998).

Today, one of the most important mechanisms to mitigate default risk in derivatives contracts is probably close-out netting. It allows market participants to net claims and obligations related to derivatives with a counterparty in case the counterparty defaults. Close-out netting tends to reduce credit exposure of derivatives portfolios significantly. A recent report by the International Swaps and Derivatives Association suggests that for large market participants, close-out netting reduces exposure by up to 90%. Legal and economics aspects of close-out netting are described in detail by Bergman, Bliss, Johnson, and Kaufman (2003) and Bliss and Kaufmann (2004). Bergman, Bliss, Johnson, and Kaufman (2003) point out that close-

out netting might provide considerable benefits for large participants, but that it may be disadvantageous for small participants. In addition, close-out netting may have adverse effects on systemic risk, as Bergman, Bliss, Johnson, and Kaufman (2003) suggested.

The other important mechanism to mitigate default risk is margining. As we described in the Introduction, margining is supposed to reduce default risk in derivatives by increasing the minimum delivery rate of a counterparty and thereby reducing credit exposure. Margining is an essential risk management tool for derivatives clearinghouses, as documented by Moody's Investor Service (1998) and Knott and Mills (2002), among others. As the International Swaps and Derivatives Association (2006) showed (see Footnote 3), it is increasingly employed by market participants in relation to their over-the-counter activities.

We mentioned earlier that margins may impose costs on market participants and thus be reflected in market activity and in prices. Variation margin changes the cash flows and the value of a futures contract. Cox, Ingersoll, and Ross (1981) argued that variation margin of futures contracts can therefore be regarded as a stochastic dividend. Margin may also impose a funding constraint on market participants. It is thus conceivable that margin requirements affect agents' trading behavior and market activity. Cuoco and Liu (2000) incorporated margin requirements into the portfolio optimization problem of a single agent and compute the effects on portfolio weights, albeit for asset trading. An adverse effect on market liquidity was already noted by Telser (1981). The theoretical discussion remains inconclusive, though. While Anderson (1981) argued that opportunity costs of collateral are probably too low for margining to have any effects on liquidity, Kalavathi and Shanker (1991) claimed that opportunity costs can in fact be significant in terms of yield foregone. The effect of margins on prices was analyzed by Brunnermeier and Petersen (2005) in an equilibrium model, albeit for asset trading. They found that under certain conditions, margins may have adverse effects on asset demand and supply and result in price spirals.

The effects of margins on market activity have also been investigated empirically. Analyzing changes in the initial margin at the Chicago Mercantile Exchange and the Chicago Board of Trade, Hartzmark (1986) found that an increase in the initial margin resulted in a decline of open interest and of volume. Similarly, Hardouvelis and Kim (1995) found that an increase in the initial margin for contracts traded on the New York Commodity and Mercantile Exchange and the Chicago Board of Trade reduced trading volumes sig-

nificantly. Margins also seem to be reflected in prices. Johannes and Sundareshan (2006) showed that swap rates in over-the-counter markets increase with an increase in the cost of collateral.

Empirical evidence thus suggests that margining might result in inefficiencies. Some of these inefficiencies can be lifted by the establishment of a central counterparty, typically provided by an exchange clearinghouse. If market participants trade through a central counterparty, original bilateral contracts are extinguished and replaced by new contracts with the central counterparty. A central counterparty is typically of a very high credit quality, partly due to its stringent risk management approach. Thus, bilateral credit risk of variable quality is replaced with a high quality credit risk exposure to the central counterparty. Furthermore, a central counterparty allows for multilateral netting of contracts. Koepl and Monnet (2006) develop a model where a central counterparty is necessary to implement efficient trade when trades are time-critical, liquidity is limited and there is limited enforcement of trades. They also show that the efficiency of central counterparties depends on their governance structure. Jackson and Manning (2006) present a model where multilateral netting by a central counterparty provides for a substantial reduction in default risk. This reduction results from the additional netting benefits of a central counterparty over bilateral netting, the higher dispersion of losses, and the diversification of the central counterparty across an array of imperfectly correlated assets.

A central counterparty also provides informational advantages over bilateral trading. Central counterparties and their economic implications are also discussed by Bliss and Papathanassiou (2006). One might argue that the substitution of a central counterparty for the many original counterparties may increase systemic risk. Although clearinghouses might default, they do so rather rarely. Recent clearinghouse failures occurred in Paris (1973), Kuala Lumpur (1983) and Hong Kong (1987), as discussed in Knott and Mills (2002). Central counterparties are typically considered of very high credit quality as a report by Moody's Investor Service (1998) showed.

Nevertheless, systemic risk associated to central counterparties should not be neglected. Brimmer (1989) and Bernanke (1990) analyzed systemic risk in clearing and settlement systems during the 1987 crash. The latter investigation pointed out that several clearinghouses, without government intervention, would have become illiquid.

The stability of a clearinghouse, and its systemic effects, may be weakened by moral hazard. Moser (1998) and Knott and Mills (2002) pointed out that the clearinghouse's customers—as a consequence of

multilateral netting—may be encouraged to take on more risk thereby increasing the default risk of the clearinghouse.

Additional approaches to mitigating default risk in derivatives markets includes self-regulation and loss-sharing arrangements, as described by Moser (1998). Kroszner (1999) argued that recent innovations in the legal system, well-functioning rating agencies, as well as the development of risk models might allow market participants to reach the same level of efficiency in over-the-counter markets that were previously only possible with a central counterparty. Recent events, however, like the recapitalization of LTCM and the bankruptcy of Enron and their respective consequences, cast some doubt on this view, in particular, in the context of systemic risk.

## 2. Economic Setting

The objective in the following sections is to investigate how different types of margins affect a bank’s utility, trading volume, default rates, and default severity. We consider a bank facing a standard utility maximization problem. The bank initially holds a certain amount of a numeraire asset and an illiquid contract. It may hedge the illiquid contract by taking a position in another, liquid contract. The payoffs of the two contracts we consider are imperfectly correlated, though. Therefore, the bank cannot perfectly hedge its holding and may default. The examples we have in mind relate to a bank holding a position in a long-term contract and that may only hedge by trading a short-term contract; or, a bank holding a defaultable bond that may only hedge the interest rate risk related to the bond position by trading an interest-rate swap.

More particularly, we consider an economy with two risky (reference) assets,  $S^1$  and  $S^2$ , generating payoffs  $\xi^i \in \{u^i, m^i, d^i\}$ ,  $i = 1, 2$ , per period, and a risk-free asset (“cash”) paying zero interest. We require  $u^i > m^i \geq 0 > d^i$ . The risky assets are expected to generate the following profits between  $t$  and  $t + 1$ :

$$(S_{t+1}^1, S_{t+1}^2) = \begin{cases} (S_t^1 + u^1, S_t^2 + u^2) & \text{with probability } p_1, \\ (S_t^1 + m^1, S_t^2 + m^2) & \text{with probability } p_2, \\ (S_t^1 + d^1, S_t^2 + d^2) & \text{with probability } p_3, \end{cases}$$



such that  $p_1 + p_2 + p_3 = 1$ , where  $S_t^1$  and  $S_t^2$  denote the values of the two processes at time  $t$ , respectively. We assume  $S_0^1 = S_0^2 = 0$ . The model has  $T$  steps. We denote the difference in payoffs between  $t$  and  $t + 1$  by  $\Delta S_t^i = S_{t+1}^i - S_t^i$ .

A bank may trade contracts written on the payoffs generated by the risky assets. A contract on  $S^i$  yields  $\Delta S_t^i$  per period. We assume that banks set the prices of the contracts to zero when they trade those contracts. In other words, there is no exchange of cash flows at contract initiation. Trading in the contracts takes place between times 0 and  $T - 1$ , and contracts are settled in cash at time  $T$ . This means that any liability arising from a contract entered into between 0 and  $T - 1$ , only falls due for payment at time  $T$ . The contracts are both, a potential asset and a potential liability.

Banks initially hold an amount  $c$  of cash and one contract on  $S^2$ . From now on we assume that banks cannot trade contracts on  $S^2$ , or, in other words, that contracts on  $S^2$  are illiquid. However, they may trade contracts written on  $S^1$ .<sup>8</sup> We denote the number of units of contract 1, the contract on  $S^1$ , held by the bank by  $\alpha = (\alpha_0, \dots, \alpha_{T-1})$ ;  $\alpha_T \equiv 0$ . A trading strategy  $\alpha$  thus generates contractual obligations at time  $T$  of  $\sum_{t=0}^{T-1} \alpha_t \Delta S_t^1$ .

As obligations in relation to positions in  $S^1$  and  $S^2$  only fall due at time  $T$ , these positions do not generate any cash flows between times 0 and  $T - 1$ . We denote the cash flows generated by positions in  $S^1$  and  $S^2$  by  $C_t^1$  and  $C_t^2$ , respectively. We have

$$C_t^1 = \begin{cases} \sum_{s=1}^T \alpha_{s-1} \Delta S_{s-1}^1 & \text{for } t = T, \\ 0 & \text{for all } t \in \{0, \dots, T-1\}. \end{cases}$$

and similarly,

$$C_t^2 = \begin{cases} S_T^2 & \text{for } t = T, \\ 0 & \text{for all } t \in \{0, \dots, T-1\}. \end{cases}$$

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<sup>8</sup>We interpret this setting as one where banks are locked into a position in one contract and try to hedge their initial position with another contract. This liquid hedging contract may become subject to various types of margin requirements. The setting is motivated by the fact, as mentioned in the Introduction, that margin requirements tend to change over time. For example, exchanges may increase margin requirements during periods of market stress. Similarly, margin requirements for over-the-counter derivatives contracts have been increasing over the last several years, possibly as a reaction to default events in those markets; cf. Footnote 3. Many banks may thus find themselves in a situation where they are locked into a long-term position and their hedging activities suddenly become subject to margin requirements.

We assume that banks are subject to limited liability. As long as a bank has sufficient assets to meet its obligations, it fully delivers. The bank defaults on its contractual obligations whenever it holds an insufficient amount of assets to meet those obligations. We now specify the default mechanism.

Let  $A(\alpha)$  denote a bank's total assets available for distribution in the event of a default, given a position  $\alpha$ . Similarly, let  $L^i(\alpha)$ , denote the bank's liabilities in relation to contract  $i$ , and let  $L(\alpha)$  denote the bank's total liabilities. At time  $T$ , we have

$$L_T^1(\alpha) = \left[ \sum_{t=0}^{T-1} \alpha_t \Delta S_t^1 \right]^{-}, \quad L_T^2(\alpha) = \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^{-},$$

and  $L_T(\alpha) \equiv L_T^1(\alpha) + L_T^2(\alpha)$ , where  $x^- \equiv \max\{-x, 0\}$  denotes the negative part of  $x$ . On the other hand, the bank's assets are given by

$$A_T(\alpha) = c + \left[ \sum_{t=0}^{T-1} \alpha_t \Delta S_t^1 \right]^+ + \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^+,$$

where  $x^+ \equiv \max\{x, 0\}$  denotes the positive part of  $x$ . Note that the above equation implies that the bank's counterparty does not default. We assume that in the event of a default, the bank's assets are distributed pro-rata to its creditors.<sup>9</sup> We denote the deliveries on the bank's liabilities in relation to contract  $i$  at time  $T$ , given a position in  $S^1$  of  $\alpha$ , by  $D_T^i(\alpha)$ . We set  $D_T^i(\alpha) = 0$  whenever  $L_T(\alpha) = 0$ .

**Lemma 1** *We have*

$$D_T^i(\alpha) \equiv \begin{cases} L_T^i(\alpha) \min\left\{\frac{A_T(\alpha)}{L_T(\alpha)}, 1\right\} & \text{if } L_T(\alpha) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

for  $i = 1, 2$ . Furthermore,  $A(\alpha)$ ,  $L(\alpha)$ , and  $D_T(\alpha)$  are convex functions of  $\alpha$ .

**Proof.** We consider the case of  $L_T(\alpha) > 0$ . If  $L_T^i(\alpha) = 0$  then  $D_T^i(\alpha) = 0$ . On the other hand, whenever  $A_T(\alpha) \geq L_T(\alpha)$ ,  $D_T^i(\alpha) = L_T^i(\alpha)$ . If  $0 < A_T(\alpha) < L_T(\alpha)$ ,  $D_T^i(\alpha) = (L_T^i(\alpha)/L_T(\alpha))A_T(\alpha)$ . ■

Clearly,  $D^i$  ranges from 0 to  $L^i$ . Two comments in relation to the bank's default are in order. First, the bank defaults "mechanically", whenever its capital drops below zero. Second, it always delivers up to its capacity; in other words, the delivery rate is standard, meaning that the bank has no direct discretion over the amount it delivers on its liabilities. In other words, there is no strategic renegotiation of contractual

<sup>9</sup>Obviously, this will change in the presence of margin requirements.

obligations. Moreover, it may be worth noting that the “probability” of the bank defaulting as well as the loss in case of default are determined by  $\alpha$ .

The introduction of limited liability obviously establishes a lower bound on the bank’s deliveries at zero and provides the bank with an option to default. It thus reduces the expected value of the bank’s liabilities. To balance the bank’s delivery on his obligations, we assume that it is subject to a default penalty proportional to the amount it defaults on. In other words, the bank incurs a penalty  $\lambda z^+$ , where  $z$  is the difference between the bank’s obligations (liabilities due) and its actual delivery on those obligations, and  $\lambda \geq 0$ .

### 3. Margins

A bank faces a trade-off between a lower delivery and a higher default penalty. Even in the presence of a default penalty, it may be optimal for a bank to risk to default. We now describe how margin requirements affect the bank’s deliveries, and the constraints they may impose on the bank’s trading strategy.

The primary rationale for the introduction of margin is typically the reduction of credit exposure in relation to a contract or a position. This is, representatively, illustrated in International Swaps and Derivatives Association (2005). We differentiate two types of credit exposure. By *current exposure*, we mean the exposure that has been generated by a holding in a contract up to the present, that is,  $[\sum_{s=0}^{t-1} \alpha_s \Delta S_s]^+$ . By *potential future exposure*, on the other hand, we mean the exposure that may be created by holding a contract between the present and contract expiry,  $[\sum_{s=t}^{T-1} \alpha_s \Delta S_s]^+$ . In the following, we discuss two types of margin requirements, called *initial* and *variation* margin, that are supposed to reduce those two types of exposures.

#### 3.1. Initial Margin

Initial margin is typically linked to potential future exposure. When a contract is subject to initial margin, the bank has to post  $\Phi \geq 0$  units of cash per unit of contract traded, as collateral. By attaching a certain amount of collateral to a contract at contract initiation, the counterparty is ensured to receive at least the collateral should the bank default on its obligations.

In the following, we illustrate the mechanics of initial margin. We consider the bank's liabilities in relation to its position in contract 1, that is,  $L_T^1(\alpha) = [\sum_{t=0}^{T-1} \alpha_t \Delta S_t^1]^-$ . If the obligation is smaller than the collateral posted,  $L_T^1(\alpha) < |\alpha_{T-1}| \Phi$ , the bank fully delivers, that is, it delivers  $L_T^1(\alpha)$ . If the obligation is larger than the margin requirement,  $L_T^1(\alpha) \geq |\alpha_{T-1}| \Phi$ , the bank delivers the collateral posted,  $|\alpha_{T-1}| \Phi$ , plus a pro-rata share of any excess assets available for distribution. The assets are given by

$$A_T(\alpha) = c + \max\{-|\alpha_{T-1}| \Phi, -L_T^1(\alpha)\} + \left[ \sum_{t=0}^{T-1} \alpha_t \Delta S_t^1 \right]^+ + \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^+. \quad (1)$$

The assets comprise the initial holding in cash,  $c$ , and the profits generated by the positions in  $S^1$  and  $S^2$ . They are reduced by the fraction of the liabilities in relation to contract 1 which are secured by collateral as those assets are not available for distribution anymore.

**Lemma 2** *In the presence of initial margin (only), the delivery on  $L_T^1(\alpha)$ , the liabilities in relation to the position in  $S^1$ , is given by*

$$D_T^1(\alpha) \equiv \begin{cases} \min \left\{ |\alpha_{T-1}| \Phi + \max \left\{ 0, (L_T^1(\alpha) - |\alpha_{T-1}| \Phi) \frac{A_T(\alpha)}{L_T^1(\alpha)} \right\}, L_T^1(\alpha) \right\} & \text{if } L_T^1(\alpha) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for  $L_T^2(1)$  and  $D_T^2(\alpha)$ .

The above result is similar to Lemma 1. Initial margin effectively attaches collateral to the trading position in  $S^1$ . It raises the minimum delivery rate per unit of contract to  $\Phi$ . Initial margin requires the bank to fund every unit of contract held with  $\Phi$  units of cash. In other words, it shifts risk, and in case of default also transfers wealth, from the counterparties of the position in  $S^2$  and from the bank's owners to the counterparty of the position in  $S^1$ .<sup>10</sup>

Initial margin also "locks up" part of the bank's cash holding. Since there are no intermediate cash flows between 0 and  $T - 1$  (and we assume that the bank cannot borrow against its holdings in  $S^1$  and  $S^2$ ), initial margin imposes the following constraint on the bank's trading strategy:  $|\alpha_t| \Phi \leq c$ , for all  $t \in \{0, \dots, T - 1\}$ .

<sup>10</sup>Cooper and Mello (1991) and Richard J. Rendleman (1992) already pointed out that swaps may transfer wealth from shareholders and debtholders of a firm to the counterparties to the swap.

According to the analysis above, by raising the lower bound of the minimum delivery rate, initial margin achieves its objective: it reduces potential future exposure in a contract, *ceteris paribus*. The latter, however, hinges on the assumption that exposure per contract is exogenous and thus independent of the margin requirement  $\Phi$  in particular.<sup>11</sup>

### 3.2. Variation Margin

We now turn to variation margin. Over time, a position in a contract may build up current exposure exceeding the initial margin held. This will leave part of the exposure uncovered. The main purpose of variation margin, supposedly, is to settle changes in contract value at frequent intervals in order to eliminate current exposure. It requires the bank to post  $\alpha_{t-1}\Delta S_{t-1}^1$  units of cash at each settlement date  $t \in \{1, \dots, T\}$ . This means that any changes in the value of the portfolio of contracts on  $S^1$  are now settled periodically throughout the life of the contract.

Variation margin thus changes the cash flow pattern of the position in  $S^1$ . It also potentially changes the timing of default since the bank may now default at any time between dates 1 and  $T$ . In the presence of variation margin, the bank holds  $c + C_t^1$  units of cash at time  $t$ , where  $C_t^1$  is now given by

$$C_t^1 = \begin{cases} \sum_{s=1}^t [\alpha_{s-1}\Delta S_{s-1}^1 - D_s^1(\alpha)] & \text{for } t = \{1, \dots, T\}, \\ 0 & \text{for } t = 0. \end{cases} \quad (2)$$

Note the difference between the above equation and Equation (1). The contract on  $S^1$  is now settled periodically, as reflected by the term  $C^1$ .

In combination with initial margin, the bank's funding constraint becomes  $|\alpha_t|\Phi \leq c + C_t^1$ , for all  $t \in \{1, \dots, T\}$ . Any profits generated by the position  $\alpha$  relax the funding constraint, whereas losses tighten it. Let  $L_t^1(\alpha)$  denote the liabilities related to the contract on  $S^1$  incurred by the bank between times  $t-1$  and  $t$ , that is,  $L_t^1(\alpha) = [\alpha_{t-1}\Delta S_{t-1}^1]^-$ . These liabilities are immediately due at time  $t$ . If  $L_t^1(\alpha) < |\alpha_{t-1}|\Phi$ , the bank

<sup>11</sup>The analysis changes if the price of the contract is a function of  $\Phi$ . One could conceive that the price per contract increases with  $\Phi$ , to account for the costs imposed by the margin requirement. Then the exposure per unit of contract will no longer be  $[\xi]^+$  but  $[\xi - p]^+$ , where  $p$  denotes the contract price. Suppose  $\xi > 0$ . The contract's payoff in such a state will decrease with  $\Phi$ . However, if  $\xi < 0$ , the (negative) payoff will increase. This means that the liability in relation to the contract will increase as  $\Phi$  increases, and so will the counterparty's exposure. In the remainder, we will focus on the case where  $p$  is independent of  $\Phi$ .

delivers fully, that is,  $L_t^1(\alpha)$ . If  $L_t^1(\alpha) \geq |\alpha_{t-1}| \Phi$ , the bank may default and only delivers  $|\alpha_{t-1}| \Phi$  at present. Its position in the contract is then closed out, that is,  $(\alpha_{t+1}, \dots, T) \equiv 0$ . The position in the contract on  $S^2$ , though, stays until maturity date  $T$ . Thus, in addition to  $|\alpha_{t-1}| \Phi$ , the bank might also deliver a pro-rata share of any excess assets available for distribution at time  $T$ . As a result, in case of default, the bank immediately delivers up to the collateral attached to the position in  $S^1$ , and it might deliver additional cash at time  $T$  when the contract on  $S^2$  falls due.<sup>12</sup>

The assets available for distribution at time  $T$  to both, the counterparty to contract on  $S^1$  and the counterparty to the contract on  $S^2$ , are given by

$$A_T(\alpha) = c + C_T^1 + \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^+.$$

**Lemma 3** *Let  $0 < \tau \leq \infty$  denote the time of the bank's default and let  $s_t \equiv \min\{\tau, t\}$ . In the presence of both initial and variation margin, the delivery on  $L_{s_t}^1(\alpha)$ , the liability in relation to the position in  $S^1$ , is then given by*

$$D_t^1(\alpha) = \min \{ |\alpha_{t-1}| \Phi, L_t^1(\alpha) \}$$

and

$$D_T^1(\alpha) = [L_{s_t}^1(\alpha) - D_{s_t}^1(\alpha)]^+ \frac{A_T(\alpha)}{L_T(\alpha)},$$

and similarly for  $L_T^2(\alpha)$ .

Whereas all deliveries on contract 2 take place at time  $T$ , deliveries on contract 1 are split into deliveries at every time  $t$ ,  $D_t^1$ , and another final delivery at time  $T$  when contract 2 matures,  $D_T^1$ . In combination with initial margin, variation margin further increases the minimum delivery rate on the position in  $S^1$ . Changes in the value of a position are immediately set off by variation margin. If the bank defaults, credit exposure is limited to the change in contract value during the period immediately prior to default.

In this section, we described the mechanics of initial and variation margin under the premise of an otherwise constant economic setting. Under those circumstances, margin does indeed reduce credit exposure. In the remainder, we will investigate to what extent the “ceteris paribus” assumption can be justified. More

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<sup>12</sup>This default mechanism is common in over-the-counter markets, as described for example in ISDA (2005). An alternative default mechanism provides for the immediate close-out of all holdings of the agent at the time of default.

precisely, we will analyze the effects of margin on a bank's trading strategy, its default rates, its losses in case of default, and its expected utility wealth.

## 4. Unilateral Default Risk

We analyze the effects of margin in two steps. In this section, we consider a bank that may default but is not exposed to default risk itself. In other words, we consider a bank that trades with a (default-)risk free counterparty. In the subsequent section, we consider two banks trading with each other that may both default. This approach allows us to separate direct and indirect effects of margins on the banks' optimization problem. We will show that there exist circumstances under which the costs for a bank engendered by a margin requirement exceed its benefits.

### 4.1. Optimization Problem

We assume that the bank has a strictly concave, nondecreasing utility function  $u$ . In the presence of a default penalty, its utility is of the form  $u(E) - \lambda z^+$ , where  $E$  denotes the bank's equity,  $\lambda > 0$  denotes the default penalty, and  $z$  denotes the amount the bank defaulted on.

The bank maximizes its expected utility of terminal equity by optimally choosing  $\alpha$ , the number of units of the contract on  $S^1$ . A bank thus faces the following optimization problem:

$$\mathcal{P}_0 : \quad \max_{\alpha} \mathbf{E}[U(\alpha; \Phi)] = J(\Phi) \quad (3)$$

subject to

$$\begin{aligned} U(\alpha; \Phi) = & u \left( c + \left[ \sum_{t=0}^{T-1} \alpha_t \Delta S_t^1 \right]^+ + \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^+ - D_T^1(\alpha; \Phi) - D_T^2(\alpha; \Phi) \right) \\ & - \lambda \left( \left[ \sum_{t=0}^{T-1} \alpha_t \Delta S_t^1 \right]^- + \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^- - D_T^1(\alpha; \Phi) - D_T^2(\alpha; \Phi) \right). \end{aligned} \quad (4)$$

The bank's equity at the terminal time  $T$  thus comprises its initial capital  $c$  and the assets generated by its holdings in  $S^1$  and  $S^2$ , respectively. It is reduced by the deliveries on the obligations in relation to its holdings in  $S^1$  and  $S^2$ , respectively.  $D_T^1(\alpha; \Phi)$  and  $D_T^2(\alpha; \Phi)$  are given by Lemma 1.

To maximize expected utility, the bank will try to equalize the marginal utilities in the various states by implementing an appropriate strategy  $\alpha$ . If the default penalty  $\lambda$  is strictly positive, the bank, in choosing the optimal strategy  $\alpha$ , does not only face a trade-off between the marginal utilities in the different states. It also faces a trade-off between a change in utility from terminal equity and a change in the default penalty *within* the different states.

**Proposition 1** *There exists a unique solution  $\hat{\alpha}$  to  $\mathcal{P}_0$ .*

The result follows from the the strict concavity of the objective function (3). Note that the introduction of default in the agent's optimization problem renders equity and utility non-linear functions of  $\alpha$ . The objective function is thus not everywhere differentiable. Constraints imposed subsequently by margin requirements will introduce further non-linearities as well as path-dependence of equity and utility. These complexities make an analytic evaluation of the bank's optimization problem cumbersome. We will therefore evaluate the problem numerically.

## 4.2. Parameterization of Numerical Evaluation

In our numerical evaluation,  $S^1$  and  $S^2$  follow a trinomial process with the following parameters:

$$u_1 = \mu_1 h + \sigma_1 h, \quad m_1 = \mu_1 h, \quad d_1 = \mu_1 h - \sigma_1 h,$$

$$u_2 = \mu_2 h + \sigma_2 \left( \frac{3}{2} \rho + \frac{1}{2} \sqrt{1 - \rho^2} \right) h, \quad m_2 = \mu_2 h - \sigma_2 \frac{3}{2} \sqrt{1 - \rho^2} h, \quad d_2 = \mu_2 h - \sigma_2 \left( \frac{3}{2} \rho - \frac{1}{2} \sqrt{1 - \rho^2} \right) h,$$

where  $h = T/N$ . States have equal probabilities  $p_1 = p_2 = p_3 = 1/3$ . We use the following parameter values:  $\mu_1 = \mu_2 = 0.1$ ,  $\sigma_1 = \sigma_2 = 1.0$ ,  $c = 0.3$ ,  $\lambda = 0.8$ ,  $S_0^1 = S_0^2 = 0$ ,  $T = 1$ , and  $N = 2$ . The bank has an exponential utility function  $u(x) = 1 - e^{-x}$ .

Given the strictly concave utility function, the bank will try to hedge its endowment by entering into a short position in  $S^1$ . However, since  $S^1$  and  $S^2$  are imperfectly correlated, the hedge will not be perfect. As



a consequence, there may be states where the bank defaults, either on its obligations in relation to  $S^1$  or to  $S^2$  or both.

### 4.3. Initial Margin

In the remainder, we will see how margin requirements, supposed to reduce any loss in relation to a position in  $S^1$ , affect the bank's equity, its default rate, trading volume, its losses in case of default as well as expected utility. We now investigate the effects of initial margin. We first restate the optimization problem in Equations (3) and (4) and then evaluate it numerically.

If the bank has to post initial margin when holding a position in contract 1, the optimization problem becomes

$$\mathcal{P}_1 : \quad \max_{\alpha} \mathbf{E}[U(\alpha; \Phi)] = J(\Phi) \quad (5)$$

subject to

$$U(\alpha; \Phi) = u \left( c + \left[ \sum_{t=0}^{T-1} \alpha_t \Delta S_t^1 \right]^+ + \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^+ - D_T^1(\alpha; \Phi) - D_T^2(\alpha; \Phi) \right) \quad (6)$$

$$- \lambda \left( \left[ \sum_{t=0}^{T-1} \alpha_t \Delta S_t^1 \right]^- + \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^- - D_T^1(\alpha; \Phi) - D_T^2(\alpha; \Phi) \right),$$

$$|\alpha_t| \Phi \leq c, \quad \forall t \in \{0, \dots, T-1\}, \quad (7)$$

where  $D_T^1(\alpha; \Phi)$  and  $D_T^2(\alpha; \Phi)$  are given by Lemma 2. Equation (7) reflects the funding constraint imposed by the initial margin requirement. At any time  $t \in \{0, \dots, T-1\}$ , the bank's position in the contract has to be smaller than its "trading capacity"  $c/\Phi$ . Equation (7) can also be interpreted such that the bank has to finance every unit of the contract traded with  $\Phi$  units of its initial capital. Initial margin does not change the timing of cash flows, that is, settlement of positions and exchange of cash only takes place in the terminal period. Thus, the bank might default in the terminal period only.

In optimization problem  $\mathcal{P}_1$ , initial margin affects the bank's optimal trading strategy  $\alpha$  in two ways. Initial margin establishes upper and lower bounds on the bank's position in the contract,  $-c/\Phi \leq \alpha_t \leq c/\Phi$ ,  $t \in \{0, \dots, T-1\}$ . These bounds remain constant over time. In addition, initial margin may affect the

**Table 1**

**Effects of changes in the initial margin requirement,  $\Phi$ , on bank's trading strategy in the absence of variation margin.**

The table below shows the results of the bank's optimization problem  $\mathcal{P}_1$  for varying values of initial margin,  $\Phi$ . We vary one parameter at a time setting the other parameters to their default values.  $\hat{\alpha}_t$  denotes the bank's position in the contract on  $S^1$  at time  $t = 0, 1, 2, 3$ .  $vol = |\hat{\alpha}_0| + 1/3(\sum_{t=1}^3 |\hat{\alpha}_t|)$  denotes the bank's average trading volume over time.  $\#d^i$  and  $l^i$  denote the number of defaults on and the loss-given-default in relation to contracts 1 and 2, respectively.  $DR$  denotes the expectation of the loss-given-default (probability of default times loss-given-default), that is, the default risk the bank constitutes.  $J$  denotes the bank's expected utility.

$\Phi$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	vol	$\#d^1$	$\#d^2$	$l^1$	$l^2$	$DR$	$J$
0.0	-0.966121	-0.966121	-0.966121	-0.966122	0.966122	0	0	0.000	0.000	0.000	0.286396
0.1	-0.966121	-0.966121	-0.966121	-0.966121	0.966121	0	0	0.000	0.000	0.000	0.286396
0.2	-0.966121	-0.966121	-0.966121	-0.966121	0.966121	0	0	0.000	0.000	0.000	0.286396
0.3	-0.966121	-0.966121	-0.966121	-0.966121	0.966121	0	0	0.000	0.000	0.000	0.286396
0.4	-0.750000	-0.750000	-0.750000	-0.750000	0.750000	0	1	0.000	0.185	0.021	0.282806
0.5	-0.600000	-0.600000	-0.600000	-0.600000	0.600000	0	1	0.000	0.320	0.036	0.276607
0.6	-0.500000	-0.500000	-0.500000	-0.500000	0.500000	0	1	0.000	0.410	0.046	0.270929
0.7	-0.428571	-0.428571	-0.428571	-0.428571	0.428571	0	1	0.000	0.474	0.053	0.266158
0.8	-0.375000	-0.375000	-0.375000	-0.375000	0.375000	0	3	0.000	0.532	0.177	0.262205
0.9	-0.333333	-0.333333	-0.333333	-0.333333	0.333333	0	3	0.000	0.603	0.201	0.258960
1.0	-0.300000	-0.300000	-0.300000	-0.300000	0.300000	0	3	0.000	0.660	0.220	0.256275

delivery on the contract, and thus its payoff. This in turn changes the value of the default option and thus of the contract and may affect  $\alpha$  as well.

Since the objective function in (5) is strictly concave and the constraints in (7) are convex, optimization problem  $\mathcal{P}_1$  has a unique solution  $\hat{\alpha}$ . When evaluating  $\mathcal{P}_1$ , we use the parameterization of Section 4.2. We will vary the margin requirement per unit of contract held,  $\Phi$ , from 0 to  $\sigma_1 = 1.0$ , the maximum loss per unit of contract. The results of the numerical evaluation are presented in Table 1.

The bank's trading capacity, that is, its maximum number of contracts held, is unlimited when the margin requirement  $\Phi$  is zero. It drops to 0.3 when  $\Phi = 1.0$ . The optimal number of contracts held in the unconstrained case ( $\Phi = 0$ ) is -0.966. So any margin requirement greater than 0.311 constrains the bank's trading strategy. We see in Table 1 that for margin requirements greater than 0.3, the number of contracts decreases while the marginal utility of an increase of  $\alpha_t$  decreases for any  $t \in \{0, 1, 2, 3\}$ . Trading volume decreases due to the decrease in  $\alpha$ . At the same time, the number of defaults, the loss-given-default, and thus default risk increase when  $\Phi$  is increased.

An increase in the margin requirement on the bank's position in contract 1 increasingly constrains its ability to hedge its holding in contract 2. As soon as the constraint becomes binding, the bank can no longer hedge its liabilities in relation to contract 2 sufficiently and defaults in some states. The number of defaults as well as the loss-given-default increases, increasing the default risk of the bank. In other words, an increase in the margin requirement on contract 1, supposed to reduce the loss-given-default on contract 1, may increase the default risk in contract 2. From this perspective, the margin requirement can be viewed as an externality, shifting wealth from the bank and from the counterparty in contract 1 to the counterparty in contract 2.

#### 4.4. Variation Margin

We now come to the evaluation of the effects associated with variation margin. As we saw in Section 3.2, variation margin changes the timing of cash flows and thus of obligations related to a position in contract 1. This means that variation margin might change not only the loss-given-default, but also the timing of default. Additionally, variation margin changes the funding constraint.

In the presence of variation margin, the bank's optimization problem  $\mathcal{P}_0$  can be re-stated as follows:

$$\mathcal{P}_2 : \quad \max_{\alpha} \mathbf{E}[U(\alpha; \Phi)] = J(\Phi) \quad (8)$$

subject to

$$U(\alpha; \Phi) = u \left( c + \left[ \sum_{t=0}^{T-1} \alpha_t^1 \Delta S_t^1 \right]^+ + \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^+ - D^1(\alpha; \Phi) - D^2(\alpha; \Phi) \right) \quad (9)$$

$$- \lambda \left( \left[ \alpha_{t-1}^1 \Delta S_{t-1}^1 \right]^+ + \left[ S_T^2 \right]^+ - D^1(\alpha; \Phi) - D^2(\alpha; \Phi) \right)$$

$$|\alpha_t^1| \Phi \leq c + C_t^1, \quad \forall t \in \{0, \dots, T-1\}, \quad (10)$$

$D^1(\alpha; \Phi)$  and  $D^2(\alpha; \Phi)$  are given by Lemma 2, and  $C^1$  is given by (2). Problem  $\mathcal{P}_2$  has a unique solution  $\hat{\alpha}$ .

Variation margin changes the amount of cash available at the different points in time. A certain position in contract 1 at time 0 results in a cash flow, positive or negative, at time 1. Trading capacity at time 1 now equals  $(c + C_t^1)/\Phi$ ,  $0 < t \leq T$ . A position  $\alpha_0$  at time 0 not only affects utility and default penalties directly

by generating a profit or loss at time 1; it also affects the optimal positions at  $t > 0$  by altering the future trading capacities. This means that variation margin renders the bank's optimal strategy  $\alpha$  and thus equity  $E$  path-dependent.

Table 2 shows the results of the numerical evaluation of the bank's optimization problem in the presence of variation margin,  $\mathcal{P}_2$ . Note that the case  $\Phi = 0$  resembles the situation where the bank has to post variation margin only and no initial margin. The cases where  $\Phi > 0$  reflect situations where the bank is subject to both, initial and variation margin.

Observing the trading strategies of the bank,  $\alpha$ , for the different values of  $\Phi$  in Table 2, one notices that as  $\Phi$  increases, the bank shifts its holdings between the different states. For low values of  $\Phi$ , the bank does not invest in contract 1 at  $t = 0$  avoiding potential default at the end of the first period. The cost of default of a holding of contract 1 thus outweighs the benefits from hedging. An increase in  $\Phi$  gradually constrains the bank's position in contract 1. In other words, the benefits from a position in contract 1 during the first period increasingly outweigh the costs of the constraint. As the bank goes short in the contract at  $t = 0$ , its position generates cash in the down state at  $t = 3$ . This in turn allows the bank to increase its short position at  $t = 3$ , improving its hedge. Its position thus generates more cash at the terminal time, allowing the bank to deliver more on its liabilities related to contract 2, and thus decreasing the default penalty. In our case, the bank never benefits from variation margin, though.

Table 2

Effects of changes in the initial margin requirement,  $\Phi$ , on bank's trading strategy in the presence of variation margin.

The table below shows the results of the bank's optimization problem  $\mathcal{P}_2$  for varying values of initial margin,  $\Phi$ . We vary one parameter at a time setting the other parameters to their default values.  $\hat{\alpha}_t$  denotes the bank's position in the contract on  $S^1$  at time  $t = 0, 1, 2, 3$ .  $vol = |\hat{\alpha}_0| + 1/3(\sum_{t=1}^3 |\hat{\alpha}_t|)$  denotes the bank's average trading volume over time.  $\#d^i$  and  $l^i$  denote the number of defaults on and the loss-given-default in relation to contracts 1 and 2, respectively.  $DR$  denotes the expectation of the loss-given-default (probability of default times loss-given-default), that is, the default risk the bank constitutes.  $J$  denotes the bank's expected utility.

Values in brackets show the differences between the results of the similar optimization problem without variation margin,  $\mathcal{P}_1$ , as reported in Table 1, and optimization problem  $\mathcal{P}_2$ .

$\Phi$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	vol	$\#d^1$	$\#d^2$	$l^1$	$l^2$	DR	J
0.0	0.000000 (0.966121)	-0.966121 (0.000000)	-0.966121 (0.000000)	-0.148266 (0.817856)	0.693503 (-0.272619)	2 (2)	2 (2)	0.003000 (0.003000)	0.953000 (0.953000)	0.425000 (0.425000)	0.254114 (-0.032282)
0.1	0.000000 (0.966121)	-0.966121 (0.000000)	-0.966121 (0.000000)	-0.147623 (0.818498)	0.693289 (-0.272832)	1 (1)	2 (2)	0.000000 (0.000000)	0.961000 (0.961000)	0.320000 (0.320000)	0.254113 (-0.032283)
0.2	0.000000 (0.966121)	-0.966121 (0.000000)	-0.966121 (0.000000)	-0.147623 (0.818498)	0.693289 (-0.272832)	1 (1)	2 (2)	0.000000 (0.000000)	0.961000 (0.961000)	0.320000 (0.320000)	0.254113 (-0.032283)
0.3	-0.081372 (0.884749)	-0.850818 (0.115303)	-0.966121 (0.000000)	-0.213723 (0.752398)	0.676888 (-0.289233)	1 (1)	2 (2)	0.020000 (0.020000)	0.864000 (0.864000)	0.295000 (0.295000)	0.254379 (-0.032017)
0.4	-0.263304 (0.486696)	-0.387957 (0.362043)	-0.717087 (0.032913)	-0.361021 (0.388979)	0.488688 (-0.261312)	1 (1)	2 (2)	0.039000 (0.039000)	0.649000 (0.464000)	0.229000 (0.208444)	0.255420 (-0.027386)
0.5	-0.457380 (0.142620)	-0.096882 (0.503118)	-0.554262 (0.045738)	-0.517319 (0.082681)	0.629819 (0.029819)	1 (1)	2 (2)	0.023000 (0.023000)	0.422000 (0.102000)	0.148000 (0.112444)	0.259434 (-0.017173)
0.6	-0.500000 (0.000000)	-0.041667 (0.458333)	-0.458333 (0.041667)	-0.566609 (-0.066609)	0.688870 (0.188870)	0 (0)	1 (1)	0.000000 (0.000000)	0.380000 (-0.030000)	0.042000 (-0.003556)	0.261588 (-0.009341)
0.7	-0.428571 (0.000000)	-0.091837 (0.336734)	-0.397959 (0.030612)	-0.494176 (-0.065605)	0.572889 (0.144318)	0 (0)	2 (2)	0.000000 (0.000000)	0.456000 (-0.018286)	0.101000 (0.048302)	0.259379 (-0.006779)
0.8	-0.375000 (0.000000)	-0.117187 (0.257813)	-0.351562 (0.023438)	-0.451085 (-0.076085)	0.494112 (0.119112)	0 (0)	3 (3)	0.000000 (0.000000)	0.534000 (0.001500)	0.178000 (0.000500)	0.257051 (-0.005154)
0.9	-0.333333 (0.000000)	-0.129630 (0.203703)	-0.314815 (0.018518)	-0.417521 (-0.084188)	0.435470 (0.102137)	0 (0)	3 (3)	0.000000 (0.000000)	0.597000 (-0.006334)	0.199000 (-0.002111)	0.254918 (-0.004042)
1.0	-0.300000 (0.000000)	-0.135000 (0.165000)	-0.285000 (0.015000)	-0.390642 (-0.090642)	0.390214 (0.090214)	0 (0)	3 (3)	0.000000 (0.000000)	0.648000 (-0.012000)	0.216000 (-0.004000)	0.253021 (-0.003254)

On the other hand, margin clearly serves its purpose. It eliminates default risk in relation to the bank's position in contract 1. However, this elimination comes at non-trivial costs. We saw that initial margin already imposes significant costs on the bank resulting in higher default rates, higher overall loss-given-default, a drop in trading volume, as well as a decrease in expected utility. These effects set in as soon as the initial margin requirement constrains the bank's portfolio choice. They increase with an increase in the initial margin requirement. Variation margin, while offering potential benefits to the bank in terms of higher trading capacity in the second period, imposes additional overall costs. Compared to the case without variation margin, default risk increases considerably for low values of  $\Phi$ . Although the additional costs imposed by variation margin decrease with an increase in  $\Phi$ , they never vanish and the bank's expected utility is always lower than in the case without variation margin. Although margin serves its purpose in relation to contract 1, and thus benefits the counterparty to contract 1, it does so at high costs to the bank and, noticeably, to the counterparty to contract 2. As we mentioned previously, this means that margins might impose an externality on the counterparties to contract 2 and to the shareholders of the bank.

## 5. Bilateral Default Risk

So far, we have investigated the optimization problem of a single bank trading with a (default) risk-free counterparty. In this setting, margin requirements imposed costs on the bank but they did not yield significant benefits to the bank under consideration. Most importantly, the margin requirements did not affect the counterparty. In particular, the counterparty never defaulted. We now relax the assumption of a risk-free counterparty. More precisely, we introduce a second bank that is similar to bank 1 except that it holds a unit *short* position in contract 2. This means that bank 2's holding in contract 2 is symmetric to bank 1's holding. Similar to bank 1, bank 2 may hedge its holding by trading in contract 1. Banks trade contract 1 only with each other. Bank 2 may default and will be subject to the same margin requirements as bank 1. As opposed to the case discussed in the previous section, banks may now derive additional benefit from margining as it may lower counterparty default risk. Moreover, a change in the margin requirement might now have indirect, reciprocal, effects. As we saw in the previous section, whenever the margin requirement imposes a constraint on the bank, an increase in the margin requirement engenders a cost for a bank. Now that the counterparty may default, the effect of margin requirements might be ambiguous: whereas it might

still constrain the bank's trading strategy, it might at the same time reduce counterparty default risk in the contract. In other words, margin requirements now not only engender costs but they may also yield benefits to a bank. In the remainder of this section we will investigate those trade-offs.

## 5.1. Optimization Problem

We assume that both banks have the same strictly concave, non-decreasing utility function  $u$ , and that they face the same default penalty  $\lambda z^+$ , where  $\lambda > 0$  denotes the default penalty and  $z$  denotes the amount the bank defaulted on.

Before we state the banks' optimization problems, we discuss their deliveries on their obligations. Obviously, the deliveries now depend on the deliveries of the respective counterparty. Let  $L_T^{j1}(\alpha^j) = \left[ \sum_{t=0}^{T-1} \alpha_t^j \Delta S_t^1 \right]^-$  denote bank  $j$ 's obligations in relation to contract 1 at time  $T$ . The assets available for distribution to the bank's creditors are given by

$$A_T^j(\alpha^j; \alpha^k, \Phi) = c + D_T^{k1}(\alpha^k; \alpha^j, \Phi) + \left[ \sum_{t=0}^{T-1} \beta^j \Delta S_t^2 \right]^+,$$

where  $\beta^1 = 1$  and  $\beta^2 = -1$ . The assets comprise the bank's initial capital  $c$ , the delivery by the (risky) counterparty on contract 1 as well as the delivery by the (risk-free) counterparty on contract 2.

**Lemma 4** *In the absence of margin requirements, the delivery on  $L_T^{j1}(\alpha^j)$ , the liabilities of bank  $j$ ,  $j = 1, 2$ , in relation to the position in  $S^1$ , is given by*

$$D_T^{j1}(\alpha^j; \alpha^k, \Phi) \equiv \begin{cases} L_T^{j1}(\alpha^j) \min \left\{ \frac{A_T^j(\alpha^j; \alpha^k, \Phi)}{L_T^j(\alpha^j)}, 1 \right\} & \text{if } L_T^j(\alpha^j) > 0 \text{ and } t = T, \\ 0 & \text{otherwise,} \end{cases}$$

for  $(j, k) \in \{(1, 2), (2, 1)\}$ . Similarly for  $L_T^{j2}(\beta^j)$ .

Both banks maximize expected utility of terminal equity by optimally choosing  $\alpha^j$ , the number of units of the contract on  $S^1$ . We thus have the following optimization problem:

$$\mathcal{P}_3 : \begin{cases} \max_{\alpha^1} \mathbf{E} [U^1(\alpha^1; \alpha^2, \Phi)] = J^1(\alpha^1; \alpha^2) \\ \max_{\alpha^2} \mathbf{E} [U^2(\alpha^2; \alpha^1, \Phi)] = J^2(\alpha^2; \alpha^1) \end{cases} \quad (11)$$

subject to

$$U^1(\alpha^1; \alpha^2, \Phi) = u \left( c + D_T^{21}(\alpha^2; \alpha^1, \Phi) + \left[ \sum_{t=0}^{T-1} \beta^1 \Delta S_t^2 \right]^+ - D_T^{11}(\alpha^1; \alpha^2, \Phi) - D_T^{12}(\alpha^1; \alpha^2, \Phi) \right) \\ - \lambda \left( \left[ \sum_{t=0}^{T-1} \alpha_t^1 \Delta S_t^1 \right]^- + \left[ \sum_{t=0}^{T-1} \beta^1 \Delta S_t^2 \right]^- - D_T^{11}(\alpha^1; \alpha^2, \Phi) - D_T^{12}(\alpha^1; \alpha^2, \Phi) \right), \quad (12)$$

$$U^2(\alpha^2; \alpha^1, \Phi) = u \left( c + D_T^{11}(\alpha^1; \alpha^2, \Phi) + \left[ \sum_{t=0}^{T-1} \beta^2 \Delta S_t^2 \right]^+ - D_T^{21}(\alpha^2; \alpha^1, \Phi) - D_T^{22}(\alpha^2; \alpha^1, \Phi) \right) \\ - \lambda \left( \left[ \sum_{t=0}^{T-1} \alpha_t^2 \Delta S_t^1 \right]^- + \left[ \sum_{t=0}^{T-1} \beta^2 \Delta S_t^2 \right]^- - D_T^{21}(\alpha^2; \alpha^1, \Phi) - D_T^{22}(\alpha^2; \alpha^1, \Phi) \right), \quad (13)$$

and

$$\alpha_t^1 = -\alpha_t^2, \quad \forall t \in \{0, \dots, T\}. \quad (14)$$

A bank's equity at the terminal time  $T$  thus comprises its initial capital  $c$  and the delivery by its counterparty on its position in  $S^1$ , and its position in  $S^2$ ; it is reduced by the deliveries on the obligations in relation to its holdings in  $S^1$  and  $S^2$ , respectively.  $D^{j1}(\alpha^j; \alpha^k, \Phi)$  and  $D^{j2}(\alpha^j; \alpha^k, \Phi)$  are given by Lemma 4. The utility a bank gains from its terminal equity in a given state is potentially reduced by a default penalty. Equation (14) reflects the market clearing condition for the trades in contract 1.

**Proposition 2** *There exists a unique solution  $(\hat{\alpha}^1, \hat{\alpha}^2)$  to  $\mathcal{P}_3$ .*

The result follows from the the strict concavity of the objective functions (11). Obviously, the banks' strategies are interdependent or reciprocal. The two banks have full information on each other, meaning that their strategies are fully revealing.



## 5.2. Parameterization of Numerical Evaluation

We use the same parameterization for the numerical evaluation as in Section 4.2. The additional parameter  $\beta^j$ ,  $j = 1, 2$ , reflecting bank  $j$ 's endowment in the second contract, takes values  $+1$  for bank 1 and  $-1$  for bank 2.

Both banks will try to hedge their holdings by entering into positions in  $S^1$ . Bank 1 will enter into a short position in contract 1, as previously. Bank 2, with a symmetric endowment in contract 2, will take a long position in contract 1. The sensitivities of the optimal solution to the various parameters, such as  $c$ ,  $\mu/\sigma$ ,  $\lambda$ , are similar to those presented in Section 4.2.

## 5.3. Initial Margin

We now turn to the evaluation of the effects of initial margin on the banks' optimization problem. As in the case of unilateral default risk, initial margin changes the delivery rates of the banks. The following result states the delivery rates of the banks when the counterparty to contract 1 may default.

Let  $L_T^{j1}(\alpha^j) = \left[ \sum_{t=0}^{T-1} \alpha_t^j \Delta S_t^1 \right]^-$  denote bank  $j$ 's obligations in relation to contract 1 at time  $T$ . If  $L_T^{j1}(\alpha^j) < |\alpha_{T-1}^j| \Phi$ , the bank delivers  $L_T^{j1}(\alpha^j)$ . If  $L_T^{j1}(\alpha^j) \geq |\alpha_{T-1}^j| \Phi$ , the bank delivers  $|\alpha_{T-1}^j| \Phi$  plus a pro-rata share of any excess assets available for distribution. The assets are given by

$$A_T^j(\alpha^j; \alpha^k, \Phi) = c + \max\{-|\alpha_{T-1}^j| \Phi, -L_T^{j1}(\alpha^j)\} + D_T^{j1}(\alpha^k; \alpha^j, \Phi) + \left[ \sum_{t=0}^{T-1} \beta^j \Delta S_t^2 \right]^+.$$

The delivery on the bank's position in  $S^1$  is now no longer risk-free and may thus be less than the contractual entitlement  $\left[ \sum_{t=0}^{T-1} \alpha_t^j \Delta S_t^1 \right]^+$ .

**Lemma 5** *If banks have to post initial margin, the delivery on  $L_T^{j1}(\alpha^j)$ , the liability of bank  $j$ ,  $j = 1, 2$ , in relation to the position in  $S^1$ , is given by*

$$D_T^{j1}(\alpha^j; \alpha^k, \Phi) \equiv \begin{cases} \min \left\{ |\alpha_{T-1}^j| \Phi + \max \left\{ 0, (L_T^{j1}(\alpha^j) - |\alpha_{T-1}^j| \Phi) \frac{A_T^j(\alpha^j; \alpha^k, \Phi)}{L_T^{j1}(\alpha^j)} \right\}, L_T^{j1}(\alpha^j) \right\} & \text{if } L_T^{j1}(\alpha^j) > 0 \text{ and } t = T, \\ 0 & \text{otherwise,} \end{cases}$$

for  $(j,k) \in \{(1,2), (2,1)\}$ . Similarly for  $L_T^{j2}(\beta^j)$ .

We expect that margin has not only direct effects on the two banks but also indirect effects. For example, the holding in the contract might not only be constrained by a bank's funding limit but also by the other bank's trading strategy through the market clearing condition. Moreover, a higher margin requirement, by constraining the counterparty's ability to hedge its holding in  $S^1$ , may reduce its ability to perform on its obligations related to  $S^1$ . The banks' optimization problem in the presence of initial margin, which we denote by  $\mathcal{P}_4$ , is similar to problem  $\mathcal{P}_3$  in Equations (11) to (14) except for the following funding constraint imposed by initial margin:

$$|\alpha_t^j| \Phi \leq c, \quad \forall t \in \{0, \dots, T-1\}, \quad j = 1, 2. \quad (15)$$

The delivery rates of the banks,  $D_T^{ji}(\alpha^j; \alpha^k, \Phi)$ ,  $i = 1, 2$  are given by Lemma 5. There exists a unique solution  $\hat{\alpha}$  to  $\mathcal{P}_4$  since the objective functions of the agents, given by Equation (11), are strictly concave, and the constraints (12) to (15) are convex.

To evaluate  $\mathcal{P}_4$ , we use the parameterization of Section 5.2. We vary the initial margin requirement per unit of contract held,  $\Phi$ , from 0 to  $\sigma_1 = 1.0$ , the maximum loss per unit of contract. The results of the numerical evaluation are shown in Table 3.

In case of bilateral default risk, a bank may further benefit from margin since margin reduces the bank's potential and actual exposure to its counterparty to contract 1. In the current set-up, bank 1 benefits from this effect. Its expected utility increases when  $\Phi$  increases, for low values of  $\Phi$ . For high values of  $\Phi$ , the costs again exceed the benefits of margin and bank 1's expected utility drops. For bank 2, the costs of margin always exceed its benefits, and bank 2's expected utility falls when  $\Phi$  is increased. Default risk of both banks increases with  $\Phi$ , and trading volume drops. We can conclude that the effects of initial margin in case of bilateral default risk are similar to those in case of unilateral default risk.

**Table 3**

**Effects of changes in the initial margin requirement,  $\Phi$ , on banks' trading strategy in the absence of variation margin.**

The table below shows the results of the banks' optimization problem  $\mathcal{P}_4$  for varying values of initial margin,  $\Phi$ .  $\hat{\alpha}_t^j$  denotes bank  $j$ 's position in the contract on  $S^1$  at time  $t = 0, 1, 2, 3$ ,  $j = 1, 2$ .  $vol = |\hat{\alpha}_0^j| + 1/3(\sum_{t=1}^3 |\hat{\alpha}_t^j|)$  denotes the banks' average trading volume over time.  $\#d^{ji}$  and  $l^{ji}$  denote the number of defaults on and the loss-given-default in relation to contracts 1 and 2 by bank  $j$ , respectively.  $DR^j$  denotes the expectation of the loss-given-default (probability of default times loss-given-default), that is, the default risk the bank constitutes.  $J^j$  denotes the bank's expected utility.

$\Phi$	$j$	$\hat{\alpha}_0^j$	$\hat{\alpha}_1^j$	$\hat{\alpha}_2^j$	$\hat{\alpha}_3^j$	vol	$\#d^{j1}$	$\#d^{j2}$	$l^{j1}$	$l^{j2}$	$DR^j$	$J^j$
0.0	1	-1.499139	-1.478607	-1.504437	-1.509362	1.511156	0	0	0.000	0.000	0.000	0.251255
	2	1.499139	1.478607	1.504437	1.509362	1.511156	0	0	0.000	0.000	0.000	0.251255
0.1	1	-1.474402	-1.306276	-1.506764	-1.448217	1.549959	0	0	0.000	0.000	0.000	0.257565
	2	1.474402	1.306276	1.506764	1.448217	1.549959	1	0	0.000	0.000	0.000	0.249702
0.2	1	-1.393057	-1.191758	-1.380462	-1.281730	1.501464	0	0	0.000	0.000	0.000	0.263384
	2	1.393057	1.191758	1.380462	1.281730	1.501464	2	0	0.071	0.000	0.016	0.243328
0.3	1	-1.000000	-1.000000	-1.000000	-1.000000	1.000000	0	0	0.000	0.000	0.000	0.286257
	2	1.000000	1.000000	1.000000	1.000000	1.000000	0	0	0.000	0.000	0.000	0.211618
0.4	1	-0.750000	-0.750000	-0.750000	-0.750000	0.750000	0	1	0.000	0.185	0.021	0.282806
	2	0.750000	0.750000	0.750000	0.750000	0.750000	0	3	0.000	0.345	0.115	0.169816
0.5	1	-0.600000	-0.600000	-0.600000	-0.600000	0.600000	0	1	0.000	0.320	0.036	0.276607
	2	0.600000	0.600000	0.600000	0.600000	0.600000	0	3	0.000	0.690	0.230	0.142060
0.6	1	-0.500000	-0.500000	-0.500000	-0.500000	0.500000	0	1	0.000	0.410	0.046	0.270929
	2	0.500000	0.500000	0.500000	0.500000	0.500000	0	3	0.000	0.920	0.307	0.122800
0.7	1	-0.428571	-0.428571	-0.428571	-0.428571	0.428571	0	1	0.000	0.474	0.053	0.266158
	2	0.428571	0.428571	0.428571	0.428571	0.428571	0	3	0.000	1.084	0.361	0.108697
0.8	1	-0.375000	-0.375000	-0.375000	-0.375000	0.375000	0	3	0.000	0.533	0.178	0.262205
	2	0.375000	0.375000	0.375000	0.375000	0.375000	0	3	0.000	1.208	0.403	0.097941
0.9	1	-0.333333	-0.333333	-0.333333	-0.333333	0.333333	0	3	0.000	0.603	0.201	0.258960
	2	0.333333	0.333333	0.333333	0.333333	0.333333	0	3	0.000	1.303	0.434	0.089473
1.0	1	-0.300000	-0.300000	-0.300000	-0.300000	0.300000	0	3	0.000	0.660	0.220	0.256275
	2	0.300000	0.300000	0.300000	0.300000	0.300000	0	3	0.000	1.380	0.460	0.082636

## 5.4. Variation Margin

Let us now turn to the evaluation of variation margin in case of bilateral default risk. First, we discuss the banks' delivery rates in the presence of variation margin. A bank holds  $c + C_t^{j1}$  units of the risk-free asset at time  $t$ , where  $C_t^{j1}$  is given by

$$C_t^{j1} = \begin{cases} \sum_{s=1}^t \left( D_s^{k1}(\alpha^k; \alpha^j, \Phi) - D_s^{j1}(\alpha^j; \alpha^k, \Phi) \right) & \text{for } t = \{1, \dots, T\}, \\ 0 & \text{for } t = 0, \end{cases} \quad (16)$$

where  $(j, k) \in \{(1, 2), (2, 1)\}$ . In combination with initial margin, bank  $j$ 's funding constraint becomes  $|\alpha_t^j| \Phi \leq c + C_t^{j1}$ , for all  $t \in \{0, \dots, T-1\}$ . Let  $L_t^{j1}(\alpha^j) = [\alpha_{t-1}^j \Delta S_{t-1}^1]^-$ . If  $L_t^{j1}(\alpha^j) < |\alpha_{t-1}^j| \Phi$ , the bank delivers  $L_t^{j1}(\alpha^j)$ . If  $L_t^{j1}(\alpha^j) \geq |\alpha_{t-1}^j| \Phi$ , the bank delivers  $|\alpha_{t-1}^j| \Phi$  immediately plus a pro-rata share of any excess assets available for distribution at the terminal time  $T$ . The assets are given by

$$A_t^j(\alpha^j; \alpha^k, \Phi) = c + C_t^{j1} + \mathbb{1}_{\{t=T\}} \left[ \sum_{t=0}^{T-1} \Delta S_t^2 \right]^+.$$

A bank's assets comprise its initial capital, the accumulated cash flow in relation to its position in contract 1 until the time of default, and, if  $t = T$ , the delivery on its holding in contract 2.

**Lemma 6** *Let  $0 < \tau \leq \infty$  denote the time of the bank's default and  $s_t \equiv \min\{\tau, t\}$ . In the presence of both initial and variation margin, the delivery on  $L_s^{j1}(\alpha^j)$ , the liability in relation to the position in  $S^1$ , is given by*

$$D_t^{j1}(\alpha^j) = \min \left\{ |\alpha_{t-1}^j| \Phi, L_t^{j1}(\alpha^j) \right\}$$

and

$$D_T^{j1}(\alpha^j) = [L_{s_t}^{j1}(\alpha^j) - D_{s_t}^{j1}(\alpha^j)]^+ \frac{A_T^j(\alpha^j)}{L_T^j(\alpha^j)},$$

for  $j = 1, 2$ . Similarly for  $L_T^{j2}(\beta^j)$ .

The funding constraint (10) becomes

$$|\alpha_t^j| \Phi \leq c + C_t^{j1}, \quad \forall t \in \{0, \dots, T-1\}, \quad j = 1, 2. \quad (17)$$

We denote the banks' optimization problem with constraint (17) substituted for constraint (15) by  $\mathcal{P}_5$ . This problem has a unique solution  $(\hat{\alpha}^1, \hat{\alpha}^2)$ . Table 4 presents the results of the numerical evaluation.

We immediately notice in Table 4 that high values of  $\Phi$  completely mitigate default risk in relation to contract 1. On other hand, high values of  $\Phi$  increase default risk in relation to contract 2 implying that margin shifts wealth from the counterparty to contract 2 and from the bank to the counterparty to contract 1. Trading volume drops when  $\Phi$  is increased. As concerns expected utility, neither bank 1 nor bank 2 sufficiently benefit from margining for the benefits to outweigh the costs.

In Section 4, where we investigated the case of unilateral default risk, we found that the costs of margin tend to exceed its benefits, and that they always do when margin levels,  $\Phi$ , are high, that is, when they are close to the maximum exposure in a contract. In this section, we investigated the case of bilateral default risk where margin has additional potential benefits for a bank. However, even in this case we found that the costs of margin tend to exceed its benefits.

Of course, our analysis hinges on the assumption that margin does actually impose costs on the banks. If collateral to meet margin requirements never becomes scarce, the funding constraint will never be binding, and thus margin will not impose any costs on a bank. Many banks may indeed have large amounts of collateral at their disposal. However, if the size of (net) derivatives positions increases as it has been in the past, collateral might indeed become scarcer. Furthermore, collateral tends to become scarcer in times of market crises when asset values tend to deteriorate and counterparties tend to reduce to set of securities accepted as collateral to high quality assets. Thus, during such crises, when counterparties increase their margin requirements, those effects might even further increase the costs of margining.

Table 4

Effects of changes in the initial margin requirement,  $\Phi$ , on banks' trading strategy in the presence of variation margin.

The table below shows the results of the banks' optimization problem  $\mathcal{P}_5$  for varying values of initial margin,  $\Phi$ .  $\hat{\alpha}_t^j$  denotes bank  $j$ 's position in the contract on  $S^1$  at time  $t = 0, 1, 2, 3$ ,  $j = 1, 2$ .  $vol = \left| \hat{\alpha}_0^j + 1/3(\sum_{r=1}^3 \hat{\alpha}_t^r) \right|$  denotes the banks' average trading volume over time.  $\#d^{j,t}$  and  $l^{j,t}$  denote the number of defaults on and the loss-given-default in relation to contracts 1 and 2 by bank  $j$ , respectively.  $DR^j$  denotes the expectation of the loss-given-default (probability of default times loss-given-default), that is, the default risk the bank constitutes.  $J^j$  denotes the bank's expected utility.

Values in brackets show the differences between the results of the similar optimization problem without variation margin,  $\mathcal{P}_3$ , as reported in Table 3, and optimization problem  $\mathcal{P}_4$ .

$\Phi$	$j$	$\hat{\alpha}_0^j$	$\hat{\alpha}_1^j$	$\hat{\alpha}_2^j$	$\hat{\alpha}_3^j$	vol	$\#d^{j,0}$	$\#d^{j,1}$	$\#d^{j,2}$	$l^{j,0}$	$l^{j,1}$	$l^{j,2}$	$DR^j$	$J^j$
0.0	1	0.000000	-2.452494	-1.539202	-0.727277	1.572991	4	2	2	0.014000	0.710000	0.483000	0.186696	
	2	0.000000	(-0.973887)	(-0.034765)	(0.783085)	(0.061835)	(4)	(2)	(2)	(0.014000)	(0.710000)	(0.483000)	(-0.064559)	
0.1	1	0.000000	2.452494	1.539202	0.727277	1.572991	2	2	2	0.829000	0.457000	0.572000	0.122585	
	2	0.000000	(-0.973887)	(-0.034765)	(-0.782085)	(-0.061835)	(2)	(2)	(2)	(0.829000)	(0.457000)	(0.572000)	(-0.128670)	
0.2	1	0.000000	-2.507192	-1.483431	-0.615385	1.535336	1	2	2	0.000000	0.786000	0.262000	0.184257	
	2	0.000000	(-1.200916)	(0.023333)	(0.832832)	(-0.014623)	(1)	(2)	(2)	(0.000000)	(0.786000)	(0.262000)	(-0.073308)	
0.3	1	0.000000	2.507192	1.483431	0.615385	1.535336	3	2	2	0.890000	0.476000	0.758000	0.120239	
	2	0.000000	(-1.474402)	(-0.023333)	(-0.832832)	(-0.014623)	(2)	(2)	(2)	(0.890000)	(0.476000)	(0.758000)	(-0.129463)	
0.4	1	0.000000	-1.500000	-1.473333	-0.533334	1.168889	2	2	2	0.000000	0.817000	0.363000	0.219187	
	2	0.000000	(-0.308242)	(-0.092871)	(0.748396)	(-0.325275)	(2)	(2)	(2)	(0.000000)	(0.817000)	(0.363000)	(-0.044197)	
0.5	1	0.000000	1.500000	1.473333	0.533334	1.168889	3	3	3	0.488000	0.765000	0.835000	0.105380	
	2	0.000000	(-0.308242)	(-0.092871)	(0.748396)	(0.325275)	(-1)	(-3)	(-3)	(-0.417000)	(-0.765000)	(-0.835000)	(-0.137948)	
0.6	1	0.000000	-0.998591	-0.998872	-0.470860	0.823107	2	2	2	0.000000	0.840000	0.373000	0.245024	
	2	0.000000	(0.001409)	(0.000128)	(0.529140)	(-0.176893)	(2)	(2)	(2)	(0.000000)	(0.840000)	(0.373000)	(-0.041233)	
0.7	1	0.000000	0.998591	0.998872	0.470860	0.823107	2	3	3	0.150000	1.065000	0.675000	0.089875	
	2	0.000000	(-0.001409)	(-0.000128)	(-0.529140)	(-0.176893)	(2)	(3)	(3)	(0.150000)	(1.065000)	(0.675000)	(-0.121743)	
0.8	1	0.000000	-0.453473	-0.723043	-0.507387	0.561301	1	2	2	0.032000	0.659000	0.230000	0.251040	
	2	0.000000	(0.296527)	(0.026957)	(0.242613)	(-0.188699)	(1)	(1)	(1)	(0.032000)	(0.659000)	(0.230000)	(-0.031766)	
0.9	1	0.000000	0.453473	0.723043	0.507387	0.561301	2	4	4	0.018000	1.250000	0.845000	0.086842	
	2	0.000000	(-0.296527)	(-0.026957)	(-0.242613)	(-0.188699)	(2)	(1)	(1)	(0.018000)	(0.905000)	(0.730000)	(-0.082974)	
1.0	1	0.000000	-0.127542	-0.557049	-0.213443	0.644698	1	1	1	0.021000	0.571000	0.132000	0.258435	
	2	0.000000	(-0.474588)	(-0.042951)	(0.386557)	(0.044698)	(1)	(0)	(0)	(0.021000)	(0.571000)	(0.096000)	(-0.018172)	
1.1	1	0.000000	0.127542	0.557049	0.213443	0.644698	0	3	3	0.000000	1.236000	0.412000	0.090025	
	2	0.000000	(-0.474588)	(-0.042951)	(-0.386557)	(-0.044698)	(0)	(0)	(0)	(0.000000)	(0.546000)	(0.182000)	(-0.052035)	
1.2	1	0.000000	-0.041667	-0.458333	-0.125000	0.791667	0	1	1	0.000000	0.579000	0.064000	0.259350	
	2	0.000000	(-0.458333)	(0.041667)	(0.375000)	(-0.291667)	(0)	(0)	(0)	(0.000000)	(0.169000)	(0.018000)	(-0.011579)	
1.3	1	0.000000	0.041667	0.458333	0.125000	0.791667	0	3	3	0.000000	1.218000	0.406000	0.092056	
	2	0.000000	(-0.458333)	(-0.041667)	(-0.375000)	(-0.291667)	(0)	(0)	(0)	(0.000000)	(0.298000)	(0.099000)	(-0.030744)	
1.4	1	0.000000	-0.091837	-0.397959	-0.153061	0.642857	0	1	1	0.000000	0.598000	0.066000	0.257988	
	2	0.000000	(0.336734)	(0.030612)	(0.275510)	(-0.214286)	(0)	(0)	(0)	(0.000000)	(0.124000)	(0.013000)	(-0.008170)	
1.5	1	0.000000	0.091837	0.397959	0.153061	0.642857	0	3	3	0.000000	1.303000	0.454000	0.086631	
	2	0.000000	(-0.336734)	(-0.030612)	(-0.275510)	(-0.214286)	(0)	(0)	(0)	(0.000000)	(0.219000)	(0.073000)	(-0.022066)	
1.6	1	0.000000	-0.117187	-0.351562	-0.164062	0.539062	0	3	3	0.000000	0.636000	0.212000	0.256090	
	2	0.000000	(0.257813)	(0.023438)	(0.210938)	(-0.164062)	(0)	(0)	(0)	(0.000000)	(0.103000)	(0.034000)	(-0.006115)	
1.7	1	0.000000	0.117187	0.351562	0.164062	0.539062	0	3	3	0.000000	1.375000	0.458000	0.081290	
	2	0.000000	(-0.257813)	(-0.023438)	(-0.210938)	(-0.164062)	(0)	(0)	(0)	(0.000000)	(0.167000)	(0.055000)	(-0.016651)	
1.8	1	0.000000	-0.129630	-0.314815	-0.166667	0.462963	0	3	3	0.000000	0.686000	0.229000	0.254179	
	2	0.000000	(0.203703)	(0.018518)	(0.166666)	(-0.129630)	(0)	(0)	(0)	(0.000000)	(0.083000)	(0.028000)	(-0.004781)	
1.9	1	0.000000	0.129630	0.314815	0.166667	0.462963	0	3	3	0.000000	1.436000	0.479000	0.076439	
	2	0.000000	(-0.203703)	(-0.018518)	(-0.166666)	(-0.129630)	(0)	(0)	(0)	(0.000000)	(0.133000)	(0.045000)	(-0.013034)	
2.0	1	0.000000	-0.135000	-0.285000	-0.165000	0.405000	0	3	3	0.000000	0.728000	0.243000	0.252421	
	2	0.000000	(0.165000)	(0.015000)	(-0.135000)	(-0.105000)	(0)	(0)	(0)	(0.000000)	(0.068000)	(0.023000)	(-0.003854)	
2.1	1	0.000000	0.135000	0.285000	0.165000	0.405000	0	3	3	0.000000	1.487000	0.496000	0.072145	
	2	0.000000	(-0.165000)	(-0.015000)	(-0.135000)	(-0.105000)	(0)	(0)	(0)	(0.000000)	(0.107000)	(0.036000)	(-0.010491)	

As we saw in the preceding sections, in case of low margin rates,  $\Phi$ , margin may indeed serve its purpose by mitigating the default risk in a derivatives contract. It comes at a cost, though. Particularly in case of higher margin rates, the costs of margining outweigh its benefits, lowering trading volume, bank capital, and welfare. Note that margin levels in OTC markets have been increasing and are currently equal to 59% of exposure on average.<sup>13</sup> Margins for exchange-traded contracts are typically much higher, between 95% and 97% of the value-at-risk of a position.<sup>14</sup>

## 6. Multilateral Default Risk

So far, we have been analyzing a situation with two banks trading, exclusively, with each other.<sup>15</sup> Such a setting potentially underestimates the constraints imposed by margins. Banks realistically trade with more than one counterparty. We now briefly describe how margin affects banks in such a multilateral setting.

We imagine a tripartite trading relationship with bilateral default risk. Suppose bank 1 holds a unit long position in contract 1 with bank 2,  $\alpha^{12} = 1$ , and a unit short position with bank 3,  $\alpha^{13} = -1$ . Bank 1 thus is “market-neutral” since the contractual payments from the two positions exactly offset each other. Since either counterparty may default, though, bank 1’s default risk is positive. Bank 1 therefore posts collateral to both counterparties, and the margin requirement amounts to  $(|\alpha^{12}| + |\alpha^{13}|) \Phi$ . Note the difference with respect to the setting in Sections 3 to 5 where the bank would have posted  $|\alpha^{12} + \alpha^{13}| \Phi$  as we assumed that the bank holds its entire position with one counterparty. Splitting a position among many counterparties thus tightens the funding constraint imposed by the initial margin requirement.

It is important to note that variation margin is also affected by such a splitting of positions. At any time, one of bank 1’s two positions will entitle the bank to a receipt of variation margin, whereas the other position will oblige it to a payment. The receipt of variation margin is obviously subject to default risk. Thus, whereas bank 1 would not be exposed to any default risk if it held its entire position with only one counterparty, a splitting of the position increases default risk.

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<sup>13</sup>More detailed statistics are available in International Swaps and Derivatives Association (2006).

<sup>14</sup>For details and related literature, we refer the reader to Knott and Mills (2002).

<sup>15</sup>Alternatively, our setting could be interpreted as one with two groups of banks where a bank of one group only trades with exactly one bank of the other group.

To reduce the burden of initial margin for bank 1, it might agree with its counterparties to re-use collateral. Bank 1 may then use the collateral received from bank 2 to meet its margin requirement from bank 3, and vice versa.<sup>16</sup> This practice does not necessarily eliminate bank 1's default risk as initial margin might be smaller than potential future exposure to either of its two positions. It only reduces some of the inefficiencies of margining.

In order to also reduce default risk, participants might pool their positions to diversify their default risk. If all positions in a market were executed with and pooled in a central counterparty, the default risk of any market participant might be significantly reduced. Note that the net position of every market participant remains the same, and the margin requirement remains unchanged compared to the case of re-hypothecation. However, since all positions are held with a single, highly diversified counterparty, default risk might be significantly reduced. Assuming a default probability of the central counterparty of zero, bank 1 in the example above, could eliminate its default risk.

It is hypothetical that such a central counterparty would be able to completely diversify away its default risk. More realistically, it will always be exposed to some default risk, albeit smaller than the aggregate default risk in case of bilateral relationships. Typically, the potential losses of a central counterparty are shared among its customers, often according to the size of their net positions. Some of the risk of a central counterparty may also be absorbed by a lender-of-last-resort. In a setting with incomplete information, which we do not consider here, a central counterparty may provide additional benefits, for example, by centralizing information on market participants' positions and their default risk.

## 7. Dynamic Market Model

In the previous sections, we analyzed default risk mitigation mechanisms in case of a small number of banks. In this section, we present a simulation model that allows us to analyze effects of default risk mitigation mechanisms in a market with a large number of heterogeneous banks. In this multi-period, dynamic model, banks holding some exogenous random position in an illiquid contract trade another, liquid contract with each other to hedge the price (market) risk of their initial holding, as in the previous sections. Upon delivery

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<sup>16</sup>The re-use of collateral is typically called *re-hypothecation*; cf. International Swaps and Derivatives Association (2006).



of the hedging contract, banks may default. Step by step, we introduce initial and variation margin as well as a central counterparty and investigate their effects on banks' expected utility, capital, default rates, loss-given-default, and trading volume.

Building on the analysis developed in Sections 2 to 6, the aim in conducting the simulations is to introduce additional characteristic features of derivatives markets in order to gain further insight into the effects of margining. First, we take into account the high level of concentration among derivatives market participants. Secondly, we consider that many market participants tend to have increasingly significant credit exposures in relation to derivatives contracts. Thirdly, banks mainly pledge cash as collateral and, typically, not risky assets such as other derivatives contracts. And lastly, default risk exposures to a central counterparty have a zero capital requirement.<sup>17</sup>

## 7.1. Economic Setting

We consider an economy with a real sector and a financial sector comprising  $N$  banks indexed by  $i = 1, \dots, N$ . Time is measured in discrete intervals  $t = 0, \dots, T$ . Banks are exposed to a stochastic default-free short-term rate. The interest rate,  $r$ , in the continuous-time limit, satisfies the following continuous-time stochastic process:

$$d \ln r_t = \left( u_t + \frac{1}{\sigma_t} \frac{\partial \sigma}{\partial t} \ln r_t dt \right) + \sigma_{r,t} dW_t, \quad (18)$$

where  $u_t$  and  $\sigma_{r,t}$  are the (time-dependent) drift and diffusion coefficients, respectively, and  $W$  represents a standard Brownian motion. Equation (18) resembles the model of Black, Derman, and Toy (1990).<sup>18</sup>

<sup>17</sup>These stylized facts refer to data reported in OCC (2006) and International Swaps and Derivatives Association (2006).

<sup>18</sup>Both, interest rate and hazard rate process can easily be calibrated to actual term structures as described in Black, Derman, and Toy (1990) and Skinner and Diaz (2001). For the calibration we use the following functional forms. The interest rate at time  $t$  is given by

$$r_t = r_{t-1} \exp \left( u_t \Delta T + \sigma_{r,t} \sqrt{\Delta T} \right), \quad (19)$$

where  $\Delta T$  is the time interval. The hazard rate at time  $t$ , conditional upon no prior default,

$$h_t = h_{t-1} \exp \left( v_t \Delta T + \rho_{h,r} \frac{\sigma_h}{\sigma_r} r_t \Delta T \right), \quad (20)$$

where  $v_t$  and  $\sigma_h$  denote the drift and diffusion coefficient, respectively, of the hazard rate, and  $\rho_{r,t}$  denotes the correlation between interest rate and hazard rate. The hazard rate process in Equation (20) extends the Black-Derman-Toy model in a straightforward manner. We choose it for reasons of consistency with our interest rate model.

At time  $t = 0$ , banks are endowed with a certain amount of cash  $c$ . The amount varies across banks, as specified further down. The equity of bank  $i$  is denoted by  $E^i$ . At the beginning of every period  $t$ , banks receive a demand for a bond  $B$  with maturity  $t + T_B$ ,  $T_B \in \mathbb{N}$ , from the real sector, that is, from their clients.

Client demand is uniformly distributed within  $[-lE_t^i, lE_t^i]$  with  $l \in \mathbb{R}^+$ . Note that client demand might either be positive (lending) or negative (borrowing).

Clients might default on their obligations. We assume that the hazard rate is the same for all clients.<sup>19</sup> We denote the clients' hazard rate by  $h^c$ . For the time being, we also assume that all banks have the same rating level, which we denote by  $h$ .<sup>20</sup>

The bond traded with clients is either a fixed-rate or a floating-rate bond. For half of the banks ( $i \bmod 2 = 1$ ), all lending is in floating-rate bonds whereas all borrowing is in fixed-rate bonds, and vice versa for the other half ( $i \bmod 2 = 0$ ). The value at time  $t$  of a defaultable bond maturing at time  $t + 1$  is given by

$$B_t(t + 1) = \mathbf{E}_t^{\mathbb{Q}} \left[ e^{-r_t \Delta T} (h_t^c \omega_{t+1} + (1 - h_t^c) B_{t+1}(t + 1)) \right], \quad (21)$$

where  $\omega_{t+1}$  is the recovery amount and  $B_{t+1}(t + 1)$  is the promised payout at maturity  $t + 1$ . In other words, the bond promises to pay  $B_{t+1}(t + 1)$  at maturity  $t + 1$ , but the promise may be broken at hazard rate  $h_t^c$ . If default occurs at time  $t$ , an amount  $\omega_{t+1}$  is paid at  $t + 1$ , conditional upon no prior default.  $\mathbb{Q}$  denotes the risk-neutral probability measure. We assume that a risk-neutral probability measure exists and that all banks choose the same measure  $\mathbb{Q}$  when valuing the bond.<sup>21</sup> Under the risk-neutral probability measure  $\mathbb{Q}$ , these cash flows can be discounted at the risk-free interest rate, as shown by Duffie and Singleton (1999). If the bond has a time to maturity of more than one period and default occurs prior to maturity, we will assume that  $\omega$  is invested at the risk-free rate  $r$  until the bond's maturity.

We assume that all bonds have a notional amount of one unit of cash. A bank lending an amount  $x$  to a client will thus enter into a position of  $\alpha = \lfloor x/B \rfloor$  units of the bond, where  $\lfloor x \rfloor$  denotes the integer part of  $x$ . A bank entering, at time  $t$ , into a position of  $\alpha$  units of the bond contract with maturity  $t + T_B$  will exchange

<sup>19</sup>Cf. Section 7.

<sup>20</sup>This assumption can be interpreted such that all banks (or their derivatives trading units) have a similar credit rating which, we believe, fairly reflects the current situation in derivatives markets.

<sup>21</sup>We cannot assume that  $\mathbb{Q}$  is unique since this would imply market completeness. However, we will later assume that banks cannot hedge default risk, implying an incomplete market.

the notional principal of  $-\alpha B_t(t + T_B)$  at initiation and of  $\alpha B_{t+T_B}(t + T_B)$  at expiry, and will make or receive interest payments in all periods  $t \in \{t + 1, \dots, t + T_B\}$ .<sup>22</sup>

Client demand exposes the banks to both, interest rate risk and default risk. Default risk cannot be hedged, rendering the market incomplete.<sup>23</sup> Interest rate risk can be hedged by trading in a swap contract. By entering into a swap contract, a bank agrees to pay the agreed swap rate and to receive the current interest rate (long position), or vice versa (short position). More precisely, by executing at time  $t$  a swap contract maturing at time  $t = T_S$ ,  $T_S \in \mathbb{Z}$ , counterparties agree to exchange fixed interest payments at an agreed swap rate,  $s(t + T_S)$ , against floating interest rates. We assume that the swap contract has the same notional principal as the bond, that principals are not exchanged, and that the swap has the same time to maturity as the bond, that is,  $T_S = T_B$ . The swap contract is subject to default risk.

In order to trade in the swap, a bank submits an order to the market. An order has the form  $(\alpha^i, s^i)$ , where  $\alpha^i$  is the number of contracts the agent wants to trade and  $s^i$  is the swap rate. The swap rate  $s^i$  determines the payoff of the contract, which can be both, positive and negative. When a bank submits an order, it sets the swap rate  $s^i$  such that the contract has a value of zero.

Orders are submitted sequentially but in random order, that is, the order in which banks submit orders in the different periods (and in the different simulation runs) changes. A trade takes place whenever two orders match. Our trading mechanism thus resembles a discriminatory continuous double auction. Such a trading mechanism is used in most of today's security markets. It determines the structure of contractual relationships between agents.

Every swap position might be subject to a margin requirement of  $\phi \in \mathbb{R}^+$  units of cash per contract traded. This (initial) margin has to be posted at the date when a bank initiates a position. Contracts might also be subject to variation margin. In this case, the change in the value of a bank's position between two dates is settled in cash. Bank  $i$  holding  $\alpha_{t-1}^i$  swap contracts at date  $t - 1$  has a variation margin requirement of  $\alpha_{t-1}^i (S_t(\bar{s}_t) - S_{t-1}(\bar{s}_{t-1})) + \alpha_t^i (S_t(\bar{s}_t) - S_t(s_t^i))$  at date  $t$ , where  $S(s)$  denotes the value of the contract given swap rate  $s$ , and  $\bar{s}$  denotes the settlement price. In other words, the value of the position of the previous

<sup>22</sup> Note that  $B_t(t + T_B)$  reflects the default risk of the clients. Therefore, the payments at contract initiation include a compensation for the default risk in the bond.

<sup>23</sup> One might argue that this assumption is unrealistic due to the existence of credit derivatives. However, supply of such derivatives might be insufficient to cover demand. More importantly, credit derivatives are also subject to default risk. We therefore believe our assumption to be reasonable.

period is set to zero (first term) and so is the value of the contracts traded in the current period (second term). The second term is necessary since contracts might be traded at a price different from the current period's settlement price and therefore already have to be "marked to market" in the period in which they are traded. This means that variation margin eliminates current exposure of a position.

By default, contracts are traded bilaterally, that is, directly between banks. The value of a swap contract is zero at initiation and zero at expiry.<sup>24</sup> A swap contract results in periodical cash flows comprising the "coupon" payment (differential between current interest rate and swap rate) and variation margin (differential between previous period's and current period's contract value). Obviously, margining changes the cash flow pattern.

To summarize, banks are exposed to an interest rate and counterparty risk exposure through client demand for a bond. Clients default on their obligations with hazard rate  $h^c$ . Banks hedge their interest rate exposure by trading in a swap contract. The market is characterized by a time horizon  $T$ , the number of banks  $N$ , the interest rate process  $r$ , the client's hazard rate process  $h^c$ , the margin requirements  $\Phi$ , and the risk limit  $l$ . Banks are characterized by their initial holding in cash  $c$ .

We point out at this stage that all these parameters can be calibrated to quantities observed in actual markets. We will do so in our simulations in order to resemble actual markets as closely as possible.

## 7.2. Banks' Optimization Problem

An time  $t = 0$ , a bank holds  $c$  units of cash. Over time, it builds up a portfolio of long and short positions in the bond. At time  $t$ , the cash holding of a bank is given by

$$c_t = (1 + r_{t-1})c_{t-1} - \alpha_t^c(t + T_B)B_t(t + T_B) + \alpha_t^c(t)B_t(t) + \sum_{\tau=t}^{t+T_B} \Psi(\alpha_t^c(\tau), B_t(\tau)), \quad (22)$$

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<sup>24</sup>The value of the swap obviously depends on the hazard rate of the counterparty. As we assume a single hazard rate across banks at the moment we drop the reference to the counterparty, i.e.,  $S^i(\cdot) = S(\cdot)$  for all  $i = 1, \dots, N$ .

where  $\alpha^c$  reflects the aggregate position with the client sector and  $\psi(\cdot)$  denotes the interest payment of a particular position.  $\psi$  is given by

$$\psi(\alpha_t^c(\tau), B_t(\tau)) = \begin{cases} \alpha_t^c(\tau)^+ r_t - \alpha_t^c(\tau)^- s_{t+T_B-\tau}(\tau) & \text{if } i \bmod 2 = 1, \\ \alpha_t^c(\tau)^+ s_{t+T_B-\tau}(\tau) - \alpha_t^c(\tau)^- r_t & \text{if } i \bmod 2 = 0. \end{cases} \quad (23)$$

The expression above reflects the fact that half of the banks lend at fixed rates and borrow at floating rates, and vice versa for the other half. As indicated by Equation (23), fixed rate bonds entered into at time  $t$  have a coupon payment of  $s_t(t + T_B)$ , that is, the coupon is set equal to the swap rate in the respective period.<sup>25</sup> Equation (22) reflects the cash flows resulting from bond contracts maturing at  $t$  and contracts entered into at  $t$  (with maturity  $t + T_B$ ) as well as interest payments. The equity of a bank at  $t$  is given by

$$E_t = c_t + \sum_{\tau=t}^{t+T_B} \alpha_t^c(\tau) B_t(\tau). \quad (24)$$

A bank's equity at time  $t$  thus comprises its cash holdings and the current value of the bond holdings not yet matured.

We assume that a bank will always try to eliminate its exposure to interest rate risk. Whenever a bank enters into a fixed-rate bond, it tries to enter into a swap position of the same quantity.<sup>26</sup> In other words, the bank's objective function yielding the optimal quantity in the swap contract is given by

$$\alpha^* = \begin{cases} (\alpha^c)^- & \text{if } i \bmod 2 = 1, \\ (\alpha^c)^+ & \text{if } i \bmod 2 = 0. \end{cases} \quad (25)$$

Note that banks' trading in the swap contract will not only be affected by client demand but also by client as well as counterparty default. The latter renders trading demand dynamic.

We now explain the implications of the various mitigation mechanisms associated with the swap contract on a bank's cash flow and its wealth. We discuss four cases: (1) no margining, (2) initial margin only, (3) initial and variation margin, as well as (4) initial and variation margin with a central counterparty. These four cases are supposed to resemble the combinations of mitigation mechanisms found in actual markets.

<sup>25</sup>This assumption is made for simplicity.

<sup>26</sup>There exist situations where complete hedging is optimal for banks. Bauer and Ryser (2004) show that this the case, e.g., in the presence of high asset volatility.

In line with empirical observations, we assume that different contracts undertaken with the same counterparty can be netted; in case of a central counterparty they can be netted across counterparties, of course.<sup>27</sup> Furthermore, we assume that only cash can be posted as collateral.<sup>28</sup>

*No margining:* In case of no margining, a swap contract results in interest (“coupon”) payments only, that is,

$$c_t = (1 + r_{t-1})c_{t-1} - \alpha_t^c(t + T_B)B_t(t + T_B) + \alpha_t^c(t)B_t(t) + \sum_{\tau=t}^{t+T_B} \psi(\alpha_t^c(\tau), B_t(\tau)) + \sum_{\tau=t}^{t+T_S} \alpha_t(\tau) [r_t - s_{t+T_S-\tau}(\tau)]. \quad (26)$$

The last term in the equation reflects the interest payments in relation to the swap contracts.<sup>29</sup> The bank’s equity is given by

$$E_t = c_t + \sum_{s=t}^{t+T_B} \alpha_t^c(s)B_t(s) + \sum_{j=1}^{N-1} \sum_{\tau=t}^{t+T_S} \alpha_t^j(\tau)S_t(\tau), \quad (27)$$

where  $\alpha^j$  denotes a position in the swap contract with bank  $j$ . The last term in the equation above reflects the net value of the swap position.

*Initial margin:* When initial margin has to be posted by banks, a swap contract may result in a cash flow at time of its initiation. We will assume that margin requirements can be offset across maturities. From now on, we will denote the total (initial) margin requirement of the bank at time  $t$  by  $\Phi_t$ . The margin requirement can be expressed as

$$\Phi_t = \sum_{j=1}^{N-1} \left| \sum_{\tau=t}^{t+T_S} \alpha_t^j(\tau)\phi \right|. \quad (28)$$

$\Phi$  is always positive. The expression above reflects the fact that margin requirements in our model are symmetric (both counterparties to a contract have the same margin requirement per unit of contract). A bank entering into  $q$  units of a contract with another bank has to deliver  $|\alpha|\phi$  units of cash as margin and will receive  $|\alpha|\phi$  units of cash from the other bank (assuming that the two banks had no contracts with each

<sup>27</sup>International Swaps and Derivatives Association (2006) reports that the largest part of over-the-counter derivatives contracts are netted.

<sup>28</sup>International Swaps and Derivatives Association (2006) reports that about 90% of collateral posted in the over-the-counter derivatives markets is cash.

<sup>29</sup>Note that for those banks where  $i \bmod 2 = 1$ ,  $q$  is negative, and vice versa for all other banks.

other before).

The cash holding of the bank at time  $t$  is now given by

$$c_t = (1 + r_{t-1})c_{t-1} - \alpha_t^c(t + T_B)B_t(t + T_B) + \alpha_t^c(t)B_t(t) + \sum_{\tau=t}^{t+T_B} \psi(\alpha_t^c(\tau), B_t(\tau)) + \sum_{\tau=t}^{t+T_S} \alpha_t(\tau) [r_t - s_{t+T_S-\tau}(\tau)] - (\Phi_t - \Phi_{t-1}). \quad (29)$$

The last term reflects the change to the cash holding due to a change in the initial margin requirement. The initial margin requirement may change as a result of a change in the bank's swap position. The bank's equity can be expressed as

$$E_t = c_t + \sum_{s=t}^{t+T_B} \alpha_t^c(s)B_t(s) + \sum_{j=1}^{N-1} \sum_{\tau=t}^{t+T_S} \alpha_t^j(\tau)S_t(\tau) + \Phi_t. \quad (30)$$

The last term in the equation above denotes the collateral delivered by the bank.

*Initial and variation margin:* In case a variation margin is charged, the changes in the swap contracts' values are settled periodically. Obviously, this significantly affects the cash flow of the bank. Its cash holding at time  $t$  is now given by

$$c_t = (1 + r_{t-1})c_{t-1} - \alpha_t^c(t + T_B)B_t(t + T_B) + \alpha_t^c(t)B_t(t) + \sum_{\tau=t}^{t+T_B} \psi(\alpha_t^c(\tau), B_t(\tau)) + \sum_{\tau=t}^{t+T_S} \alpha_t(\tau) [r_t - s_t(\tau)] + \sum_{j=1}^{N-1} \sum_{\tau=t}^{t+T_S} \alpha_t^j(\tau) (S_t(\tau) - S_{t-1}(\tau)) - (\Phi_t - \Phi_{t-1}). \quad (31)$$

The expression above reflects the assumption that variation margin is based on the expected value of the swap. The bank's equity is given by Equation (30).

*Initial margin, variation margin, and central counterparty:* In the presence of a central counterparty, the margin requirement changes. A bank only has one position in the swap contract, namely with the central counterparty.  $\Phi$  is now given by

$$\Phi_t = \left| \sum_{\tau=t}^{t+T_S} \alpha_t(\tau)\phi \right|, \quad (32)$$

and the cash holding changes accordingly to

$$\begin{aligned}
c_t &= (1 + r_{t-1})c_{t-1} - \alpha_t^c(t + T_B)B_t(t + T_B) + \alpha_t^c(t)B_t(t) \\
&\quad + \sum_{\tau=t}^{t+T_B} \Psi(\alpha_t^c(\tau), B_t(\tau)) \\
&\quad + \sum_{\tau=t}^{t+T_S} \alpha_t(\tau) [r_t - s_t(\tau)] \\
&\quad + \sum_{\tau=t}^{t+T_S} \alpha_t(\tau) (S_t(\tau) - S_{t-1}(\tau)) \\
&\quad - (\Phi_t - \Phi_{t-1}).
\end{aligned} \tag{33}$$

The bank's equity is now given by

$$E_t = c_t + \sum_{s=t}^{t+T_B} \alpha_t^c(s)B_t(s) + \sum_{\tau=t}^{t+T_S} \alpha_t(\tau)S_t(\tau) + \Phi_t. \tag{34}$$

In the presence of margining, trading in the swap contract is constrained by the bank's holding of cash as well as by the solvency constraint. Due to these constraints, the bank may not be able to implement its optimal position in the swap contract. In such a case, it will not be able to fully hedge its exposure to interest rate risk. We assume that banks default when their equity falls below zero (asset-based insolvency). In case of a default, a bank is liquidated, that is, cash and receivables are distributed proportionally to its creditors.

In our model, all banks have common knowledge about the parameters of both interest rate and hazard rate processes, that is, about  $u$ ,  $\sigma_r$ ,  $v$ ,  $\rho_{h,r}$  and  $\sigma_h$ . At any given time, banks therefore trade at the same swap rate.<sup>30</sup>

To summarize, in every period the  $N$  different banks try to hedge their interest rate risk exposure by entering a swap position. More precisely, for a given number of client bonds,  $\alpha_t^c$ , they try to solve the following optimization problem for all  $t \in \{0, T - 1\}$ :

$$\min_{\alpha_t} |\alpha_t + \alpha_t^c| \quad \text{s.t.} \tag{35}$$

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<sup>30</sup>This implies that banks do not charge liquidity premia. We will come back to this issue later on.



$$E_t(\alpha_t) \geq 0$$

$$\Phi_t(\alpha_t) \leq c_t.$$

Note that  $\alpha, \alpha^c \in \mathbb{R}$ . In other words, in every period banks seek to completely hedge their exposure to interest rate risk.

### 7.3. Simulation

We now turn to the analysis of the model described in the previous section. We will briefly discuss the simulation methodology, describe the calibration of the model as well as the different scenarios we analyze, and explain a sample run of the simulation.

As mentioned already, we use simulations to assess the implications of the model. This allows us to analyze aggregate phenomena like market liquidity and its effects on aggregate wealth with a high degree of flexibility at the level of their microeconomic foundations.<sup>31</sup>

A point of critique often brought forward against simulation is the number of free parameters and a resulting high degree of flexibility in “fitting the model to the facts”. We address this issue in two ways. First, we set up the model such that all parameters can be related to quantities observed in actual markets. We then calibrate the model such that its parameters are in line with the quantities observed.

In Sections 4 and 5 we saw that the costs of margin outweigh its benefits whenever margin rates are high. One may conceive that such is the case during a market crisis, as described previously. This is, however, the time when market participants expect to benefit from margining. In order to check the robustness of the results of Sections 4 and 5, we therefore calibrate the simulation such that it resembles a situation where the banking system is under severe and sustained stress. More precisely, we choose an environment where banks experience highly volatile interest rates as well as substantial default rates from their clients. The question we ask is to what extent margining improves the market outcome in such a situation in terms of banks’ utility of wealth, default rates, and losses given default. The set of model parameters is displayed in

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<sup>31</sup>Some of the scenarios we investigate would be inaccessible by analytical modelling approaches. These benefits, however, come at the expense of a lower degree of analytical tractability. We think, though, that in our case the benefits outweigh the costs. In the simulation we employ parts of the JUNG library as described in O’Madadhain, Fisher, Smyth, White, and Boey (2005).

**Table 5****Description of model parameters**

The following table lists the market parameters for the simulation.

Type	Parameter	Description
Market	$T$	Time horizon
	$N$	Number of banks
	$r, u, \sigma_r$	Dynamics of short-term interest rate
	$T_B$	Time to maturity of bond
	$T_S$	Time to maturity of swap
	$\phi$	Margin requirement per swap contract
Real sector	$h, v, \rho_{h,r}, \sigma_h$	Hazard rate dynamics
Banks	$m$	Initial amount of cash

**Table 6****Parameter values**

The table below shows values of the model parameters used in the simulation.

Type	Parameter	Value
Market	$T$	100
	$N$	25
	$r, u, \sigma_r$	Empirical term structure (see text)
	$T_B$	48
	$T_S$	48
	$\phi$	95%, 97%, 99% VaR of swap contract
Real sector	$h, v, \rho_{h,r}, \sigma_h$	See text
Banks	$m$	Empirical distribution (see text)

Table 5.

*Time horizon:* We fix the time horizon of the model,  $T$ , to 100 periods. With this choice we keep computational time at a reasonable level. We think of one period in the model as representing one month in calendar time.

*Term structure:* We calibrate our term structure to prices of U.S. Treasury securities in the period from January 1996 to April 2004. The level of interest rates fluctuated widely during this time period. In addition, the shape of the yield curve changed as well.

*Instrument maturities:* We fix the maturities of the two instruments, the bond and the swap, to 48 periods, that is,  $T_B = T_S = 48$ .<sup>32</sup>

*Margin:* We will consider three levels of initial margin,  $\phi$ . More precisely, we will consider scenarios where  $\phi$  is set equal to 95%, 97%, and 99% of the 1-month value-at-risk of a swap contract. As described in Knott and Mills (2002), for example, clearinghouses typically set initial margins within this range. The time horizon is usually shorter, though. As we consider the somewhat ideal case of full cross-margining (contracts of different maturities are offset against each other in the margin calculation), initial margin would typically be higher. We consider the choice of the 1-month value-at-risk as a reasonable approximation.

*Number of agents:* We fix the number of agents,  $N$ , to 25. As described in OCC (2006), the five largest U.S. insured commercial banks currently hold about 96 percent of notional outstanding of derivatives held by all members of this group of banks, and the largest 25 banks hold about 99 percent. We therefore consider the choice of 25 banks as a valid representation.

*Hazard rate of clients:* The aim is to investigate a scenario where banks are under severe stress in terms of both, interest rate risk and default risk. We therefore set  $h^c$  such that a significant amount of banks could not sustain losses without hedging. We set  $v \equiv 0$ ,  $\sigma_h \equiv 0.01$ , and  $\rho_{r,h} \equiv -0.5$ .<sup>33</sup>

*Initial amount of cash:* Banks are endowed with initial amounts of equity (all in cash) such that their distribution reflects the empirical distribution of equity of the largest derivatives dealers (OCC 2006).

We consider three sets of mitigation mechanisms. The first set, “IM”, considers the case where counterparties charge initial margin. The second set, “IM & VM”, also takes variation margin into account. Finally, the third set, “CCP”, includes the previous set as well as a central counterparty. These combinations of default risk mitigation mechanisms are the ones most often observed in practice.<sup>34</sup> Cash holdings and equity of the banks given the three different sets of mitigation mechanisms are computed according to the expressions shown above.

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<sup>32</sup>We believe that this is a reasonable approximation of the average maturity of the instruments on a bank’s balance sheet. For derivatives, cf. OCC (2006).

<sup>33</sup>For the choice of  $\sigma_h$  cf. the analysis by Skinner and Diaz (2001), and for  $\rho_{r,h}$  cf. Duffee (1998).

<sup>34</sup>Cf. Appendix A.

We now briefly describe a sample run of the simulation. At the beginning of a run, banks receive their cash endowment,  $c$ . At the beginning of every period, banks receive random client demand and enter into a position,  $\alpha^c$ , in the bond with the client at its expected value.<sup>35</sup> The size of client demand is constrained by a bank's equity as well as a solvency constraint, as described above. Subsequently, banks submit an order for the swap contract to the market. The size of the order is set such that the trade in the swap equals the number of fixed-rate bonds in the portfolio, as described above. The size of the order might be constrained by the margin requirement,  $\Phi$ , as well as by the solvency constraint. After trading, the change of the interest rate,  $r$ , is revealed and positions are settled. Settlement includes mature bond positions, mature swaps positions, coupon payments, as well as the settlement of initial and variation margins in relation to the swap positions.<sup>36</sup>

As described previously, a bank might be solvent but not have sufficient cash to make the payments due in a given period. In this case, banks are provided with liquidity in the form of a short position in a one-period bond at the current interest rate.<sup>37</sup> The provision of liquidity is constrained by a bank's equity through a solvency requirement ( $E > 0$ ). If a bank becomes insolvent, it defaults and its positions are liquidated at current market prices together with the margins this bank held and delivered.

#### 7.4. Discussion of Simulation Results

We now turn to the discussion of the simulation results stemming from the model. Each parameter configuration discussed below was simulated 100 times. Before we turn to the investigation of risk mitigation mechanisms, we briefly describe the “base case”, that is, the case where banks do not trade in the swap at all.

As already described above, banks build up a portfolio of bond positions. Half of the banks have long positions in floating-rate bonds and short positions in the fixed-rate bond (*floating-fixed portfolio*), and vice versa for the other half (*fixed-floating portfolio*). Given the interest rate environment in the simulation, banks with a floating-fixed portfolio will experience a net loss, while banks with a fixed-floating portfolio

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<sup>35</sup>Cf. Footnote 22.

<sup>36</sup>To compute the payments vector, we use an implementation of the algorithm described in Eisenberg and Noe (2001).

<sup>37</sup>We assume that liquidity is supplied exogenously, e.g., by a central bank, up to a solvency limit. Since we use actual interest rates, they will include any liquidity premium observed during a difficult market environment.

**Table 7****Overview of simulation results for a generic parameter configuration**

The table below shows the effects of the three sets of default risk mitigation mechanisms we investigate. 0 denotes the case where banks trade in the swap contract but do not employ any risk mitigation mechanisms. Initial margin is set at the 95% value-at-risk level.  $\sigma_E$ : standard deviation of  $E$ ;  $E_T$ : terminal equity;  $d$ : default rate;  $LGD$ : loss-given-default *per default*;  $V$ : volume traded. All values are averages across banks. Standard deviations are normalized.

	BC	0	IM	IM & VM	CCP
$\sigma_E$	0.492	0.397	0.401	0.498	0.498
$E_T$	166.0	174.7	169.8	162.2	161.4
$d$	0.176	0.112	0.140	0.240	0.240
$LGD$		0.227	9.94	5.74	5.62
$V$		19.0	18.6	15.4	15.4

will generate a significant profit. Absent any hedging activities, several banks with a floating-fixed portfolio will default during the simulation.

Table 7 displays simulation results for the base case (BC) as well as the case of hedging with no default risk mitigation mechanisms (0). While we include terminal equity for the sake of completeness, the more meaningful measure as concerns wealth implications is the standard deviation of equity. Since banks try to hedge all of their exposure, standard deviation is an appropriate measure to analyze the effectiveness of hedging.

Hedging reduces the standard deviation of equity by 19.3 percent. Terminal equity increases by 5.2 percent while the default rate decreases by 36.4 percent. The introduction of initial margin does not considerably reduce the benefits of hedging. The increase of the standard deviation of equity and of the default rate are not statistically significant. However, initial margin increases the average loss-given-default.<sup>38</sup> It also reduces trading volume, albeit slightly. The latter two effects are statistically significant. These results are in line with the findings previously obtained in Section 4 and 5.

The introduction of variation margin considerably deteriorates the benefits of trading in the swap contract. In terms of standard deviation of equity, it removes all the benefits of trading. It increases the standard deviation of equity by 24.0 percent and reduces trading volume by 16.7 percent. While it decreases loss-

<sup>38</sup>In the following, when we refer to loss-given-default we mean the average loss per default in relation to swap contracts. It does not include losses in relation to bonds traded with clients.

**Table 8****Effects of increased initial margin on the market outcome**

The table below shows the effects of increases of initial margin on the market outcome. **0** denotes the case where banks trade in the swap contract but do not employ any risk mitigation mechanisms. Initial margin is set at the 95% (IM95), 97% (IM97), and 99% (IM99) value-at-risk level. Case IM95 is the same as IM before.  $\sigma_E$ : standard deviation of  $E$ ;  $E_T$ : terminal equity;  $d$ : default rate;  $LGD$ : loss-given-default *per default*;  $V$ : volume traded. All values are averages across banks. Standard deviations are normalized.

	BC	0	IM95	IM97	IM99
$\sigma_E$	0.492	0.397	0.401	0.403	0.406
$E_T$	166.0	174.7	169.8	167.6	164.9
$d$	0.176	0.112	0.140	0.156	0.172
$LGD$		0.227	9.94	13.7	18.5
$V$		19.0	18.6	18.4	18.1

given-default, compared to case IM, by 42.3 percent, it increases the default rate by 71.0 percent. These results, again, are similar to those stemming from the theoretical analysis. A central counterparty, in our set-up, cannot effectively reduce the negative effects of variation margin. It does, however, reduce loss-given-default by 2.1 percent.

We now turn to a more detailed analysis of the default risk mitigation mechanisms. We consider various levels of initial margin and analyze the effects of an increased initial margin requirement on the standard deviation of equity, default risk, and market liquidity. As before, we compare trading with initial margin to the cases of no trading (BC) and to trading without any risk mitigation mechanisms (0).

As shown in Table 8, the effects of initial margin persist when its level is increased within the range commonly observed in financial markets. An increase in the level of initial margin reduces trading volume and increases both, default rates and default severity. Its effects on the standard deviation of equity are negligible, though, and not statistically significant.

Marking-to-market of positions and settlement of the differences in cash has ambiguous effects on a bank's equity. While it reduces credit exposure, variation margin entails opportunity costs in terms of foregone interest and potentially lower capacity to trade bonds and swaps. Thus, it might exacerbate market movements in either direction. Table 9 shows the effects of variation margin on the market outcome. While

**Table 9****Effects of variation margin on the market outcome**

The table below shows the effects of variation margin, given certain levels of initial margin, on the market outcome. 0 denotes the case where banks trade in the swap contract but do not employ any risk mitigation mechanisms. Initial margin is set at the 95% (IM95), 97% (IM97), and 99% (IM99) value-at-risk level.  $\sigma_E$ : standard deviation of  $E$ ;  $E_T$ : terminal equity;  $d$ : default rate;  $LGD$ : loss-given-default *per default*;  $V$ : volume traded. All values are averages across banks. Standard deviations are normalized.

The values in brackets show the differences between the current setting and the previous setting variation margin, as reported in Table 8.

	BC	0	IM95 & VM	IM97 & VM	IM99 & VM
$\sigma_W$	0.492 (0.000)	0.397 (0.000)	0.498 (0.097)	0.499 (0.096)	0.499 (0.093)
$E_T$	166.0 (0.0)	174.7 (0.0)	162.2 (-7.6)	160.5 (-7.2)	159.1 (-5.8)
$d$	0.176 (0.000)	0.112 (0.000)	0.240 (0.100)	0.260 (0.104)	0.272 (0.100)
$LGD$		0.227 (0.000)	5.74 (-2.20)	9.99 (-3.7)	16.0 (-2.5)
$V$		19.0 (0.0)	15.5 (-3.3)	15.3 (-3.1)	15.0 (-3.1)

**Table 10****Effects of a central counterparty on the market outcome**

The table below shows the effects of a central counterparty, given certain levels of initial margin and variation margin, on the market outcome. 0 denotes the case where banks trade in the swap contract but do not employ any risk mitigation mechanisms. Initial margin is set at the 95% (IM95), 97% (IM97), and 99% (IM99) value-at-risk level.  $\sigma_E$ : standard deviation of  $E$ ;  $E_T$ : terminal equity;  $d$ : default rate;  $LGD$ : loss-given-default *per default*;  $V$ : volume traded. All values are averages across banks. Standard deviations are normalized.

The values in brackets show the differences between the current setting and the previous setting without a central counterparty, as reported in Table 9.

	BC	0	IM95 & VM & CCP	IM97 & VM & CCP	IM99 & VM & CCP
$\sigma_E$	0.492 (0.000)	0.397 (0.000)	0.498 (0.001)	0.499 (0.000)	0.499 (0.000)
$E_T$	166.0 (0.0)	174.7 (0.0)	161.4 (-0.8)	160.4 (-0.1)	158.2 (-0.9)
$d$	0.176 (0.000)	0.112 (0.000)	0.240 (0.000)	0.252 (-0.008)	0.272 (0.000)
$LGD$		0.227 (0.000)	5.62 (-0.12)	9.66 (-0.33)	15.9 (-0.1)
$V$		19.0 (0.0)	15.4 (-0.1)	15.3 (-0.0)	15.0 (-0.0)

reducing the loss-given-default, it increases the standard deviation of equity and the default rate while reducing trading volume and terminal equity.

Finally, we investigate the effects of a central counterparty. The simulation results for the cases IM & VM & CCP are displayed in Table 10. One would expect a central counterparty to lift at least some of the inefficiencies introduced by initial and variation margin. In our setting, most of the inefficiencies arise from variation margin. The size of variation margin is not effected by the introduction a central counterparty. However, a central counterparty does marginally reduce default rates and losses given default, albeit not considerably.

While we do believe that those simulations capture certain key characteristics of derivatives markets, in particular, the interaction between market, credit, and liquidity risk, they bear certain limitations. We now discuss what we consider to be the most important ones. First, our assumption that banks try to completely hedge their exposure to market risk may be unrealistic. If this is so, the negative impact of default risk mitigation mechanisms might be less pronounced. Second, we consider a derivatives market consisting of banks (hedgers) only. In this market, demand and supply are usually not balanced due to the size



distribution of banks. In actual markets, any excess demand or supply might be balanced by third parties including speculators (those, however, might be subject to the same constraints). In such a case, the impact of default risk mitigation mechanisms on trading volume, and on default, might be less noticeable than in this model. On the other hand, close to one hundred percent of notional is held by the largest banks acting in derivatives markets; therefore, our model might reflect actual markets rather well. Fourth, in our model a shortfall in trading volume is reflected in traded quantities only and not in prices. As we analyze capital effects, we believe that this does not affect our results in principle, though. Fifth, we ignore information effects of margining and of a central counterparty. Margining might provide more timely information about the financial strength of a counterparty. A central counterparty pools information about positions of market participants and is often in a superior position to manage counterparty risk than single counterparties. The benefits of such information effects from the perspective of a single market participant are probably ambiguous, however. Finally, and most importantly, in evaluating default risk mitigation mechanisms, we ignore externalities of derivatives markets and of the banking system. By such externalities, we mean, for example, contagion of defaults in the derivatives market into other areas of the financial system or negative effects of such defaults on the real sector. Taking information effects and externalities (public or social costs) into account would probably change any cost-benefit analysis of such mechanisms. Such analysis, however, is beyond the scope of this investigation.

## **8. Conclusion**

In this study, we examined several types of risk mitigation mechanisms, and in particular margining, employed in derivatives markets to reduce default risk. Exchanges and clearinghouses that have been making extensive use of margining have indeed been experiencing very low default rates, and losses in cases of default were small. One might infer, therefore, that margining is a panacea for the management of default risk. It should be noted, though, that clearinghouses employ a range of other mechanisms in addition to margining that may play an important rôle in reducing default risk. Maybe most importantly, clearinghouses centralize information about the distribution of positions in a given market and thus are better positioned to manage default risk than market participants.

Both, the theoretical and the simulation model presented suggest that margin requirements may reduce trading volume, capital, and welfare while increasing default rates and losses given default when margin rates are high and collateral is scarce. This may particularly be the case during a market crisis. The relationship between margin requirements on one side and trading volume, default rates, losses given default and capital on the other can be tested empirically. Hartzmark (1986), Hardouvelis and Kim (1995) and others investigated the relationship between margin requirements and open interest as well as trading volume for several derivatives exchanges. They find that an increase in margin requirements tends to have adverse effects on both open interest and trading volume. Our results offer a potential explanation of these findings. To the best of our knowledge, though, no previous empirical analysis is focused on the relationship between margin requirements and default risk.

We believe that the effectiveness of margining should not only be evaluated in terms of its effects on credit exposure but on a more general level considering its impact on agents' capital or welfare. Of particular interest is the search for default mitigation mechanisms that preserve the benefits of margining while reducing its costs. One such mechanism is rehypothecation which allows market participants to re-use the collateral they receive to serve margin requirements. Rehypothecation may significantly reduce the amount of collateral necessary to cover a given portfolio of contracts. International Swaps and Derivatives Association (2006) reports that it is already used extensively by large market participants. Rehypothecation has two major drawbacks, though. First, it does not completely eliminate credit exposure of a given position, as described in Section 6. Secondly, it exposes the collateral posted to default risk.<sup>39</sup>

Another mechanism to reduce the burden of margining is a central counterparty. A central counterparty provides the same benefits as rehypothecation. It typically reduces credit exposure even further and may at the same time further decrease the margin requirement of a given position. A major advantage of a central counterparty over other default risk mitigation mechanisms is the centralization of information about market participants' positions. This allows for a centralization of default risk management. In OTC markets, the distribution of risk is typically unknown rendering default risk management cumbersome, particularly in periods of market stress. Schinasi (2006) claimed that these information asymmetries in OTC markets constitute a major threat to the stability of the financial system. Central counterparties can effectively reduce

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<sup>39</sup>This issue is discussed in International Swaps and Derivatives Association (2005).

some of these asymmetries. A central counterparty does not come without costs, though. We believe that in many markets they would help to effectively reduce default risk and increase welfare.

Our results reinforce two points. First, the choice and the implementation of default risk mitigation mechanisms in derivatives markets is a market microstructural issue that may have a significant impact on the capital and the risk profile of a bank. Secondly, disentangling risks, such as market, liquidity and default risk, and considering these risks in isolation may hinder the efficient risk management of banks' derivatives positions.

## **A. Description of Default Risk Mitigation Mechanisms**

In the following, we describe the most important default risk mitigation mechanisms. We focus on those mitigation mechanisms that differ across contracts traded in today's derivatives markets. We analyze netting, margining, rehypothecation and central counterparties. We ignore certain other mitigation mechanisms such as minimum capital requirements, definitions of default events and enforcement procedures either because they seem to be rather similar across contracts traded or because they seem to be less relevant.

A derivatives contract with its default risk mitigation mechanisms is necessarily embedded in the legal code of the respective jurisdiction. Therefore, many of the contractual innovations discussed below were preceded by changes in corporate law. This is particularly true for certain forms of netting and margining. As an example, amendments to the U.S. Bankruptcy Code in the 1970s and 1980s assign a special status to collateral in derivative transactions and were key to the success of collateral, as described by Johannes and Sundaresan (2006). Today, the important mitigation mechanisms including the ones discussed below are supported by the legal codes of the most important jurisdictions, as described in Allen & Overy (2002).

### **A.1. The Nature of Default Risk in Derivatives Contracts**

Like any other contract, a derivatives contract is a *promise to perform*. At delivery date, a counterparty to the contract might choose not to perform on its obligations, that is, she might choose to default. This choice might indeed be optimal to one side of the contract. The right to exercise such a choice can be viewed as a nonperformance option. In granting such options, market participants recognize that the cost of absolute performance assurances can exceed the value of trading benefits and might act as barriers to trade. In other words, default can be welfare-improving, as Dubey and Shubik (1979) and others showed.

The credit exposure in relation to derivatives contracts is highly complex. It depends on the prices of the underlying assets and thus varies over time. Often, credit exposure changes rapidly and in large amounts. Furthermore, credit risk in derivatives interacts not only with market risk but also with liquidity, operational, and other types of risks. These interactions and the resulting conceptual and measurement issues are not yet well understood.

The aim of default risk mitigation mechanisms is to reduce credit exposures by reducing loss amounts when nonperformance occurs (loss given default), and to reduce the probabilities of nonperformance states (probabilities of default).

The off-balance sheet character of derivatives contracts makes it difficult indeed for counterparties to evaluate the financial health of an institution and its contingent liabilities. Data on individual exposures and their fluctuations is largely proprietary. As a result, information to assess the creditworthiness of a counterparty (probability of default) is often insufficient. In such cases, the reliance on collateral appears to be a rather valuable risk-mitigation mechanism.

To illustrate the mitigation mechanisms we will make use of the following simple example: Suppose trader A holds a long position of ten units of a forward contract with trader B, and a short position of ten units of the same contract with trader C. A thus has a “net zero” position, that is, when the price of the underlying changes, a gain in one position is exactly offset by a loss in the other position. We assume that the probability of default of each of the three traders is the same and strictly positive.<sup>40</sup>

## A.2. Netting

Netting allows a trader to offset obligations with a counterparty.<sup>41</sup> Without netting, a defaulting counterparty might “cherry pick” profitable contracts and disclaim unprofitable ones. Netting allows a trader to cancel offsetting transactions and “net” their values thus reducing credit exposure. The most common form of netting employed in derivatives markets is *close-out netting*, allowing for the netting of contractual obligations in the event of a counterparty default. In the remainder, whenever we speak of netting we will mean close-out netting.

Suppose that trader A in the example above has the initial long position of 10 units with trader B, as well as additional, offsetting short position of 10 units with the same trader. A is now “net zero” with B. Let us assume further that the long position is now seasoned and has a value of 10. Accordingly, the short position

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<sup>40</sup>Later on, we will gradually add various default risk mitigation mechanisms to the forward contract. A forward contract subject to *both* initial and variation margin *and* cleared by a central counterparty is commonly called a *futures* contract. We will stick with the term *forward* although at some stage the contract will have metamorphosed into a futures contract.

<sup>41</sup>Of the mitigation mechanisms discussed here, margining was the first to be employed by traders. Netting and central counterparties were introduced subsequently. Cf. Moser (1994).

is worth -10 to A. Assume now that B defaults. Without netting, the creditors of B might cherry-pick the short position (with positive value to B). A would thus have to fully deliver on the short position and might receive nothing from the long position. With a netting provision in place, A would cancel both transactions and “net” their respective values, reducing her obligation to 0.

Netting of obligations with a single counterparty is called *bilateral netting*. A further reduction of credit exposure might be achieved by *multilateral netting*, that is, netting across counterparties. Let us return to the initial set-up of our example where trader A holds a long position of ten units with trader B and an offsetting short position of ten units with trader C. If B defaults, A’s “net zero” position is turned into a short position. With multilateral netting in place, A would be able to cancel both transactions and retain its initial net zero position. Multilateral netting is only possible with a central counterparty, though, as described below.

By reducing the contractual obligations resulting from a position of several transactions, netting reduces the loss-given-default of a position. As netting affects the credit risk borne by a trader, it also affects her solvency, and thus her probability of default.

### **A.3. Margining**

A further mechanism to reduce default risk is margining. Its purpose is to establish a lower bound for the delivery rate in case of a counterparty default. Margin is supposed to cover not just current exposure but also potential future exposure and replacement costs of contracts. A growing number of derivatives contracts are subject to margin requirements. In the following, we will distinguish two types of margin: *initial margin* and *variation margin*.

Initial margin is charged at contract initiation. Its size is usually related to the potential future exposure of the contract and is often based on the contract’s value-at-risk. Initial margin is increasingly supplemented by variation margin. In this case, the contracts are regularly marked to market and the differences in value are settled in cash. This difference is called variation margin. It sets the current exposure of a contract to zero. Variation margin is usually settled daily or even more frequently.

If netting of contracts is provided for, margin requirements are based on the net position with a single counterparty (in case of bilateral netting) or across counterparties (in case of a central counterparty).

Initial margin may be covered by a variety of collaterals. As the value of collateral (except cash) fluctuates, collateral is typically subject to a haircut. Variation margin has to be covered by cash. Both, initial and variation margin might thus lead to (funding) liquidity issues.<sup>42</sup>

Let us return to our example. Trader A would have to post collateral to both, trader B and trader C although she holds a balanced position. At the same time, she will receive collateral from both counterparties. To avoid such situations, counterparties today provide each other with the right to re-use collateral, called *rehypothecation*. In this case, the collateral received from one counterparty can be used to cover a margin requirement of another counterparty. The amount of collateral required to support a portfolio of positions is thus decreased, often dramatically. In our example, trader A would be allowed to re-use the collateral received from B to cover the margin requirement of C and vice versa. Thus, A would not need any collateral of its own for her position.

We point out at this stage that minimum capital requirements like those in the Basel accord can also be viewed as a form of margining.

#### **A.4. Central Counterparties**

Default risk can be further reduced by the use of a central counterparty. A central counterparty, as part of a derivatives clearinghouse, intermediates contracts, that is, it becomes the legal counterparty to every contract in a given market.<sup>43</sup> A central counterparty enables offsetting of obligations. It occurs when the aggregated claims against any counterparty are netted against the aggregate of the counterparty's claims against all other counterparties—what we called multilateral netting above. The current liabilities of the central counterparty and its members are the net of these obligations.

A central counterparty typically has a balanced market position (except when one or more members are in default) but has current credit exposure. Credit risk arises because a change in the price of the underlying

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<sup>42</sup>Variation margin is one of the two features by which futures contracts are differentiated from forward contracts. The other difference is the type of trading mechanism and clearing. Whereas futures contracts are traded on exchanges and cleared by a central counterparty, forward contracts are traded over the counter and are usually not cleared.

<sup>43</sup>Traditionally, central counterparties were only employed in relation to exchange-traded contracts. More recently, however, central counterparties are also available for certain OTC contracts. The most prominent example is SwapClear operated by LCH-ClearNet.

asset could cause one counterparty to owe a considerable amount on its position, particularly if the contract is highly leveraged.

If one of its members defaults, the clearinghouse usually has the right to liquidate the member's position as well as the member's margin. The member's margin might not be sufficient to cover the central counterparty's losses in case of default. To protect itself from this risk, a central counterparty typically requires its members to enter into a loss-sharing agreement in the form of a *default fund*. Additional losses are often covered by insurance.

A central counterparty is usually one of several services offered by a derivatives clearinghouse. Due to broad diversification, high margin requirements, and other credit enhancements, central counterparties are typically of very high credit quality. Additional services offered by derivatives clearinghouses include contract management, collateral management, and payment processing. These services often lower administrative and processing costs of contracts. In addition, clearinghouses, by way of their central counterparty, provide anonymity. Most trading mechanisms, including exchanges and brokers, nowadays provide anonymity at the trading stage (pre-trade anonymity). However, only central counterparties also provide anonymity at the settlement stage (post-trade anonymity).

The benefits offered by clearinghouses do not come for free but are subject to fees. Many derivatives clearinghouses are run as non-profit organizations or are subject to competition so that it can be assumed that the fees are priced near their costs.

There is a subtle issue in relation to central counterparties that is often ignored. It is related to the question whether the benefits of multilateral netting can be achieved by bilateral netting and rehypothecation. To answer this question, let us reconsider our simple example. We assume that the traders have charged each other with initial and variation margin. This means that the positions have no current credit exposure as it has been offset by variation margin. Let us now assume that trader B defaults. In this case, A is left with an open position with C. In other words, A is committed to honoring her obligations with C. This position carries potential future exposure and replacement cost risk. A might try to replace the defaulted contract with B by trading in the market. However, such a contract might not be available, for example, during a



market crisis.<sup>44</sup> A will have to cover future losses arising from the contract with C by the initial margin received from B and, if this is not sufficient, her own resources. In case of multilateral netting, such a situation would not arise since a default of the central counterparty would cancel the entire (net) position. Only if the margins received by A covered the entire potential future exposure would the resulting default risk be the same (zero) in both cases. However, margin rarely covers the entire potential future exposure as this would be too costly.<sup>45</sup>

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<sup>44</sup>Such a situation appears somewhat artificial. However, it might very well occur during a market crisis (it did occur, e.g., after the terrorist attacks on September 11, 2001). Market participants might refrain from trading for several reasons, one being the disruption of the trading infrastructure. Clearing in particular and risk mitigation mechanisms in general are about extreme events and should thus be evaluated in such settings.

<sup>45</sup>The potential future exposure of many derivatives contracts is, theoretically, infinite.

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