Strategic Loan Modification: An Options-Based Response to Strategic Default

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Abstract

A key tool in battling the current foreclosure crisis is loan modification. This paper presents a model for the optimal principal reset in a loan modification, thereby maximizing the loan value to the lender bank and minimizing the likelihood of foreclosure by the homeowner. Reducing the loan-to-value (LTV) ratio will reduce the present value of future payments on the loan, but will also reduce the probability of strategic default, thereby saving deadweight foreclosure costs. The optimal trade-off of these two countervailing effects will pinpoint the optimal LTV at which the loan must be reset. We present a reduced-form barrier option decomposition of the loan value that makes the optimization of LTV easy to implement. An extension of the model is shown to account for coupons and varying growth rate assumptions about house prices. The model in this paper accounts for the homeowner's ability to pay and willingness to pay, and uses the framework to model shared-appreciation mortgages (SAMs). We show that SAMs are structures that mostly improve the lender's loan value.

Key words: strategic default, foreclosure, loan modification
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1 Introduction

The housing crisis of 2008–2009 in the U.S. is unprecedented. By the end of 2009, there was a glut of 3.2 million unsold new and existing single-family homes, amounting to seven months inventory.[2] It is anticipated that another 2.4 million foreclosures will be added to this supply in 2010 alone, driving prices down further. Economists predict that the average national decline in home values since 2006 will be 40%. About 16 million people, one-third of all homeowners with a mortgage, are estimated to have negative equity in their homes as of end–2009.

All efforts to stem the tide of foreclosures appear to have failed. In a recent paper, Das (2009) developed a model to show that the current approach taken by lenders and regulators, i.e., to reduce monthly payments by writing down interest rates, extending maturity, or forebearing principal, actually increases the propensity for homeowners to default. This leads to heavy societal costs—foreclosure discounts are estimated to be greater than 20% of home value on average—see Pennington-Cross (2004) for an estimate in good economic times of 22%. A better solution is to write down principal, resulting in lower foreclosure rates, mitigation of the deadweight costs of foreclosure,[3] and an overall higher economic value of the loan to the lender, after accounting for the borrower's option to default—see Goodman (2010) for an excellent analysis of why the negative equity problem must be tackled head on with principal modifications. The recent introduction of the HAMP-PRA (Principal Reduction Alternative) scheme by the Federal government adds the principal modification quiver to the arsenal aimed at stemming foreclosure.

The intuition behind principal forgiveness is based on analyzing the option to default held by the homeowner. This option is an American (Bermudan-style) put, allowing the borrower to put the home back to the lender. It is in-the-money when the value of the home (the underlying) is less than the loan balance (the option strike), i.e., when there is negative equity in the home. To keep the monthly payment fixed at some reduced level, it is usually preferable to write down principal because it makes this option less in-the-money (unless the rate is above market, when it makes sense to also reduce the rate). Other approaches, such as reducing the loan rate below market, require higher principal balances given that the monthly payment is held fixed, taking the option further in-the-money. Likewise, extending maturity also makes the option more valuable, as options tend

[2] New York Times, January 4, 2010—"This Year's Housing Crisis".

[3] This deadweight cost, also known as the "foreclosure discount" comprises damage repairs to restore the house to a sale-able condition, a distress sale discount, brokerage commissions and direct selling costs, taxes, insurance, and property management, and interest on capital for the holding period.

to increase in value when their maturities increase, especially when the option to default is in-the-money. Given the huge deadweight cost of foreclosure, minimizing the homeowner's propensity to default increases the economic value of the loan (the default-adjusted expected present value of the modified loan's payments), even after writing down principal.

That writing down principal is optimal is becoming self-evident. The New York Times editorial page (01/04/2010) expressed the essence of this most effectively:

> *The best way to modify an underwater loan is to reduce the principal balance, lowering the monthly payment and restoring equity. But for the most part, lenders have refused to reduce principal because it would force them to take an immediate loss on the loan. Lenders also have vehemently and successfully resisted Congressional efforts to change the law so that bankruptcy courts could reduce the mortgage balances for bankrupt borrowers.*
>
> *The administration decided not to press lenders to grant principal reductions in the flawed belief that simply making payments more affordable would be enough to forestall foreclosures. It hasn't. The administration also didn't fight for the bankruptcy fix when it was before Congress last year despite President Obama's campaign promise to do so.*
>
> *The economy is hard pressed to function, let alone thrive, when house prices are falling. As home equity erodes, consumer spending falls and foreclosures increase. Lenders lose the ability and willingness to extend credit and employers are disinclined to hire. True economic recovery is all but impossible.*
>
> *To avert the worst, the White House should alter its loan-modification effort to emphasize principal reduction. Job creation should also be a priority so that rising unemployment does not cause more defaults.*

If we accept that principal write-downs are the optimal way to modify distressed loans, then lenders' reluctance to take write-offs appears to be more of an accounting issue than an economic one. Even so, it is possible to write down the principal in stages, so as to avoid an abrupt accounting hit, as well as link these staged writed-downs to continued borrower payments, thereby providing the borrower with incentives to keep making payments.

Nevertheless, to optimize the loan's economic value, we *need a model* to determine the write-down amount—this paper presents a model for the optimal principal reset in a loan modification. The focus is on setting the loan-to-value (LTV) ratio to a level that maximizes the lender's default-adjusted loan value after the modification.

Homes with negative equity have LTV greater than one, and it stands to reason that the LTV will need to be lowered below one or at least close enough to one that the borrower's cost of defaulting exceeds the excess of LTV over one. Reducing the LTV will reduce the present value of future payments on the loan, but will also reduce the probability of default, thereby saving deadweight foreclosure costs. The optimal trade-off of these two countervailing effects will pinpoint the optimal LTV at which the loan must be reset. We present a reduced-form barrier option decomposition of loan value in closed form that makes the optimization of LTV easy to implement.

The model in this paper accounts for the homeowner's ability to pay and willingness to pay. Many borrowers end up in foreclosure because they have diminished financial capacity, resulting in a low ability to pay on their monthly loan commitments. We term these "helpless" defaults. But there are other borrowers who have the financial capacity to pay but choose not to, and exercise their option to default. Defaults where there is ability but not the willingness to pay have been termed "strategic" or "ruthless" by bankers, and are estimated to account for 26% of foreclosures—see Guiso, Sapienza and Zingales (2009). Our optimal modification model accounts for both situations, helpless and strategic default.

It is not easy for lenders to distinguish between loan modification requests from homeowners with low ability to pay or low willingness to pay. Lenders are also reluctant to write down principal. In order to mitigate these issues, an innovative modified loan structure, known as a shared-appreciation mortgage (or SAM) has come into vogue. In a SAM, the lender writes down principal, but in return takes a share of the appreciation in the home. In this structure, the homeowner gives up a part share in the home that is realized by the lender if and when home values recover. In essence, the lender effects a debt-equity swap, writing down some of the debt in exchange for equity. The SAM makes the principal write-down more palatable for the lender, and also avoids encouraging new strategic defaulters to mimic helpless defaulters because they now have to part with some of the upside in home values.

We show that a SAM may be decomposed into three components, thereby making it easy to value. First, the value of the loan conditional on no default, i.e., the non-default value of the loan. We show that this portion of the loan is a *down-and-out cash-or-nothing call option*. Second, a component that comprises the expected value of the loan conditional on default, i.e., the default recovery value of the loan. We show that this portion of the loan is equal to a *down-and-out call option rebate*. Third, the loan value has a component for the shared appreciation delivered by the SAM. This portion is equal to a *down-and-out vanilla call option*. We obtain all these three component values in closed-form and make valuation of SAMs facile. In addition to the closed-form solution, we also present a tree-based model that is flexible and can handle staged views of the evolution of home prices so that the model may be tuned to reflect the views of market participants.

The contributions of the paper are as follows:

(1) We develop two models (a closed-form equation and a tree model) for optimizing loan modications in the presence of strategic default. The model also accounts for non-tradability (illiquidity) of the underlying home, and the ability and willingness to pay by the borrower. We use the model to examine the sensitivity of home values to various loan parameters.
(2) Loan modications with shared appreciation are shown to mitigate the risks of strategic default, because lenders can modify the loan to lower LTVs than without shared appreciation, yet maximize the value of the loan via the shared appreciation component. Therefore, we show that loan modifications with shared appreciation by the lender are recommended.
(3) The model may be used for a range of home price growth scenarios, in particular mean-reversion in home prices and di?erent growth rates for di?erent periods, thereby allowing the modeler to determine the optimal loan modication under various future price assumptions in the housing markets.

In short, the paper provides a comprehensive analysis of how principal forgiveness may be applied to optimally solving the negative equity aspect of the housing crisis.

The paper proceeds as follows. Section 2 introduces the framework we use for the dynamics of home values and the notation for the paper. Section 3 develops the barrier formulation of the model. This approach allows us to accommodate the borrower's willingness to pay, thereby accounting for strategic defaulters. It also accounts for the appreciation share in a SAM, and its impact on the default barrier. Section 4 presents the basic intuition for the decomposition of the loan value using barrier formulations, and derives the closed-form loan value described in the previous paragraph. This section also discusses the fundamental partial differential equation driving the pricing in the model. This is necessary in order to account for the non-tradeability of the underlying asset. In Section 5 we present some numerical examples to demonstrate the implementation of the model. We compare the optimal modification with and without the shared appreciation feature. Section 6 extends the one horizon model to two horizons to allow for changes in the housing market and to provide a means by which mean reversion in house prices may be incorporated in the model. This extension also handles coupon interest on the loan. Section 7 provides further discussion and extensions of the model.

2 Model

Our reduced-form model values a home loan over an investor horizon, that may, for example, range from one to five years. It is reduced-form in that it does not model all the cashflows of the mortgage, only the expected value of the loan principal over non-default or foreclosure states—coupon cashflows are also handled in the extended model in Section 6. The model also includes shared-appreciation rights that may be held by the lender.

Home value H_t is a stochastic process. This may be quite general; we assume it is a geometric Brownian motion, i.e.,

$$dH_t = \mu H_t \, dt + \sigma H_t \, dZ_t \tag{1}$$

where the drift is μ and the volatility is proportional to σ, with scalar Wiener process Z_t. Since options are involved, discounting under the risk-neutral probability measure will be required, and r is the applicable risk-free rate of interest. Later in this paper, we will adjust the drift μ to account for premia that are related to housing price risk and the non-tradability/illiquidity of the underlying asset. We also generalize μ to vary with time, i.e., denoted μ_t.

Let the horizon of the optimization be T years for loan balance L. At the end of the T years, the borrower may be in foreclosure, in which case the foreclosure present value is a fraction $\phi \in (0,1)$ of the home value. The deadweight cost of foreclosure is the fraction $(1-\phi)$ of the home's value.

Foreclosure occurs when the borrower exercises the option to default, i.e., "puts" the home back to the lender at a strike price equal to the loan balance—see Merton (1974); Kau and Keenan (1999); Deng, Quigley, and van Order (2000); Ambrose, Capone and Deng (2001); Das (2009)—all papers dealing with the option to default. The option to default is in-the-money at time t when the home value is less than the loan balance ($H < L$), i.e., when there is negative equity in the home. In such cases the loan-to-value (LTV) ratio is greater than unity. Given that the borrower has a put option to default, the previously cited work shows that there is an LTV at which it is optimal for the borrower to default. Equivalently, there is a default home-value level D (the default barrier) at which the borrower decides to default. Normalizing initial home value $H_0 = 1$, we express $D < 1$ as the fraction of initial home value below which the borrower defaults—in this setting when $H_0 = 1$, L is the loan-to-value ratio (LTV). We do not need to assume that this is the same for all borrowers with identical homes; indeed, borrowers exhibit widely varying default behavior. We therefore, assume that D comes from an econometric model, and in this paper we take it to be a function of the borrower's willingness to pay, as well as the terms and conditions of the loan modification.

A borrower who has negative equity and does not default retains the probable future appreciation of home value above the loan balance, an amount $H - L > 0$. In a shared appreciation mortgage (SAM) the lender retains the rights to a share θ of the pre-specified appreciation of house value H above a strike level K. This might be, for instance, a share of positive equity, i.e., $H - L$, when $K = L$. Or, a share of future appreciation above the current home value, i.e., when $K = H_0$. Further, we require that $K \geq L$, else there is no equity to pay the SAM.

In return for shared appreciation, the lender offers better terms to the borrower, for example, a lower interest rate on the loan, or reduced principal. The borrower's incentive to default is a function of this appreciation share θ taken by the lender. Ceteris paribus, the larger θ is, the greater incentive to default, because the borrower has less upside to look forward to if he continues to make loan service payments. In other words we may write the default barrier as a function of theta, i.e., $D(\theta)$, where $dD/d\theta > 0$. As we will see in the next section, the default barrier will also be a function of the initial LTV, and so we write it as $D(L, \theta)$.

3 Default Barrier D and Share θ

We now provide a simple structure for the function $D(\theta)$. We then use this function to analyze how loan value changes with θ, the lender's stake in the shared-appreciation mortgage. Our functional form is as follows:

$$D(L, \theta; \gamma) = L \exp[-\gamma(1 - \theta)] \tag{2}$$

where the parameter $\gamma \in (0, \infty)$ is the borrower's willingness to make good on loan service. The following properties of the willingness to pay parameter immediately follow from the function specification above:

(1) The greater the willingness to pay (γ), the lower is the trigger default level of home value D, i.e., the borrower is less likely to default.
(2) When $\gamma = \infty$, the willingness to pay is infinite, the default level $D = 0$. The borrower never defaults unless the home value goes to zero.
(3) When $\gamma = 0$, there is no willingness to pay and the default level is $D = L$, i.e., the borrower defaults the moment the home value drops infinitesimally below the loan amount.

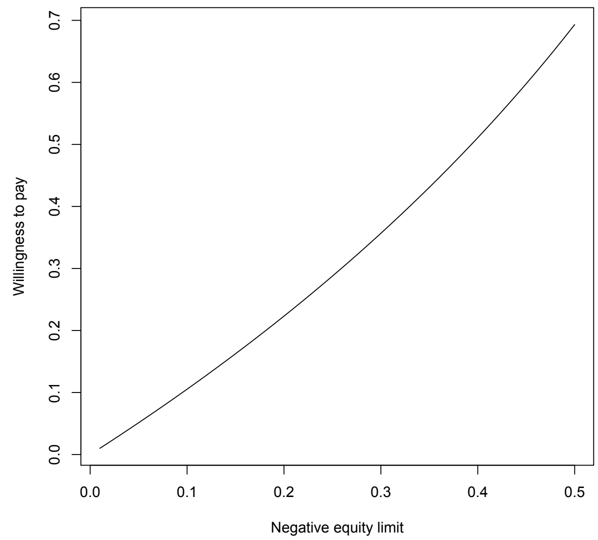

Fig. 1. Relationship of a borrowers willingness to pay to the negative equity limit (NEL). The NEL of the loan is the amount of negative equity the borrower will sustain before deciding to default. All numbers are based on a home value of unity. The greater the NEL, the higher the willingness to pay.

Therefore, this function is a natural choice for the borrower's default boundary. The parameter γ may be a function of macro-economic variables and borrower specific factors. It completely specifies the default boundary. Note that γ is specific to each borrower and that there is a distribution of willingness to pay (γ) within the borrower population.

Instead of the willingness to pay parameter γ, it is sometimes better to think in terms of the borrower's negative equity limit (NEL), which we denote E^-. The negative equity at default is $(L - D)$: substituting this into equation (2) above, setting $\theta = 0$, and re-arranging, we may compute the borrower's willingness to pay. Figure 1 shows the relationship between willingness to pay and the borrower's negative equity limit. The two quantities are positively related to each other.

Therefore, if we know that a borrower is likely to default when his negative equity becomes E^-, then we can use the relation in Figure 1 to infer the parameter γ for the willingness to pay.

In addition to this parameter for the willingness to pay, the model requires an estimate of house price growth μ, and volatility σ, and an estimate of the fractional value of the home recovered on foreclosure ϕ. As we will see later, we will also need a parameter that corrects for the risk-adjusted (i.e., expected change and premium) house price appreciation or depreciation relative to a benchmark, denoted λ, the price of risk. Therefore, the only parameters in the model that need to be estimated are $\{\gamma, \phi, \mu, \sigma, \lambda\}$ standing for willingness to pay, foreclosure recovery rate, house value growth rate, house price volatility, and risk premium, respectively. Hence, the model is parsimonious.

The following properties of the default barrier function are based on the lender's share in the mortgage appreciation:

(1) The greater the lender's share (θ), the higher is the default level of home value D. The likelihood of default is therefore greater.
(2) When the lender share $\theta = 0$, the default level is $Le^{-\gamma}$.
(3) When $\theta = 1$, the default level is $D = L$. The borrower defaults the moment there is negative equity.

We see that D is increasing and convex in θ, and decreasing and convex in γ. Next, we derive the value of the loan in closed-form after imposing the default barrier.

4 A Barrier Option Decomposition of Mortgage Value

In this section, we show that the value of the loan to the lender may be expressed in closed-form as a portfolio of options. The borrower defaults whenever the value of the home touches the default level $D(L, \theta)$. We assume that the lender (or the entity that buys the loan from a lender) is interested in maximizing the value of the loan at some horizon T. For example, a hedge fund that invests in distressed home loans may have a horizon of one year, over which they expect to resell the loans, anticipating that these loans will have appreciated in value by then. Alternately, lenders may wish to think of the horizon as the time it will take for the housing market to turn around. Given this horizon T, there are three components of mortgage value to the lender.

(1) *Non-Default Value*: When the borrower has not defaulted by time T, the lender recovers the principal of the loan L. (For simplicity, we normalize the initial price of the stochastic home value to $H_0 = 1$; this implies that the loan amount L is the LTV of the loan.) This component is equivalent to a "down-and-out cash-or-nothing call" option with a down barrier of D and a payment of L. Under the risk-neutral measure, we may write this as

$$Le^{-rT} \int_{D(L,\theta)}^{\infty} p(H_T | H_t > D, \forall t < T) \, dH_T \quad (3)$$

where $p(H_T | H_t > D, \forall t < T)$ is the probability density function of the terminal home value conditional on no interim default. Note that the conditional probability function contains the parameter D that is the lower limit of the definite integral in equation (3). Hence, changes in the willingness to pay coefficient γ will impact the default barrier D, as well as the conditional probability density of no default. As the willingness to pay declines, this component of loan value declines as well. In order to account for interest payments that are being received and reinvested until the horizon, the equation may also be applied without discounting, i.e., by excluding the term e^{-rT}.

(2) *Default Value*: If the home value H touches the barrier D, then default occurs, and the lender receives a fraction ϕ of the value of the home $H = D$, i.e., ϕD. This is akin to the "rebate" on a down-and-out call option. We may write this as

$$\phi D \int_0^T e^{-rt} f(t; D)\, dt \tag{4}$$

where $f(t; D)$ is the first-passage time density for $H_t = D$. As the willingness to pay declines, the barrier D rises and the first-passage time becomes shorter, thereby increasing this component of the loan value.

(3) *Shared Appreciation*: If there is no default, then the lender shares in the appreciation above a strike level K, which might be the value of the loan L (or some other level). This is akin to holding a "down-and-out call" option with a down barrier of D and a strike price of K. This option is written as

$$e^{-rT} \int_K^\infty (H_T - K)\, p(H_T | H_t > D, \forall t < T)\, dH_T \tag{5}$$

The lender takes a fraction θ of this call option. Here we might choose $K = 1$ as one common choice for the appreciation strike, but the loan could be restructured with any other strike level as well.

4.1 Housing growth rates, non-tradability, and premia

We now return to the issue of risk premia in the stochastic process for home values in equation (1) and provide a full characterization of it here. We first note that the fundamental asset-pricing partial differential equation (PDE) for any derivative (including a mortgage) $F(H, t)$ that is a function of the home value H is given by

$$\frac{\partial F}{\partial H}[\mu - \lambda\sigma] H + \frac{1}{2}\frac{\partial^2 F}{\partial H^2}\sigma^2 H^2 + \frac{\partial F}{\partial t} = rF \tag{6}$$

where λ is the price per unit risk for housing prices and non-tradeability. This risk premium arises on account of housing price risk that cannot be diversified. This PDE may be solved subject to the relevant boundary conditions. In this case these boundary conditions comprise a payment of principal at horizon T on the event of no-default, a foreclosure amount if the default barrier is breached, and an appreciation share if the home price appreciates sufficiently to make the appreciation share in-the-money.

For completeness, though not required in the implementation of the paper, we analyze the situation when the underlying asset is not continuously traded and dynamic trading of a hedge portfolio is not possible. Garman (1976) has shown that the exact PDE takes the following form:

$$\frac{\partial F}{\partial H}[g - \beta(\mu^* - r)]H + \frac{1}{2}\frac{\partial^2 F}{\partial H^2}\sigma^2 H^2 + \frac{\partial F}{\partial t} = rF \tag{7}$$

where g is the growth rate of the home value, β is the coefficient in a regression of home value on a benchmark for the housing sector, and μ^* is the expected rate of return on the benchmark. Comparing the coefficients on $\frac{\partial F}{\partial H}$ in the equations above, we see that, by inspection,

$$g = \mu, \qquad \beta(\mu^* - r) = \lambda\sigma \tag{8}$$

which offers one approach at eliciting the price of housing risk (λ) from data. We also define

$$R = \mu - \lambda\sigma$$

This variable R is the risk-adjusted return on home values, and is distinct from the risk-free rate r. In the next subsection, we present the solution to the partial differential equation (6) subject to the conditions in equations (3), (4), and (5).

4.2 Solution

Using standard mathematics for barrier options, as in Derman and Kani (1997), we obtain the value of the mortgage in closed-form as follows:

$$\begin{aligned}
\text{LOANVAL} \equiv & \, V(H, L, K, r, T, \phi, \theta, \mu, \lambda, \sigma, \gamma) \\
= & \, Le^{-rT}\left[N(d_2') - (D/H)^{2(R/\sigma^2)-1} \cdot N(d_{2b}')\right] \\
& + \phi D\left[(D/H)^{b_1} \cdot N(a_1) + (D/H)^{b_2} \cdot N(a_2)\right] \\
& + \theta\left[C_{SAM}(H, K) - D^{2(R/\sigma^2)-1} \cdot C_{SAM}(D^2/H, K)\right]
\end{aligned} \tag{9}$$

where

$$
\begin{aligned}
D &= L\exp[-\gamma(1-\theta)] \\
d_2' &= \frac{\ln(H/D) + (R - 0.5\sigma^2)T}{\sigma\sqrt{T}} \\
d_{2b}' &= \frac{\ln(D/H) + (R - 0.5\sigma^2)T}{\sigma\sqrt{T}} \\
a_1 &= \frac{\ln(D/H) + \sqrt{2r\sigma^2 + (R-0.5\sigma^2)^2} \cdot T}{\sigma\sqrt{T}} \\
a_2 &= \frac{\ln(D/H) - \sqrt{2r\sigma^2 + (R-0.5\sigma^2)^2} \cdot T}{\sigma\sqrt{T}} \\
b_1 &= \frac{(R-0.5\sigma^2) + \sqrt{2r\sigma^2 + (R-0.5\sigma^2)^2}}{\sigma^2} \\
b_2 &= \frac{(R-0.5\sigma^2) - \sqrt{2r\sigma^2 + (R-0.5\sigma^2)^2}}{\sigma^2} \\
C_{SAM}(x,y) &= xe^{-(r-R)T}N(d_1') - ye^{-rT}N(d_1' - \sigma\sqrt{T}) \\
d_1' &= \frac{\ln(x/y) + (R + 0.5\sigma^2)T}{\sigma\sqrt{T}}
\end{aligned}
$$

As stated earlier, the first term in equation (9) may be applied without discounting to account for interest received on the loan. This formula may easily be implemented even on a spreadsheet. We present numerical examples and analysis of the closed-form model in Section 5 to understand the different aspects of the optimal loan modification. Because the loan value is available in closed-form, it is easy to find the level of LTV (L) that optimizes loan value V.

5 Implementation

We first examine the optimal LTV for the modified loan by implementing the model for a set of standard parameters. Our implementation covers both cases, with shared appreciation in the new loan structure, and without. The first term of equation (9) is applied without discounting in all examples in this section—this makes no material qualitative difference to the results. The plot of the loan value at various modified LTV levels is shown in Figure 2. There are some clear results that we see here. First, the optimal LTV is about 0.98 for loans with shared appreciation, though it is slightly higher at 1.03 for the no-SAM loan. Given the appreciation share, it is optimal for the lender to take down the LTV a little more in the case of a SAM. Second, SAMs are mostly superior to the modified loans without SAMs—the optimized loan value is higher for SAMs, especially in the

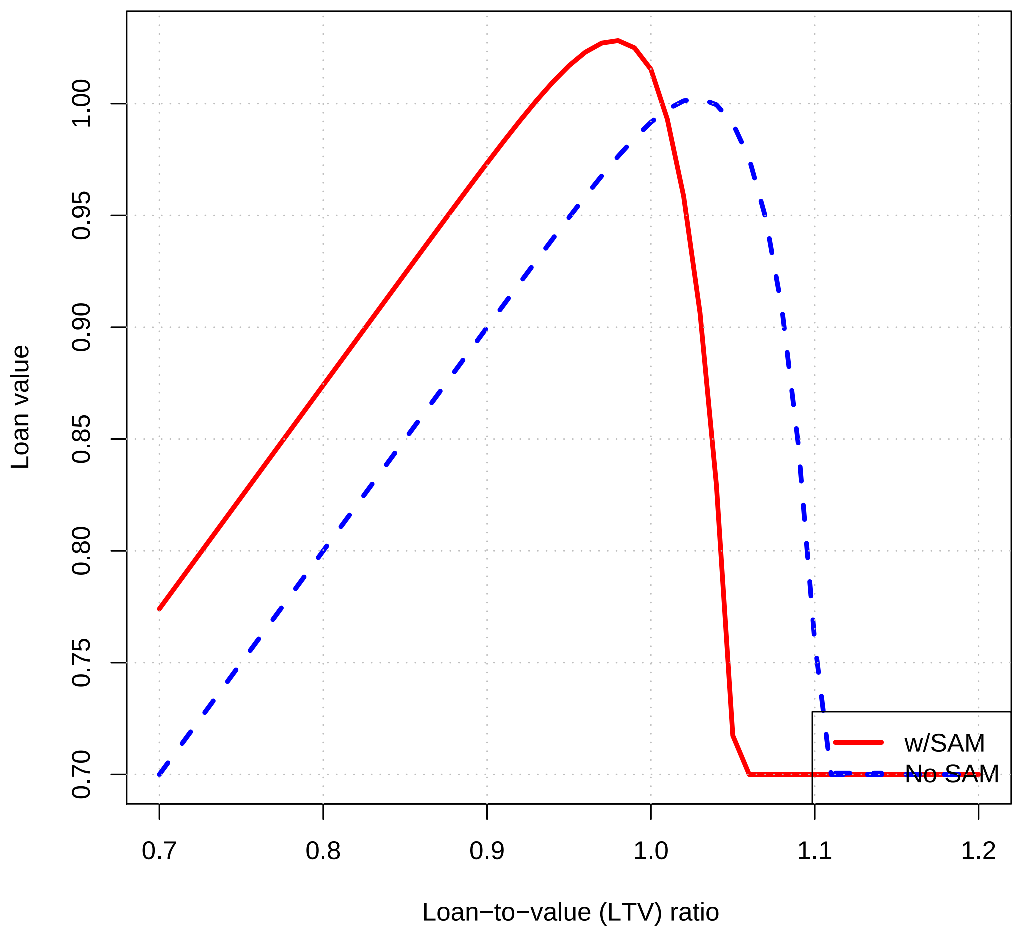

Fig. 2. Loan value as LTV is varied for loans with and without appreciation sharing. The parameters for the plot are as follows: willingness to pay coefficient $\gamma = 0.1$, home price volatility $\sigma = 0.04$, foreclosure fraction $\phi = 0.7$, risk-free rate $r = 0.02$, the house value growth rate $\mu = 0.04$, price of risk $\lambda = 0.25$, and the horizon of the model $T = 5$ years. The appreciation share fraction is $\theta = 0.50$ for the case when a SAM is applied, and $\theta = 0$ when there is no share appreciation.

relevant region where LTV is less than one.

We note that when the LTV is set too high the default barrier is greater than the home value $(D > H)$, and results in immediate default with foreclosure recovery value, shown in the flat right tail of the loan value in Figure 2.

Figure 3 shows the loan values for shared appreciation mortgages when the foreclosure recovery percentage is $\phi = \{0.5, 0.7\}$. As expected , the loan value is higher when ϕ is higher. The optimal LTV is 0.97 when $\phi = 0.5$ and 0.98 when

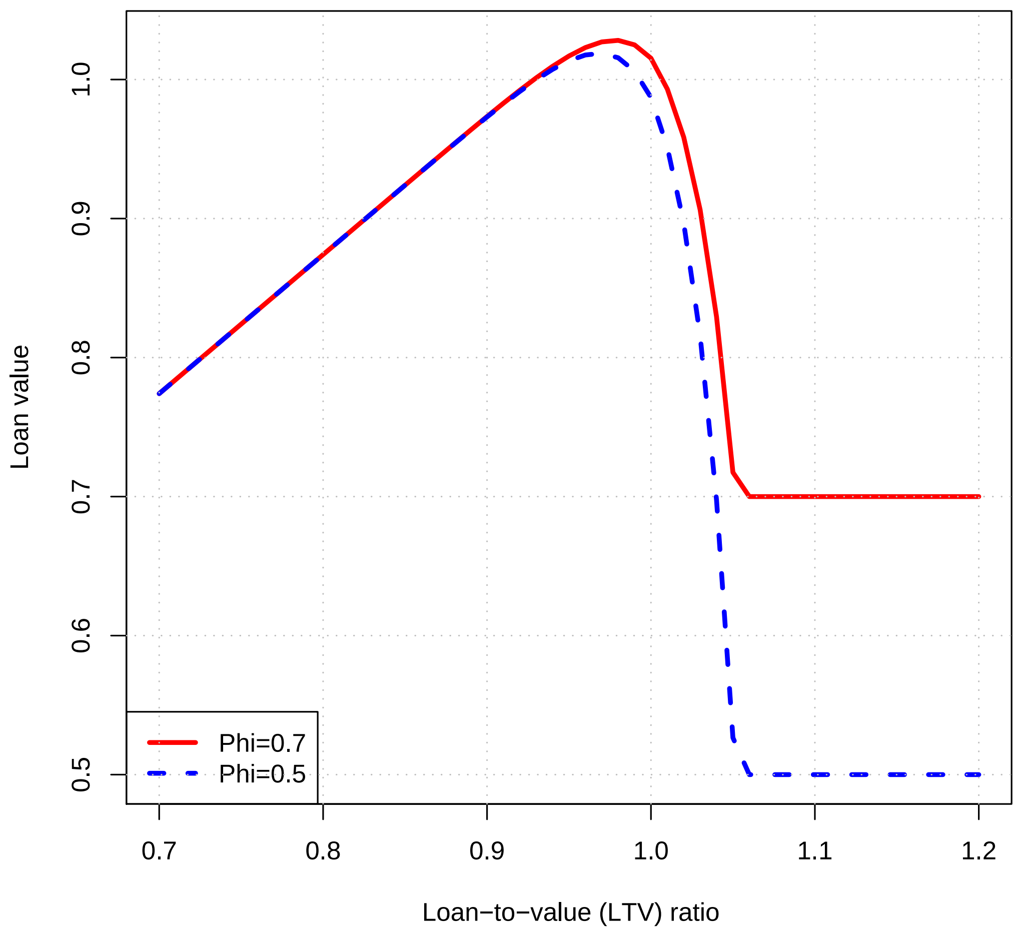

Fig. 3. Loan value as LTV is varied for loans with SAMs and the foreclosure recovery rate is varied across $\{\phi = 0.5, \phi = 0.7\}$. Both cases are with appreciation sharing. The parameters for the plot are as follows: willingness to pay coefficient $\gamma = 0.1$, home price volatility $\sigma = 0.04$, risk-free rate $r = 0.02$, the house value growth rate $\mu = 0.04$, price of risk $\lambda = 0.25$, and the horizon of the model $T = 5$ years. The appreciation share fraction is $\theta = 0.50$.

$\phi = 0.7$.

In Figure 4 we see that as home price volatility increases, the value of the loan decreases because the value of the default option to the borrower increases. For the base case, i.e., when volatility $\sigma = 0.04$, the optimal LTV is 0.99. When the volatility increases to $\sigma = 0.10$, the optimal LTV drops to 0.92. As volatility increases a higher principal reduction is needed in order to optimize loan value. A similar reduction in optimal LTV occurs when considering expected growth in home values, $\mu = \{-4\%, +4\%\}$, shown in Figure 5. The optimal LTV is 0.93

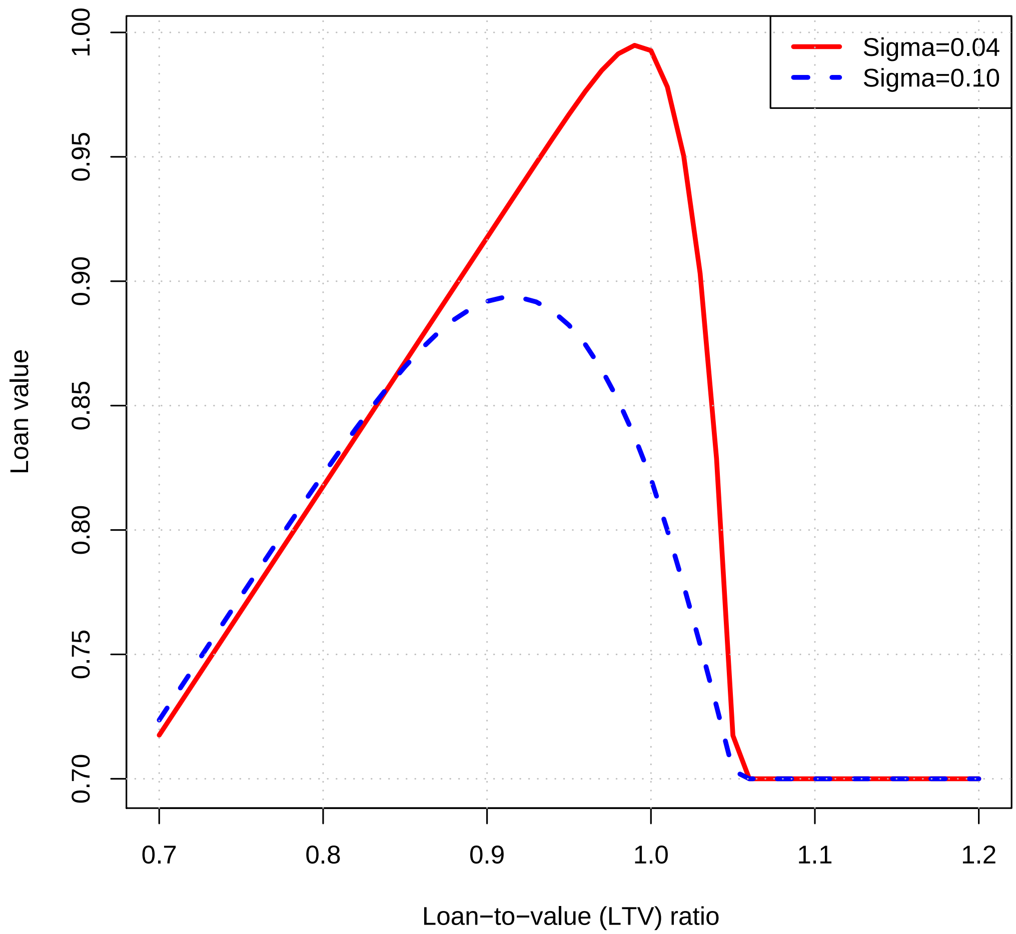

Fig. 4. Loan value as LTV is varied for loans with SAMs and housing price volatility is varied across $\{\sigma = 0.04, \sigma = 0.10\}$. Both cases are with appreciation sharing. The parameters for the plot are as follows: willingness to pay coefficient $\gamma = 0.1$, foreclosure percentage $\phi = 0.7$, risk-free rate $r = 0.02$, the house value growth rate $\mu = 0.04$, price of risk $\lambda = 0.25$, and the horizon of the model $T = 1$ year. The appreciation share fraction is $\theta = 0.50$.

when growth is negative, and 0.99 when it is positive—weaker conditions in the housing market require greater principal forgiveness.

Next, we examine the willingness to pay parameter γ. Ceteris paribus, as the willingness to pay declines, the probability that the borrower will default increases. This will have three implications. One, the value of the loan declines as the willingness to pay falls—see Figure 6. Two, the difference in loan values as γ changes is greater when LTV is high than when it is low. This is intuitively expected because the probability of defaut is greater at higher LTV, resulting in a bigger change

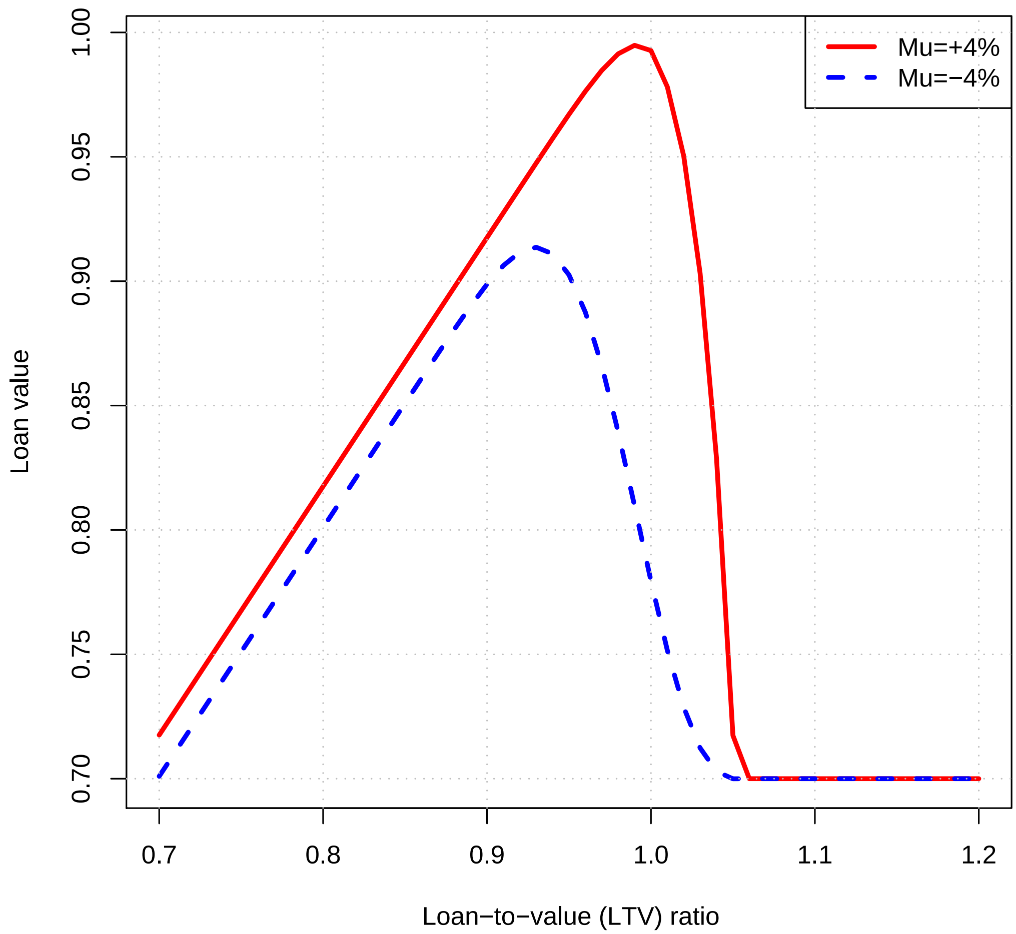

Fig. 5. Loan value as LTV is varied for loans with SAMs and housing value growth rate is varied across $\{\mu = -0.04, \mu = +0.04\}$. Both cases are with appreciation sharing. The parameters for the plot are as follows: willingness to pay coefficient $\gamma = 0.1$, foreclosure percentage $\phi = 0.7$, risk-free rate $r = 0.02$, housing price volatility $\sigma = 0.04$, price of risk $\lambda = 0.25$, and the horizon of the model $T = 1$ year. The appreciation share fraction is $\theta = 0.50$.

in expected deadweight costs of foreclosure for every unit of willingness to pay. Third, and possibly the most important, the optimal LTV changes substantially with the willingness to pay. In Figure 6 when $\gamma = 0.01$, the optimal LTV is around 0.95. And optimal LTV at $\gamma = \{0.1, 0.2\}$ is 0.99 and 1.04, respectively. Assessing a borrower's willingness to pay matters.

Figure 7 shows that the optimal loan value is higher when the appreciation share (θ) is substantial, even though an increase in appreciation share results in a lower willingness to refrain from defaulting on the loan. At $\theta = 0.7$, the optimal LTV for

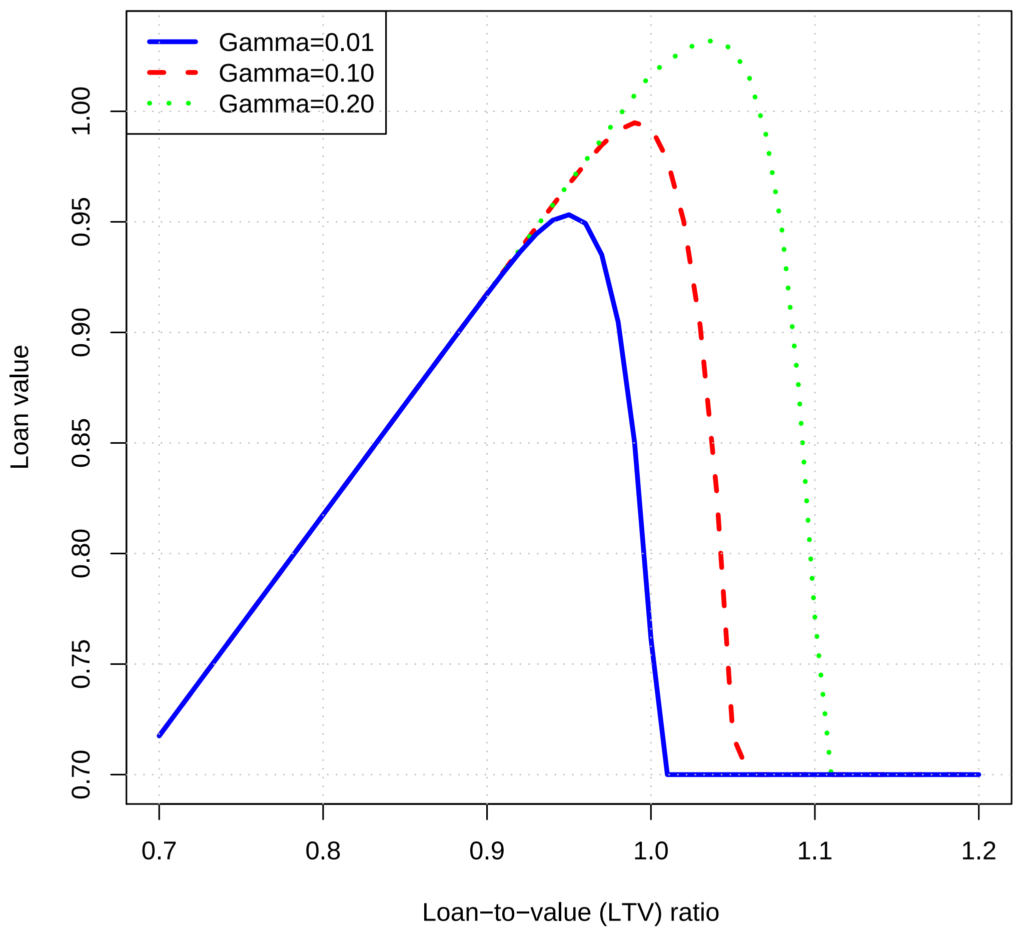

Fig. 6. Loan value as LTV is varied for loans with SAMs and willingness to pay is varied across $\{\gamma = 0.01, \gamma = 0.10, \gamma = 0.20\}$. All cases are with appreciation sharing. The parameters for the plot are as follows: the house value growth rate $\mu = 0.04$, price of risk $\lambda = 0.25$, foreclosure percentage $\phi = 0.7$, risk-free rate $r = 0.02$, housing price volatility $\sigma = 0.04$, and the horizon of the model $T = 1$ year. The appreciation share fraction is $\theta = 0.50$.

the modified loan falls to 0.96, whereas the optimal LTV is 0.98 when $\theta = 0.5$. In the former case, the optimized loan is worth 0.954 versus 0.941 in the latter case.

6 A Two-Horizon Model

The preceding model is parsimonious, and provides intuitive and appealing results. In this section, we extend this single-horizon model to two horizons, both of any

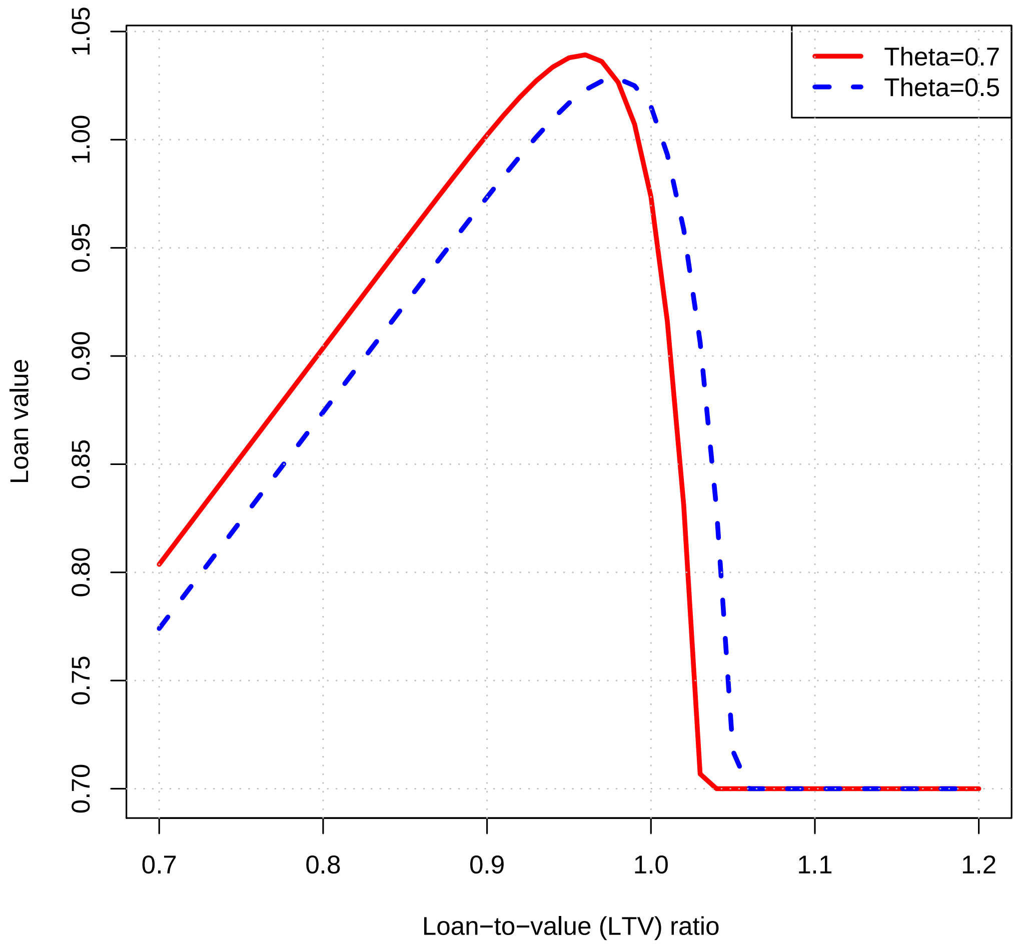

Fig. 7. Loan value as LTV is varied for loans with SAMs and appreciation is varied across $\{\theta = 0.70, \theta = 0.50\}$. The parameters for the plot are as follows: willingness to pay coefficient $\gamma = 0.1$, foreclosure percentage $\phi = 0.7$, risk-free rate $r = 0.02$, housing price volatility $\sigma = 0.04$, and the horizon of the model $T = 5$ years. The growth rate in home values is $\mu = 0.04$, and the price of risk $\lambda = 0.25$.

length. The tree approach we use is described below and is completely general and allows multiple periods, but we focus on only two periods in this section.

Having two horizons τ and $T > \tau$ accommodates different growth rates in home values in two consecutive periods. For instance, the model may use a negative growth rate for $\tau = 1$ year, and then a positive growth rate for the remaining period from τ to $T = 3$ years, i.e., for a second period of length $T - \tau = 2$ years. Such an approach also implicitly incorporates mean-reversion in the model because the growth rates in each period may be chosen to be opposite in sign.

The formula in equation (9) may be implemented computationally using a binomial tree instead. Doing so enables the use of different home price growth rates in the two periods of the extended model. (It also offers a way to check the single period model as well as a special case—indeed, the tree we develop reproduces the prices from the formula in equation (9).)

We use a simple Cox, Ross and Rubinstein (1979) binomial tree—the reader may refer to the original paper for the technology which is too widely known to repeat here. The model is adjusted for the price of housing risk. For parsimony, we present the barebones equations that will be applied in the model; this will allow a researcher familiar with the model to apply it immediately. The binomial tree has time step h and the probability of an up move on the tree is q.

$$\begin{aligned}
u &= \exp(\sigma\sqrt{h}) \qquad \text{Up move factor} \\
d &= \exp(-\sigma\sqrt{h}) \qquad \text{Down move factor} \\
\mathcal{R} &= \exp(rh) \qquad \text{Risk-neutral drift} \\
\delta &= r - \mu(t) + \lambda\sigma \qquad \text{Adjustment for the growth rate and price of risk} \\
q &= \frac{\mathcal{R}e^{-\delta h} - d}{u - d} \qquad \text{Probability of an up move}
\end{aligned}$$

The probability q incorporates the effects of the risk premium for illiquidity of the housing market. Note that $\mu(t)$ has been written as a deterministic function of time, as it is allowed to vary. The loan is valued by backward recursion, starting at the horizon of the model T, and working back to each previous period the expected discounted loan value, eventually reaching time zero, to obtain current loan value. The terminal values of home prices H_T on the tree determine the payoffs in the event of no default with and without appreciation sharing. During backward recursion, if the home price level is below the default barrier D, we foreclose and assume that the value is ϕ times the defaulted value of the home. We note that the tree is still recombining despite the fact that the home value growth rate changes over time, as long as volatility of home values is not stochastic. Our main goal here is to allow for different growth views over the investment horizon. The model is applicable to more than two periods as well.

An important benefit of the tree approach is that it is very general and may be extended in different ways. It is easy to incorporate coupon payments on the tree each period for as long as the loan has not defaulted. Because our model is focused only on strategic default, the spread over the risk free rate used in the coupon here should be interpreted as compensation for strategic default. The model may be extended to account for non-strategic default and prepayment risk, though this is not the goal of the analysis here. In the ensuing examples, we set the coupon such that it yields a par value loan exclusive of shared appreciation when the LTV is set optimally.

To illustrate the application of the model, see Figure 8. The model has two periods of one and nine years, respectively, the first with a negative growth rate (-5%) and the second with positive growth ($+4\%$). The results are similar to the ones encountered in the single horizon problem, in that the optimal LTV for a loan with a SAM is lower than that for a loan without a SAM. The SAM case gives a higher economic value to the loan at the optimal LTV. The coupon rate on the loan has been set such that at the optimal LTV, the loan prices up close to par.

In Figure 9 we examine whether a period of negative growth followed by one of positive growth is better than the reverse situation, i.e., positive growth followed by negative growth. We see that optimal loan values are lower when the first period has negative growth. Therefore, LTV has to be set lower when there is negative growth initially in house prices.

7 Discussion

Investments in distressed home loans are increasing as the housing crisis deepens. Banks are holding more of these loans, and are attempting to modify these loans in a manner that will optimize the value of their loan books. Likewise, specialized funds that buy and modify home loans also write down principal to maximize the value of their holdings. This paper provides a simple and closed-form model of loan value that may be used to determine the optimal LTV at which a loan should be reset in order to maximize loan portfolio values.

The reduced-form model presented here is parsimonious. It is solved in closed-form and requires very few parameters: the volatility of home prices (σ), expected growth in home value (μ), the price of housing risk (λ), the percentage recovery value on foreclosure (ϕ), and the willingness to pay parameter (γ). The other parameters—the risk-free rate, appreciation share, and horizon are easy for the user to supply. The model has also been generalized to an implementation with binomial trees so that different periodic growth rates, coupons and other loan features may be analyzed on the tree.

The volatility of home prices and expected growth may be obtained from various real-estate indexes or market forecasts. There are studies that estimate expected foreclosure recovery rates. The interesting parameter to estimate is willingness to pay, and given a corpus of defaulted loans, we may extract this from an examination of the levels of negative equity at which borrowers chose to default. Developing econometric models to estimate willingness to pay is an interesting avenue for

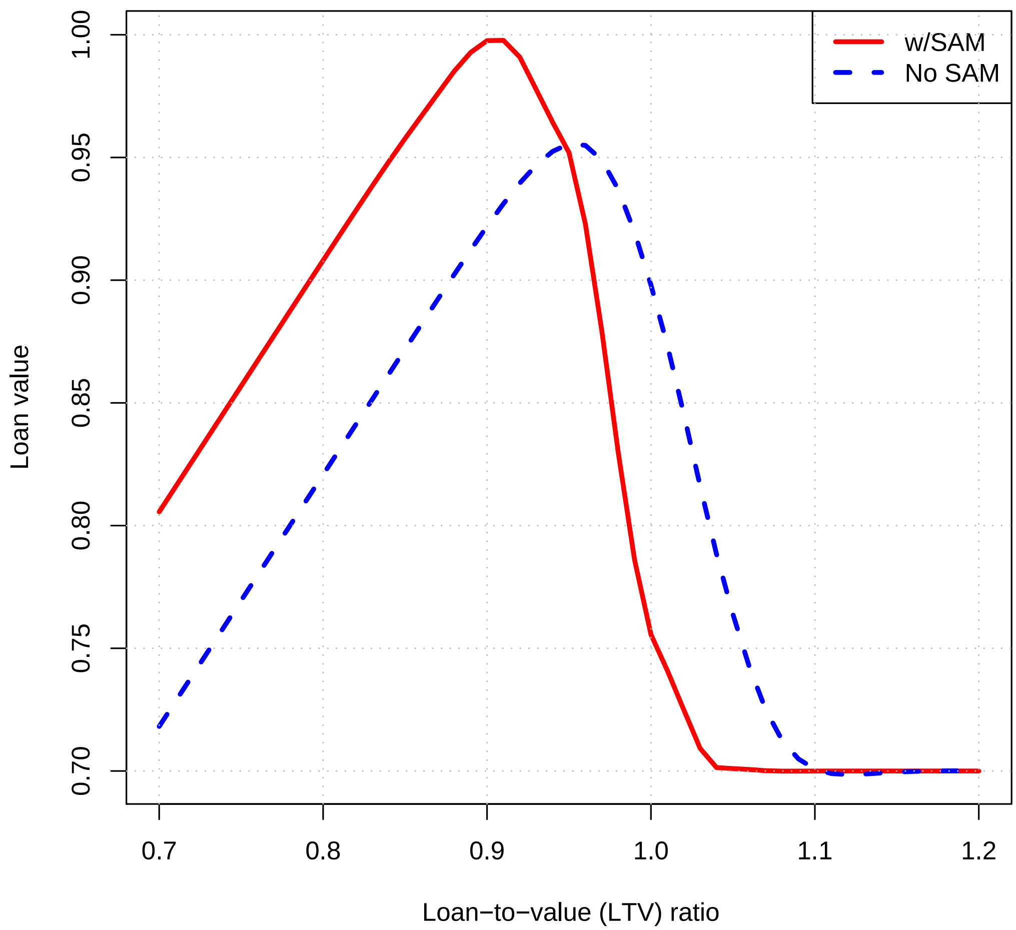

Fig. 8. Two-horizon model: Loan value as LTV is varied for loans with and without appreciation sharing. The parameters for the plot are as follows: willingness to pay coefficient $\gamma = 0.1$, home price volatility is $\sigma = 0.04$, foreclosure recovery fraction $\phi = 0.7$, risk-free rate $r = 0.03$, the house value growth rate in each period is $\mu_1 = -0.05$ and $\mu_2 = +0.04$, price of risk $\lambda = 0.25$, and the two horizons of the model are $\tau = 1$ and $T = 10$ years. The appreciation share fraction is $\theta = 0.50$ for the case when a SAM is applied, and $\theta = 0$ when there is no share appreciation. The coupon rate on the loan is 3.3%. The coupon rate on the loan has been set such that at the optimal LTV, the loan with a SAM prices up close to par.

further research as this is at the heart of any analysis of strategic default.

To summarize, the main contributions of this paper are as follows. First, we develop two models (a closed-form equation and a tree model) for optimizing loan modifications in the presence of strategic default. The model also accounts for non-tradability of the underlying home. Several comparative statics are presented

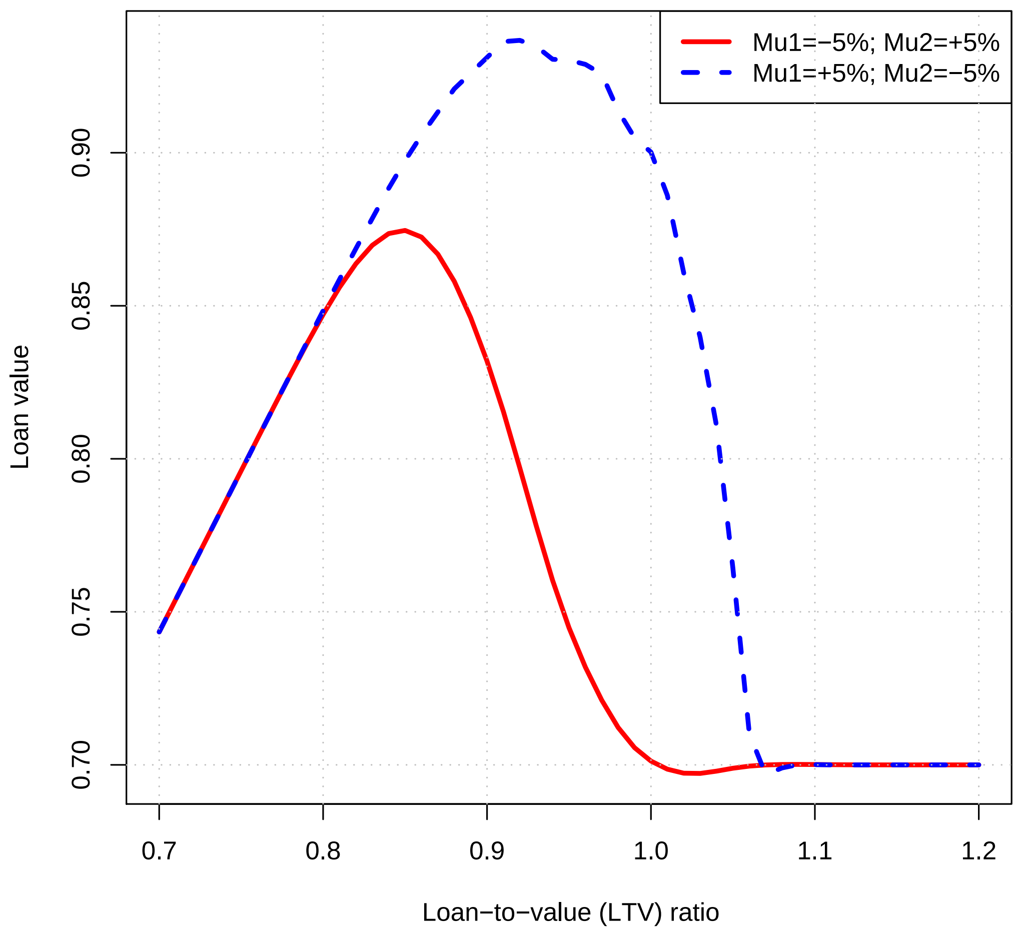

Fig. 9. Two-horizon model: Loan value as LTV is varied for loans when there are periods of positive and negative growth in home values. The parameters for the plot are as follows: willingness to pay coefficient $\gamma = 0.1$, home price volatility is $\sigma = 0.04$, foreclosure recovery fraction $\phi = 0.7$, risk-free rate $r = 0.03$, the house value growth rate in each period is $\mu_1 = -0.05$ and $\mu_2 = +0.05$ (reversed in the second case), price of risk $\lambda = 0.25$, and the two horizons of the model are $\tau = 2$ and $T = 4$ years. The appreciation share fraction is $\theta = 0.50$. The coupon rate on the loan is 4.4%. The coupon rate on the loan has been set such that at the optimal LTV, the loan for the first up then down scenario prices up close to par.

graphically to show the sensitivity of home values to various loan parameters. Second, we show that loan modifications with shared appreciation appear to mitigate the risks of strategic default, as it enables the lender to modify the loan to lower LTVs than without shared appreciation, yet maximize the value of the loan via the shared appreciation component. Third, the model may be used for a range of home price growth scenarios, in particular mean-reversion in home prices and different growth rates for different periods, thereby allowing the modeler to determine the optimal loan modification under various future price assumptions in the housing markets. Overall, the paper provides a comprehensive analysis of how principal forgiveness may be applied to optimally solving the negative equity aspect of the housing crisis.

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