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Risk-neutralizing statistical distributions: With an application to pricing reinsurance contracts on FDIC losses *

Abstract

This paper proposes methods for obtaining risk neutral distributions when only the statistical density is observed. We employ renormalized exponential tilts and estimate the tilt coefficients from related options markets. Particular emphasis is placed on reinsurance losses for which we price in closed form using the Weibull extreme value distribution. The procedure is illustrated in detail for FDIC losses.

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1 Introduction

It is well recognized that contingent claims written on the realization of an underlying uncertainty (like the realized variance on the S&P500 index or the losses of a group of hedge funds) are priced by evaluating discounted expectations of claim payouts under a risk neutral probability. This risk neutral probability generally differs from its statistical counterpart. The statistical distribution describes the likelihood of these risky outcomes and is typically estimated from historical time series data on past realizations. The risk neutral probability, on the other hand, is the market price of Arrow-Debreu securities associated with risky events. Often the risk neutral probability can be inferred from options prices as described in Breeden and Litzenberger (1978). However, options markets do not exist for a wide collection of underlying uncertainties. In contrast, there is readily available historical time series data on past realizations. The question then arises as to how one may construct the risk neutral density from the estimated statistical density for such risks, to price contingent claims written on these uncertainties. We refer to such procedures as risk neutralizing the statistical distribution.

The objective of this paper is to propose such a strategy. We obtain the risk neutral distribution by exponentially tilting and renormalizing the statistical one. This approach is broadly consistent with much of the current literature that incorporates risk aversion using either constant market prices of risk (Heath, Jarrow and Morton (1992), Heston (1993)) or constant relative risk aversion utility functions (Naik and Lee (1990)). Our specific contribution lies in showing how one may obtain the coefficient of exponential tilting by putting together an analysis of the underlying risk and information from selected traded options markets that relate to the underlying risk.

We show how tilt coefficients embedded in traded options prices can be used to price non-traded risks. Hence, we derive an explicit *tilt adjustment* that relates the degree of exponential tilting applicable to the statistical distribution of the underlying risk to the degree of exponential tilt observed in the related options market. The resulting methodology is potentially widely applicable in pricing risks embedded in loan defaults, mortgage refinancing, electricity and weather derivatives, and catastrophic losses to give a few examples. Common to the pricing of these risks is the absence of a liquid options market and the presence of high quality statistical data.

We illustrate our approach in the context of pricing contracts written on deposit insurance losses. Such an exercise is a natural experiment for our proposed approach, where we observe a long history of statistical (historical) losses on bank failures enabling us to capture the statistical distribution. In contrast, there do not exist any traded claims on these losses, hence, the risk neutral distribution can not be directly obtained. However, we can observe the risk-neutral distribution of claims traded on financial variables (e.g. a price index on bank stocks) useful in predicting these losses. We show how options written on a bank index can be used to risk neutralize the statistical distribution of annual losses faced by the Federal Deposit Insurance Corporation (FDIC).

The specific contract we analyze is the excess of loss reinsurance contract, which represents a portfolio of call options written on the aggregate loss level of an insurer. Effectively, the reinsurer sells the insurer a call-spread where the reinsurer will have to cover losses above a strike level but its commitment is capped by a stated coverage level. We obtain a closed form pricing expression for the call-spread under the assumption that the underlying losses follow a Weibull distribution, which is in the family of extreme-value distributions.

We estimate the statistical distribution of the FDIC's losses on bank failures, in the two-parameter Weibull family for annual losses incurred by the FDIC during 1986-2000. To infer the applicable tilt coefficient in the traded options markets we examine the prices of deep out-of-the-money puts and calls on bank equity index (BKX) and estimate the implied risk neutral distribution in this market. We also estimate the statistical Weibull distribution for the BKX returns and infer the implied tilting coefficient in the options markets. We then risk-neutralize the FDIC's statistical loss distribution with the level of tilting implied in the BKX options market by our derived tilt adjustment.

As a check on our methodology, we compare our estimated reinsurance prices with those of MMC Enterprise Risk (MMC). In a report submitted to the FDIC, MMC provides two rough price estimates FDIC might have to pay to a private insurer to purchase a call-spread (MMC, 2001, p. 21). Our calculations show that MMC estimates are in the vicinity of our price-estimates and thus reflect a risk neutral pricing rather than a statistical one. In another application of our findings, we estimate that the FDIC needs to charge the banking system $3.1 billion in aggregate insurance premium for loss coverage of $26.56 billion. This aggregate insurance cost represents 16.2 cents on $100 insured deposits at the level of $1.9 trillion insured deposits by the end of 2001. Given the proximity of this estimate to the effective insurance premiums assessed by the FDIC, we assert that FDIC is implicitly tilting the statistical distribution of its losses.

The paper is organized as follows. Section 2 presents the underlying framework for our analysis. Section 3 derives the Weibull option pricing model and section 4 estimates the implied risk aversion in the options market. Section 5 shows the application of reinsurance pricing. Section 6 concludes the paper.

2 Risk neutralization strategy

We recognize that the question of risk neutralizing a statistical distribution may not arise under certain conditions. For example, in the case of catastrophic loss insurance, Cummins, Lewis, and Phillips (1999), Froot (1996,1999), Cummins (1999), and Doherty (1997) argue that catastrophe risks caused by natural disasters are uncorrelated with the market portfolio and hence no change of measure is needed. In other words, prices of contingent claims on such risks equal statistically expected payoffs discounted at the risk free rate. In contrast Atlan, Geman, Madan and Yor (2004) recently show that merely being uncorrelated with the market portfolio may not be sufficient to justify the absence of a change of probability for the risk in question. Our approach, by estimating the tilt, accommodates a zero tilt as a special case.

2.1 The Exponential Tilt

Our strategy for risk neutralizing a statistical loss distribution models the expectation of the pricing kernel conditional on the risk at hand. We suppose that the quoted prices for claims of this type are free of arbitrage. In general the no-arbitrage property is equivalent to prices being equal to discounted expected payoffs under a change of probability from the statistical one to a new probability measure termed the risk neutral probability (Harrison and Kreps, 1979). Somewhat more formally, the price, w, of a claim to a state contingent cash flow $c(\omega)$ can be written as the discounted at the risk-free rate of return, r, of the expected cash flow, where expectation is taken at the risk-neutral measure, E^Q

$$w = e^{-r} E^Q \left[c(\omega)\right] \tag{1}$$

Alternatively, the expectation can be taken at the statistical measure, E^Pas follows

$$w = e^{-r} E^P \left[\Lambda(\omega) c(\omega)\right] \tag{2}$$

where $\Lambda(\omega)$ is the change of measure density. Conditioning on the level of the underlying risk, say the loss level L we may write

$$w = e^{-r} E^P \left[E^P \left[\Lambda(\omega)|L\right] c(L)\right] \tag{3}$$

where we suppose for simplicity that the claim is contingent only on the value of L. Defining by

$$g(L) = E^P \left[\Lambda(\omega)|L\right] \tag{4}$$

we see that the market price for a claim is an expected value of the tilted cash flow, tilted by $g(L)$. Formally,

$$E^Q[c(L)] = E^P[g(L)c(L)],$$

and it follows that

$$\int_0^\infty c(L) q_L(L) dL = \int_0^\infty c(L) g(L) p_L(L) dL \tag{5}$$

where q_L, p_L are respectively the risk neutral and statistical densities of the loss level. Equation (5) implies that

$$q_L(L) = g(L)p_L(L). \tag{6}$$

Further, g is a positive function of the real valued variable L and the mixture of exponentials is a spanning set of functions for all potential tilt functions g. For a local analysis of the behavior in a part of the tail, the use of a single exponential is adequate. Hence, our approach to identify $q_L(L)$ is to assume a statistical distribution for the historical loss levels, $p_L(L)$, and tilt this statistical distribution by a loss exponential tilt coefficient, α_L, and renormalizing it to obtain the risk-neutral density as follows:

$$q_L(L) = \frac{e^{\alpha_L L}}{\int_0^\infty e^{\alpha_L L} p_L(L)dL} p_L(L) \tag{7}$$

From an incomplete markets point of view the pricing kernel to be used is no longer uniquely determined and is individual specific as well. As explained in Cummins, Lewis, and Phillips (1999) the actuarial approach to pricing catastrophic insurance employs utility functions of risk averse agents to construct the risk-neutralized density. For a specific connection between equation (7) and utility theory the reader is referred to Appendix 8.1 that relates exponential tilts to a specific utility function. The coefficient of exponential tilting can then be related to a constant absolute risk aversion coefficient of an agent facing the losses.

The use of exponential tilts has a long history in finance. For one, it is now recognized that Black-Merton-Scholes option pricing results from the application of an exponential tilt to the underlying Brownian motion (Duffie, 1992). In the context of the Black-Merton-Scholes complete markets model this exponential tilt is in fact the unique complete markets solution. The idea has been subsequently used in a variety of incomplete markets contexts including Heston (1993), where the risk in the Brownian motion driving the volatility is priced by exponential tilting. More generally for a diffusion filtration it is well known (Karatzas and Shreve, 1991) that all measure changes are locally exponential tilts of the underlying Brownian motions. The method has been employed in the term structure literature. (see for example, Heath, Jarrow, and Morton, 1992). In models with jumps, Naik and Lee (1990) use exponential tilts by employing constant relative risk aversion utility functions. Furthermore, we note that in many insurance applications risk-neutral and statistical distributions are related by what is called the Esscher transform that exponentially tilts the statistical distribution to determine the risk neutral distribution (Esscher, 1932; Sondermann, 1991; Gerber and Shiu, 1996).

Our approach can also be related to the recent literature estimating the risk-neutral and statistical densities to make inferences about the implied risk-aversion coefficients (see for example, Jackwerth, 2000; Ait-Sahalia and Lo, 2000; Ait-Sahalia, Wang, and Yared, 2001; Coutant, 2001; Bakshi, Kapadia, and Madan, 2003; Bliss and Panigirtzoglou, 2004). Collectively, this literature focuses on the entire distribution of the underlying assets values. However, the contracts we focus on have payouts in the tail of the statistical distribution. Hence, we add to this literature by utilizing extreme value theory that characterizes the tail distributions of positive random variables and estimate the tilting applicable to the tail events rather than the entire distribution of the outcomes.

The risk neutralization strategy of equation (7) is completed on determining the exponential tilt coefficient α_L. We define the measure change density $y_L(L)$ by

$$y_L(L) = \frac{q_L(L)}{p_L(L)} \tag{8}$$

and note that for this risk neutralization we have

$$\alpha_L = \frac{\partial \ln (y_L(L))}{\partial L}. \tag{9}$$

2.2 Tilt coefficients for non-traded risks

We begin by noting that tilt coefficients are estimable for risks associated with liquid options markets. An analysis of the time series for the risk at hand yields the statistical distribution while from traded options we obtain the risk neutral density following Breeden and Litzenberger (1978).

Given both densities, the tilt coefficient may be estimated by regression methods. This subsection derives a relationship between tilts appropriate for non-traded risks and those observed in the options markets.

We refer to the non-traded risk under study as the risk of loss level L. The objective is to tilt the statistical loss distribution of the loss L and price a contract written on L. Let S denote the level of some financial index with a liquid options market that is related to the loss risk, in the sense that the conditional distribution of losses L given the index level S is nontrivial. Suppose that the joint density of the loss level L and some financial index S, $h(L, S)$, is such that the conditional density of L given S, $\psi(L|S)$, depends on S , i.e.

$$h(L, S) = p_S(S)\psi(L|S) \tag{10}$$

where $p_S(S)$ is the statistical density of the financial index S. The statistical density of the loss level may then be computed as

$$p_L(L) = \int_0^\infty p_S(S)\psi(L|S)dS. \tag{11}$$

A change of probability on S to $q_S(S)$ induces a change on L by

$$q_L(L) = \int_0^\infty q_S(S)\psi(L|S)dS. \tag{12}$$

Here we have supposed that there is only a change of probability on S but no change in the conditional law of L given S. Taking the ratio of (12) to (11) we observe that

$$\begin{aligned} y_L(L) &= \int_0^\infty \frac{q_S(S)}{p_S(S)} \frac{p_S(S)\psi(L|S)}{\int_0^\infty p_S(S)\psi(L|S)dS} dS & (13) \\ &= \int_0^\infty y_S(S)\psi(S|L)dS, & (14) \end{aligned}$$

where $y_S(S)$ is the statistical measure change density defined as

$$y_S(S) = \frac{q_S(S)}{p_S(S)}. \tag{15}$$

Observe that $y_L(L)$ is an average of $y_S(S)$ taken with respect to the conditional probability density

$\psi(S|L)$. By the mean value theorem, assuming continuity of y_S, there exists a function

$$S = \phi(L) \tag{16}$$

such that

$$\int_0^\infty y_S(S)\psi(S|L)dS = y_S(\phi(L)). \tag{17}$$

Equation (17) simply states that the change of measure density for loss levels is a weighted average of the change of measure density for the financial index and hence equals the latter evaluated at some level $\phi(L)$,

$$y_L(L) = y_S(\phi(L)). \tag{18}$$

The exponential tilt in L, given in equation (9) is measured by the logarithmic derivative of $y_L(L)$ and we evaluate this as

$$\begin{aligned} \frac{\partial}{\partial L}\ln(y_L(L)) &= \phi'(L)\frac{\partial}{\partial S}\ln(y_S(\phi(L)) \qquad (19) \\ \alpha_L &= \phi'(L)\alpha_S. \qquad (20) \end{aligned}$$

Hence the exponential tilt appropriate for the loss level is the options tilt (α_S) computed at $\phi(L)$ scaled by the sensitivity $\phi'(L)$.

2.3 Estimating α_L

The exponential tilt coefficient for a non-traded risk, α_L, is estimated in two stages. First, a simple model for the measure change function $y_S(S)$ in the options market provides the estimate of α_S. Next, the tilt adjustment ϕ' is obtained using a regressions model for the conditional distribution of S given L. Hence, the tilt coefficient for loss levels is estimated using equation (20).

It is customary to employ out-of-the-money call and put options in inferring risk neutral densities following Breeden and Litzenberger (1978), as these are the more liquid options. Hence for strikes below the current spot one uses puts, while for strikes above the spot we use calls. On recovering $y_S(S)$ one may graph its logarithm against S and if this is linear across the entire range of values for S the slope is the estimate for α_S.

However, one typically observes a sharp nonlinearity such that $y_S(S)$ is $U - shaped$ as is its logarithm with a minimum near the level of the current spot (see for example Carr, Geman, Madan and Yor (2002), or Jackwerth (2000). Hence the tilt coefficient is positive when $S > S_0$ or for out-of-the-money call options and negative for $S < S_0$ or for out-of-the-money put options. Thus we allow for this nonlinearity and entertain the simple measure change function

$$y_S(S) = a_P e^{-\alpha_P S}\mathbf{1}_{S<S_0} + a_C e^{\alpha_C S}\mathbf{1}_{S>S_0}. \tag{21}$$

where a_P, a_C are the tilt coefficients applicable to the put and call side respectively, which are uniquely determined from the normalizing constant assuming continuity of y_S at S_0.

To determine the coefficients of exponential tilting α_P and α_C from the traded options market, we can estimate $p(S)$ from the time series of asset returns and $q(S)$ from the prices of options that are written on these assets. The implied tilting coefficients, α_C and α_P can be estimated from the following regression:

$$\log\left(\frac{q(S)}{p(S)}\right) = \log(a_C) + \alpha_C S + \varepsilon_C,\ S > S_0 \tag{22}$$

$$\log\left(\frac{q(S)}{p(S)}\right) = \log(a_P) - \alpha_P S + \varepsilon_P,\ S < S_0 \tag{23}$$

where the error terms, ε_C and ε_P are deterministic errors of approximation in functional forms.

To estimate $\phi'(L)$, we determine $\phi(L)$ from equation (17) on first evaluating the expectation of $y_S(S)$ given L. This computation requires a specification for the conditional density of S given the loss level L. For this purpose we suppose the following simple regression model:

$$S = \alpha + \beta L + \varepsilon \tag{24}$$

with ε distributed normally with mean zero and a volatility of σ_ε.

We substitute equation (24) into equation (21) for S and evaluate the conditional expectation given L to get the left hand side of equation (17). However, we approximate by ignoring the shift in the functions on the put side or $S < S_0$ for positive β and large L and likewise we ignore the shift in the functions on the call side or $S > S_0$ and negative β.

On this calculation we obtain the approximation for equation (13) and for positive β that

$$y_S(\phi(L)) = y_L(L) \approx e^{\alpha_C(\alpha+\beta L)+\frac{1}{2}\alpha_C^2\sigma_\varepsilon^2} \tag{25}$$

while for negative β we have

$$y_S(\phi(L)) = y_L(L) \approx e^{-\alpha_P(\alpha+\beta L)+\frac{1}{2}\alpha_P^2\sigma_\varepsilon^2} \tag{26}$$

Taking logarithmic derivatives of equations (25) and (26) we have that

$$\alpha_L = \alpha_C\beta\mathbf{1}_{\beta>0} + \alpha_P|\beta|\mathbf{1}_{\beta<0}. \tag{27}$$

Comparing equation (27) with (20) we observe that tilt adjustment $\phi'(L)$ equals the absolute value of the slope estimate from the regression equation (24).

3 Pricing an excess-of-loss reinsurance contract

This section illustrates the application of the foregoing discussion in the context of pricing an excess-of-loss reinsurance contract. Such a contract is a portfolio of call options written on the aggregate loss level, L, of the insurer. The first call option is written by the reinsurer on the insurer's aggregate loss level at a strike K. As the buyer of this call, the insurer incurs losses up to K but receives from the reinsurer the loss amount L exceeding the strike K. However, the reinsurer's coverage of losses above the strike is not unlimited and payments are capped at $K+B$, where B is the the stated coverage level. This condition implies that the insurer simultaneously sells the reinsurer a call option struck at $K+B$. This second call caps the reinsurer's payout at the coverage level B. Taken together, this portfolio of two call options implies that the reinsurer sells the insurer a call-spread.

For a coverage level of B the loss contingent payoff to the call-spread, $CF(L)$, at year end can be expressed as:

$$\begin{aligned} CF(L) &= Min\left(Max\left(L-K,0\right),B\right) \\ &= Max(L-K,0) - Max(L-(K+B),0) \end{aligned} \tag{28}$$

The first call option in Equation (28) represents the call option the reinsurer sells the insurer written on the insurer's aggregate loss level, L, at a strike K. The second option ensures that the reinsurer's coverage of losses above the strike is capped at $K+B$.

Specifically, the price of this call-spread w, given an annual continuously compounded risk-free interest rate of r is given by

$$w = \lambda e^{-r} \int_0^\infty CF(L)q(L)dL. \tag{29}$$

We assume that we are dealing with an aggregate pool of losses in which there is always some payout annually and therefore that the statistical probability of a loss arrival, λ, is one. By the equivalence of risk neutral probabilities to the underlying statistical probabilities it follows that one may assume the risk neutral λ is also one.

To price the call spread, the reinsurer then just needs to identify a relevant risk-neutral probability distribution for annual loss levels, $q(L)$. This density describes the current market price of loss contingent bonds that pay one-dollar face in a year on the contingency that particular loss levels are attained. We suppose that $q(L)$ is not directly observable but $p(L)$ is. Thus, our approach of risk-neutralizing the statistical distribution is applicable. However, before we continue with the estimation of the tilt coefficient applicable to $p(L)$ we adopt a specific extreme value distribution for $q(L)$ and obtain a closed form solution for equation (29).

3.1 The Weibull call option pricing model

The focus of any reinsurance contract is on the tail of this distribution of loss levels. Essentially, the critical question is to have an adequate description of the tail behavior as the call spread contracts of interest have a zero payoff at low loss levels. For models of the tail we turn to extreme value theory that characterizes the tail distributions of positive random variables like loss levels.

There are basically three parametric classes of distributions that characterize tail behavior. These are the Frechet, Gumbell and Weibull (Embrechts, Kluppellberg, and Mikosch (1997)). We note that of the three, the Weibull describes the limiting behavior of scaled maximal losses drawn from random variables with an upper bound. In the present context the potential loss levels are bounded above by the size of assets in place and hence such a distribution might be the right choice. Indeed, Lucas, Llaassen, Spreij, and Straetmans (2001) use Weibull to describe extreme tail behavior of credit losses in terms of portfolio characteristics. These considerations lead us to proceed with the Weibull model as a candidate for the risk neutral density of losses $q(L)$.

The specific functional form for the Weibull density, $g(L;c,a)$ with parameters c and a is given by

$$g(L;c,a) = \exp\left(-\left(\frac{L}{c}\right)^{a}\right)\frac{aL^{a-1}}{c^{a}} \tag{30}$$

with mean μ and standard deviation σ

$$\mu = c\Gamma\left(1+\frac{1}{a}\right) \tag{31}$$

$$\sigma = c\sqrt{\Gamma\left(1+\frac{2}{a}\right)-\Gamma\left(1+\frac{1}{a}\right)^{2}} \tag{32}$$

where $\Gamma(x)$ is the gamma function.

The parameter c is a scaling parameter in units of dollars and a is called the shape parameter. The value of a determines the relative fatness of the tail of the distribution, with higher values of a leading to thinner tails. We see from Equations (31) and (32) that the coefficient of variation is determined by the parameter a.

We next derive in Proposition 1 the closed-form expression for the price of the call option $(Max(L-X,0)$ with strike X written on the loss level L , which is distributed Weibull.

Proposition 1 *The value of the Weibull call option, with parameters c and a, written on the loss* *level L with strike X is given by*

$$C = e^{-r}\left[L^{*}W_1 - XW_2\right]$$

$$\begin{aligned} \textit{where } L^{*} &= c\Gamma\left(1+\frac{1}{a}\right) && (33)\\ W_1 &= 1 - \textit{gammainc}\left(\left(\frac{x}{c}\right)^{a}, 1+\frac{1}{a}\right) && (34)\\ W_2 &= \exp\left(-\left(\frac{X}{c}\right)^{a}\right) && (35) \end{aligned}$$

and $L^{*}$is the expected loss level under the risk-neutral measure, Γ and gammainc are the gamma and incomplete gamma functions.

Proof in the Appendix 8.2

We note the Weibull call option formula has a similar structure to the Black-Scholes option pricing formula. The present value of the strike is multiplied by the Weibull risk neutral probability that the call is in the money. L^{*} is the risk neutral expected loss level and is multiplied by W_1, which is the probability of the call being in the money under a suitably adjusted measure. It follows that a reinsurer can obtain the theoretical expression for the call-spread, given in Equation (29), as the difference between the two call options with strikes K and $K+B$ written on loss levels that is Weibull distributed.

4 Weibull implied exponential tilt in the options market

The tilt in the options market is obtained by first constructing the statistical distribution of the stock price at a maturity matching the option maturity. Second we estimate the risk neutral distribution at a traded option maturity. To maintain consistency with our focus on pricing tail events, we choose the Weibull as the functional forms for both these densities. Finally the tilt follows on regressing the logarithm of the ratio of the two densities on the level of the stock price as per equations (22) and (23). The details for the density estimations are explained in the following two subsections.

4.1 The statistical stock price density

The measure change function given in equation (21) allows for separate tilt coefficients for stock prices below and above the current spot. We therefore study separately the statistical density on these two sides. We focus attention on returns over a prespecified horizon and let S be the final stock price while S_0 denotes the initial stock price.

We define the excess return as a positive random variable with values in the interval $[1, \infty)$ by letting

$$R_u = \frac{S}{S_0}, \; S > S_0 \tag{36}$$

$$R_d = \frac{S_0}{S}, \; S < S_0. \tag{37}$$

For such a positive random variable, bounded below by unity, the shifted Weibull distribution is an appropriate extreme value density reflecting finite moments of all orders. For the statistical density we suppose the density of R_u, R_d have the specific Weibull forms with parameters c_u^S, a_u^S and c_d^S, a_d^S with the generic form

$$f(R) = \exp\left(-\left(\frac{R-1}{c}\right)^a\right)\frac{a(R-1)^{a-1}}{c^a} \tag{38}$$

We estimate from time series data on daily returns, using tail returns, the statistical parameters for both the upside and downside by maximum likelihood.

For a comparison with the risk neutral density we have to construct statistical returns at the option maturity from the estimated daily return distribution. However, in making a comparison with risk neutral densities there is a horizon mismatch, as risk neutral densities are observed over much longer horizons than a single day. One possibility is to construct long horizon returns from daily returns using the hypothesis of identically and independently distributed returns (i.i.d.). However, such a strategy is contrary to evidence on dependence in returns as demonstrated by autocorrelations in squared returns (Engle, 1982).

We recognize, instead, that uncertainties pertaining to the distant future are rising as we move forward in time and employ instead a scaling hypothesis. Under this hypothesis we model the return at a horizon of N days as having the distribution of $\sqrt{N}$ times the daily return distribution, or define the return over N days, R_N to be in law

$$R_N - 1 \overset{law}{=} \sqrt{N}(R-1) \tag{39}$$

The variance then grows linearly with N as it would were we to add independent and identically distributed, but unlike the situation with addition of independent random variables, skewness and excess kurtosis remain constant in N. For the case of adding i.i.d. variables, skewness falls like $\frac{1}{\sqrt{N}}$ and excess kurtosis falls like $\frac{1}{N}$ as shown in Konikov and Madan (2002). In the sense of the higher moments, the uncertainty is maintained at a higher level than would be the case with summing independent and identically distributed random variables.[1]

Thus, it follows from equations (39) and (38) that the Weibull density for R_N is

$$f(R_N) = \exp\left(-\left(\frac{R_N - 1}{c\sqrt{N}}\right)^a\right)\frac{a(R_N-1)^{a-1}}{(c\sqrt{N})^a} \tag{40}$$

and with parameters $c\sqrt{N}, a$.

4.2 The risk neutral stock price density

The risk neutral stock price density for upside and downside returns are also taken to be in the Weibull family with parameters denoted by c_u^{RN}, a_u^{RN} and c_d^{RN}, a_d^{RN} respectively. These parameters are to be estimated by calibrating the model prices developed under the specific Weibull density to the prices of the out-of-the-money call and put options. For this task, following Proposition 1, we develop the Weibull call option pricing formula for up and downside returns using the Weibull density.

[1] Alternatively, one may appeal to the work of Sato (1999) who shows that the class of all limit laws of arbitrarily scaled sums of independent but not necessarily identical random variables are the laws at unit time of a scaled process of independent and generally inhomogeneous increments. This observation makes such processes relevant to the modeling of financial returns, that may easily be seen as the limit of the sum of a large number of independent effects.

For a call option of maturity t the call option value, cv,

$$cv = e^{-rt} \int_{\frac{K}{S_0}}^{\infty} (S_0 R - K) \exp\left(-\left(\frac{R-1}{c}\right)^a\right) \frac{a(R-1)^{a-1}}{c^a} dR. \tag{41}$$

Note that in equation (41), we do not impose the condition that the discounted stock price is the current stock price as we do not assert that the Weibull density applies for all levels of the stock price, but only applies in the upper right tail where the specified calls are in the money. Traditional option pricing models model the entire distribution of the underlying asset and hence must enforce the spot forward arbitrage condition requiring that

$$S_0 e^{rt} = \int_0^{\infty} S_t q(S_t) dS_t \tag{42}$$

or that the financed stock purchase has a zero price. Since we focus on the Weibull model for just the tail of the distribution, and use it to price out of the money calls and puts on the up and down side, we do not have a condition integrating across the entire range of stock prices. In fact, we impose no distributional hypothesis at all, in the center of the distribution, or the near money density.

Performing the requisite integration we obtain that

$$\begin{aligned} cv &= e^{-rt}\left[S_0\left(P_2 + c\Gamma\left(1+\frac{1}{a}\right)P_1\right) - KP_2\right] && (43) \\ P_2 &= \exp\left(-\left(\frac{\frac{K}{S_0}-1}{c}\right)^a\right) && (44) \\ P_1 &= 1 - gammainc\left(\left(\frac{\frac{K}{S_0}-1}{c}\right)^a, 1+\frac{1}{a}\right) && (45) \end{aligned}$$

We propose to estimate c_u^{RN}, a_u^{RN} using out-of-the-money calls struck at tail strikes trading in the market.

On the down side we have to evaluate the put option value pv,

$$\begin{aligned} pv &= e^{-rt} \int_{\frac{S_0}{K}}^{\infty} \left(K - \frac{S_0}{R} \right) \exp\left(- \left(\frac{R-1}{c} \right)^a \right) \frac{a(R-1)^{a-1}}{c^a} dR \\ &= K e^{-rt} \exp\left(- \left(\frac{\frac{S_0}{K} - 1}{c} \right)^a \right) \\ &\quad - S_0 e^{-rt} \int_{\frac{S_0}{K}}^{\infty} \frac{1}{R} \exp\left(- \left(\frac{(R-1)}{c} \right)^a \right) \frac{a(R-1)^{a-1}}{c^a} dR \end{aligned} \tag{46}$$

The last integral is evaluated numerically. We estimate the parameters c_d^{RN}, a_d^{RN} using downside tail strikes trading in the market.

5 Pricing the FDIC's reinsurance risk

In 1991, the Federal Deposit Insurance Corporation Act authorized the Federal Deposit Insurance Corporation (FDIC), "to obtain private reinsurance covering not more than 10 percent of any loss the Corporation incurs with respect to an insured depository institution"(12$U.S.C.A$ 1817$(b)(1)(B)$). Such authorization allows the FDIC to enter into financial contracts with the private sector that price and share bank default risk. Recently, the Options Paper produced by the FDIC (FDIC, 2000) view reinsurance as one way "to use market information to differentiate risks without imposing a particular funding structure on insured institutions."

Recently, the FDIC retained MMC Enterprise Risk (MMC) to determine the feasibility and the costs of private sector reinsurance arrangements. In a report submitted to the FDIC, MMC provides two rough price estimates for reinsuring the aggregate annual losses of the FDIC (MMC, 2001, p. 21). The specific estimates are such that the annual premium on a \$2 billion coverage at a *one* basis point (less than one chance in 10,000) risk level is \$4 million. A second price estimate states that the annual premium on a higher risk level of *one* percentage point (one chance in 100) with \$0.5 billion dollar coverage is \$10 million.

We illustrate our methodology for risk neutralizing a statistical distribution by pricing the claims quoted on by MMC. Additionally we price the aggregate loss distribution of the FDIC and quote on the level of aggregate premiums required from the banking system as a whole.

5.1 Statistical loss distribution on bank failures

Our focus is to capture the distribution of annual aggregate losses on bank failures covering the period 1986-2000. The aggregate annual loss levels of the FDIC are displayed in Table 1. Although we could increase sample observations by including years dating back to 1930s we find this manner of expanding the sample undesirable because these dated periods are not reflective of risks faced by the FDIC today. Given the relatively small sample size we estimate by the method of moments the statistical parameters c and a.

From the sample mean μ and standard deviation σ of the FDIC's annual loss experience we invert for the shape parameter a the following:

$$1 + \frac{\sigma^2}{\mu^2} = \frac{\Gamma\left(1 + \frac{2}{a}\right)}{\Gamma\left(1 + \frac{1}{a}\right)^2}. \tag{47}$$

Equation (47) is derived from Equations (31) and (32). The estimate for c follows from the equation for the mean (31) given an estimate for a.

From Table 1 we observe that the annual mean and standard deviation of annual loss levels between 1986-2000 is $\mu = \$2.106$ billion and $\sigma = \$2.497$ billion, respectively. Substituting these values in Equation (47) and using method of moments, we obtain the statistical parameters for the Weibull distribution as $c = 1.9317$ and $a = 0.8472$.

5.2 Option tilts on the BKX Index

To approximate the appropriate tilt coefficient we assume that the reinsurer examines market risk preferences on an asset that best reflects the aggregate bank risk. One proxy for such an asset is the PHLX / KBW Bank Index (BKX). BKX is a capitalization-weighted index composed of 24 geographically diverse stocks representing national money center banks and leading regional institutions. The index is evaluated annually by Keefe, Bruyette & Woods to assure that it represents the banking industry. The index was initiated on October 21, 1991 and options started trading on September 21, 1992.

In measuring the degree of exponential tilting in the options market for the bank index, BKX we must decide the region of strikes appropriate for relating these tilt coefficients to those of our loss levels. As explained in equation (13) we need to assess the concentration of the conditional density of the index given large loss levels. There are two effects to consider. First, it may be the case that loss probabilities rise in a down market and this can bring larger losses associated with a downward move in the index. Second, the magnitude of losses given default may be positively related with the size of operations and this may be positively related to the level of the index or equity.

We assess the degree of exponential tilting that occurs in pricing out-of-the-money equity call and put options at the 1, 5, and 10 percent risk levels. We do this analysis by estimating the statistical and risk neutral densities in the upper and lower tail of the returns, denoted by $p(S)$, and $q(S)$ respectively, and estimate the regression equation given in equations (22) and (23) for values of S in both tails of the statistical distribution.

We use time series data on the BKX for 1500 days ending on September 28, 2001 to obtain the statistical distribution and data on index options for every second Wednesday of each month over the year beginning in October 2000 and ending in September 2001 to estimate the risk-neutral distribution. To estimate the statistical parameters of the Weibull density we first compute the upside returns as described in the previous section. We sort these returns and extract the top and bottom 25% of returns. The Weibull model is estimated by maximum likelihood on large positive returns to yield the statistical parameters for the BKX index. The estimated parameters are $c_u^S = 0.0345$, $a_u^S = 2.5324$ and $c_d^S = 0.0335, a_d^S = 2.7593$.

The risk neutral parameters are estimated by calibrating model prices (equations (43) and (46)) to the call and put option prices with maturity of around two months with the five largest and smallest strikes trading in the market for this maturity. The calibration is done for one day in each month from October 2000 to September 2001. The results are presented in Table 2 for the BKX index, where the average $c_u^{RN} = 0.0645$, $a_u^{RN} = 0.9413$ and $c_d^{RN} = 0.0612$, $a_d^{RN} = 0.6824$.

Next, the regression equations (22) and (23) are estimated where the logarithm of the ratio of the risk neutral density to the scaled statistical density regressed on the price level in the range between 1% to 0.01% return levels in two months. The resulting slope coefficients are the associated levels of exponential tilting on the upper and lower tail of the return distribution. The results are presented in Table 3 along with the mean levels of tilting for the BKX. We observe that the mean levels of tilting to losses on the upside in BKX is 0.1739, while the corresponding figure for the downside is -0.6387.

5.3 Estimating the tilt adjustment

As described in section 2.3 tilts obtained from the options markets need to be adjusted before they can be used to tilt the statistical loss distribution. The proposed tilt adjustment requires the estimation of the regression equation (24).

To estimate the slope coefficients for the conditional distribution of the level of the index given the loss level we regressed the NASDAQ bank index on the level of $FDIC$ losses over the period $1981 - 2001$. The resulting regression for equation (24) yielded the result $\alpha = 1165.43$, $\beta = -0.1698$, with respective standard errors of 191.31, 0.0675 and an R^2 of 28.76%. We therefore employ in accordance with equation (27) the β scaled put side tilt and use $\alpha_L = 0.6387 * 0.1698 = 0.1085$.Hence, we tilt the statistical distribution of the FDIC's losses by this coefficient to obtain the risk neutral distribution.

5.4 The value of the call-spread

We can price the call spread now by using the tilt coefficient of 0.1085 to tilt the statistical distribution and obtain the risk-neutral distribution. We estimate the price of a call spread in the context of a reinsurance quote estimate given to the FDIC. Although the strike levels are not specifically indicated in the report we can easily estimate the implied strikes given the parameters of the statistical Weibull density.

Note that the probability of a loss amount exceeding the strike is given by

$$P(L > K) = \theta = 1 - F(K) \tag{48}$$

For the Weibull cumulative distribution function,

$$F(K) = 1 - \exp\left[-\left(\frac{L}{c}\right)^a\right] \tag{49}$$

equation (48) is written as

$$-\log(\theta) = \left(\frac{K}{c}\right)^a \tag{50}$$

Hence, the strike is expressed as

$$K = c\left(-\log(\theta)\right)^{\frac{1}{a}} \tag{51}$$

Substituting the estimates of c, a, and the risk level, θ, in equation (51) we obtain $K_1 = \$11.72$ billion and $K_2 = \$26.56$ billion for the high risk and low risk cases, respectively. We note that these estimates of strike levels, implied by the quoted prices, can be considered reasonable because they are well within the current \$30 billion FDIC deposit insurance fund level.

To risk neutralize the statistical loss density of the FDIC we need to exponentially tilt it by the level observed in the pricing of BKX put options scaled by the β coefficient of the regression (24), 0.1085. We recognize that exponentially tilted Weibull densities are not themselves in the Weibull class. As an alternative approach, we alter the risk neutral density such that the derivative of the logarithm of the risk neutral density accounts for the altered tilt. Specifically, using equation (7) the estimated tilting is basically the difference between the derivative of the logarithm of the risk neutral and statistical densities,

$$\frac{d\log q(L)}{dL} - \frac{d\log p(L)}{dL} = 0.1085$$

We evaluate the derivative of the logarithm of p at the two strikes of 11.72 and 26.56 to be $-.3638$ and $-.2871$. This provides us two equations

$$\begin{aligned} \frac{d\log q(L)}{dL}|_{11.72} &= -0.2553 \\ \frac{d\log q(L)}{dL}|_{26.56} &= -0.1786 \end{aligned}$$

from which we can simultaneously solve for the parameters c and a of $q(L)$ to obtain $c = 1.0442$ and $a = 0.6054$. Using these values for the risk neutral parameters we price the two call-spreads to obtain the values

$$\begin{aligned} w(.01, .5) &= 6,157,387 \\ w(.0001, 2) &= 1,394,000 \end{aligned}$$

These prices compared with those of the MMC estimates appear to indicate overpricing by MMC to the order of approximately \$4 million in each case.

We can place these estimates in perspective by pricing the call-spread statistically. In other words, assuming risk neutrality we can use the statistical mean \$2.106 billion and standard deviation \$2.497 billion as our working Weibull distribution and estimate the actuarially fair prices. Under this assumption, using equation (33), the call-spread is valued statistically at \$4.5 million and $\$150,000$ for 1 % and 0.01 % risk levels, respectively. Note that the statistical prices establish the lower bound for the reinsurance price risk. Thus, we observe that MMC estimates reflect some level of tilting in pricing rather than assuming risk neutrality on the part of the reinsurer.

5.5 Pricing the aggregate coverage

The current assessment system used by the FDIC requires the FDIC to charge at least 23 cents per \$100 deposits if the mandated reserves to insured deposits, designated reserve ratio (DRR), is below 1.25%. The Deposit Insurance Funds Act of 1996 prohibits the FDIC from assessing depository institutions as long as DRR is above 1.25%. As of December 31, 2000, 92% of all insured institutions were not paying premiums for deposit insurance (FDIC, 2001).

Our estimates of the FDIC's risk neutral loss density can also be used to compute the aggregate premium that should be collected from the insured institutions. If we accept the degree of exponential tilting outlined above, then the risk neutral density for coverage up to \$26.56 billion is:

$$q(L) = \frac{e^{0.1085L} p(L : \mu = 2.106, \sigma = 2.497)}{\int_0^{26.56} e^{0.1085L} p(L : \mu = 2.106, \sigma = 2.497) dL}. \quad (52)$$

Note that as the FDIC offers the random coverage level L each year then the aggregate premium that should be collected at the 0.01% risk level from the insured institutions is the price of this coverage and this is given in forward terms by:

$$\Pi = \int_0^{26.56} Lq(L)dL. \quad (53)$$

For the specific risk neutral distribution given in equation (52), we compute this integral at \$3.096 billion. For the level of insured deposits around \$1909.9 billion this is a premium of 16.21 cents per \$100 deposits for the year 2001, which is quite comparable with the average deposit insurance premium (19.44 cents per 100 dollars) charged by the FDIC in the period 1990-1995 when the insurance fund was below 1.25% of the insured deposits. In addition, this estimate of the aggregate deposit insurance premium is in the vicinity of those of Cooperstein, Pennacchi, and Redburn (1995), who estimate the fair premium to be in the range of $23.8 - 24.9$ cents for years 2000 and 2001.

Calculated statistically, the value of the integral in equation (53) is \$2.1032 billion. This value represents deposit insurance premiums of 11 cents per \$100 deposits. In other words, we can assert that the assessed deposit insurance premium is consistent with FDIC tilting the distribution of its historical loss experience.

To ensure that the level of the fund is a risk neutral martingale, 16.21 cents premium should be collected each year (assuming no change in statistical and the risk neutral distributions). However, statistically, the fund will have a positive expected cash flow and it is therefore expected to grow over time in line with the return commensurate with the insurance business it is engaged in (Pennacchi, 2000). The question does arise as to who gets the expected return from this activity. Although this questions begs in depth analysis, we can assert that if the fund is viewed as mutually owned by the insured institutions then the growth may be transferred to them in the form of reduced premiums and this could be the logic underlying the decision to reduce premiums to zero in certain growth situations.

6 Conclusion

This paper proposes a parsimonious approach to formulating a risk neutral distribution from an estimated statistical distribution associated with an underlying uncertainty. As a result, contingent claims on the uncertainty can be priced via the statistical density. To obtain the risk neutral density we employ a renormalized exponential tilt of the statistical density, a method often used in the literature. Our contribution lies in explicitly relating the level of this exponential tilt to those observed in related options markets that serve as a sufficient statistic for the uncertainty at hand.

More specifically, we focus attention on uncertainties related to tail loss events and develop for the purpose, option pricing models using the Weibull extreme value distribution. These are employed to obtain closed form expressions for reinsurance contract pricing. Our methods are further illustrated by pricing the risks of reinsuring the FDIC's losses.

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8 Appendix

8.1 A utility based derivation of the relation between $q(L)$ and $p(L)$.

The expected utility of an agent absorbing the loss payment L is given by

$$\begin{aligned} u &= E\left[U(W-L)\right] & (54) \\ &= \int_{W,L} U(W-L)f(W)\lambda(W)g(L|W)dWdL \\ &\quad \ldots\ldots + \int_{W} U(W)f(W)\left(1-\lambda(W)\right)dW. & (55) \end{aligned}$$

The agent's utility of the end of period wealth, $W,$ with marginal distribution, $f(W)$, in the absence of a loss, L, is $U(W)$. This state has a probability of $(1-\lambda(W))$ where, $\lambda(W) = \int_0^\infty p(W,L)dL$ is the probability of a loss given the end of period no loss wealth of W.Here, $p(W,L)$ is the joint density for a loss level L. Hence, with $\lambda(W)$ probability, the agent is exposed to losses and his utility is $U(W-L)$.In this case, the conditional density of loss is given by $g(L|W) = \frac{p(W,L)}{\lambda(W)}$.

Now suppose that the loss level is independent of the end of period no loss wealth level and that

$$g(L|W) = p(L) \tag{56}$$

the unconditional density of loss given the existence of a loss. Also suppose that $\lambda(W) = \lambda$ a constant. We may then write

$$u = \int_{W,L} U(W-L)f(W)\lambda p(L)dWdL + \int_W \left(1-\lambda\right)U(W)f(W)dW. \tag{57}$$

Suppose now that the agent is offered a contingent claim paying $c(L)$ at the end of the period. If the agent were to take a position of t units in this claim at the fair forward price of a dollars then the expected utility of the agent can be expressed as:

$$\begin{aligned} V(t) &= \int_{W,L} U(W-L+tc(L)-ta)f(W)\lambda p(L)dLdW \\ &\quad \ldots. + \int_W (1-\lambda)U(W-ta)f(W)dW & (58) \end{aligned}$$

Because the claim is fairly priced, we have that $V'(0) = 0$. Evaluating $V'(t)$ we get

$$\begin{aligned} V'(t) &= \int_{W,L} U'(W - L + tc(L) - ta) f(W) \lambda p(L)(c(L) - a) dL dW && (59) \\ & -a \int_{W} (1 - \lambda) U'(W - ta) f(W) dW && (60) \end{aligned}$$

Equating $V'(0)$ to 0 we get

$$\begin{aligned} & \int_{W,L} U'(W - L) f(W) \lambda p(L) c(L) dL dW \\ = \; & a \int_{W,L} U'(W - L) f(W) \lambda p(L) dL dW \\ & + a \int_{W} (1 - \lambda) U'(W) f(W) dW \end{aligned} \tag{61}$$

It follows that the fair forward price, a, of the contingent claim is:

$$a = \frac{\int_{W,L} U'(W - L) f(W) \lambda p(L) c(L) dL dW}{\int_{W,L} U'(W - L) f(W) \lambda p(L) dL dW} \frac{1}{1 + b} \tag{62}$$

where,

$$b = \frac{\int_{W} (1 - \lambda) U'(W) f(W) dW}{\int_{W,L} U'(W - L) f(W) \lambda p(L) dL dW} \tag{63}$$

Define

$$q(L) = \frac{\int_{W} U'(W - L) f(W) \lambda p(L) dW}{\int_{W,L} U'(W - L) f(W) \lambda p(L) dL dW} \tag{64}$$

then we can write:

$$a = \int_{L} q(L) c(L) \frac{1}{1 + b} dL \tag{65}$$

Equation (65) shows that, $q(L)$ is the risk neutral density for a loss level of L, given the existence of a loss while $(1 + b)^{-1}$ is the risk neutral probability of a loss. Hence, we can establish the relation between statistical, $p(L)$, and.risk neutral, $q(L)$, probability distributions, assuming a specific utility function. For the case of an exponential marginal utility or the case of constant absolute risk aversion, α,

$$U'(W) = \exp(-\alpha W) \tag{66}$$

equation (64) is written as

$$q(L) = \frac{\int_W \exp -\alpha(W-L) f(W) \lambda p(L) dW}{\int_{W,L} \exp -\alpha(W-L) f(W) \lambda p(L) dL dW}$$

simplifying we obtain:

$$q(L) = \frac{e^{\alpha L} p(L)}{\int_0^\infty e^{\alpha L} p(L) dL} \tag{67}$$

Note that if U' is constant and utility is linear then $b = (1-\lambda)/\lambda$ and $(1+b)^{-1} = \lambda$, which is the statistical probability of a loss. More generally we expect $U'(W-L) > U'(W)$ so b should be less than $(1-\lambda)/\lambda$ and $(1+b)^{-1}$ is greater than λ. Hence the presence of risk aversion raises the risk neutral loss probability over its statistical counterpart.

For the aggregate system we suppose that $\lambda = 1$ and there is some loss each year and hence the equation for pricing loss contingent claims in the spot market is

$$e^{-r} a = e^{-r} \int_0^\infty q(L) c(L) dL$$

where risk neutralization occurs in accordance with equation (67).

8.2 Proof of Proposition 1: Derivation of the Weibull call option model

Note that for strike X we can express the payoff to a call option written on the loss level L as follows:

$$C(X) = \int_X^\infty (L - X) f(L) dL, \tag{68}$$

where the Weibull probability density function is given by

$$f(L) = \exp\left(-\left(\frac{L}{c}\right)^a\right) \frac{aL^{a-1}}{c^a}. \tag{69}$$

Hence, equation (68) can be written as,

$$C(X) = \int_X^\infty L \exp\left(-\left(\frac{L}{c}\right)^a\right) \frac{aL^{a-1}}{c^a} dL - X \exp\left(-\left(\frac{X}{c}\right)^a\right). \tag{70}$$

The first term is simplified as follows:

$$\int_X^\infty L \exp\left(-\left(\frac{L}{c}\right)^a\right) \frac{aL^{a-1}}{c^a} dL = \frac{a}{c^a} \int_X^\infty L^a \exp\left(-\left(\frac{L}{c}\right)^a\right) dL \tag{71}$$

Letting $u = \left(\frac{L}{c}\right)^a$, $y = cu^{\frac{1}{a}}$, and $dy = \frac{c}{a}u^{\frac{1}{a}-1}$, we have

$$= \frac{a}{c^a} \int_{\left(\frac{X}{c}\right)^a}^{\infty} c^a u \left(\exp\left(-u\right)\right) \frac{c}{a} u^{\frac{1}{a}-1} du, \tag{72}$$

$$= c \int_{\left(\frac{X}{c}\right)^a}^{\infty} u^{\frac{1}{a}} \left(\exp\left(-u\right)\right) du \tag{73}$$

$$= c \int_0^{\infty} u^{\frac{1}{a}} \left(\exp\left(-u\right)\right) du - c \int_0^{\left(\frac{X}{c}\right)^a} u^{\frac{1}{a}} \left(\exp\left(-u\right)\right) du \tag{74}$$

$$= c\Gamma\left(1 + \frac{1}{a}\right) - c\Gamma\left(1 + \frac{1}{a}\right) \frac{\int_0^{\left(\frac{X}{c}\right)^a} u^{\frac{1}{a}} \left(\exp\left(-u\right)\right) du}{\Gamma\left(1 + \frac{1}{a}\right)} \tag{75}$$

Noting that,

$$gammainc(w, \gamma) = \frac{\int_0^{w} u^{\gamma-1} \left(\exp\left(-u\right)\right) du}{\Gamma(\gamma)}. \tag{76}$$

and substituting equation (75) in equation (70) and discounting it at the risk-free rate of r, we have

$$C = e^{-r} \left[c\Gamma\left(1 + \frac{1}{a}\right) \left(1 - gammainc\left(\left(\frac{x}{c}\right)^a, 1 + \frac{1}{a}\right)\right] \right. \\ \left. - X \exp\left(-\left(\frac{X}{c}\right)^a\right)\right)\right] \tag{77}$$

TABLE 1: FDIC Annual Loss Levels

Source: *Failed Bank Cost Analysis, 1986-2000*, Division of Finance, FDIC

Year	Loss (in $Billions)	Number of Bank Failures
1986	1.775	145
1987	2.023	203
1988	6.921	280
1989	6.199	207
1990	2.785	169
1991	6.148	127
1992	3.675	122
1993	0.646	41
1994	0.179	13
1995	0.085	6
1996	0.038	5
1997	0.005	1
1998	0.234	3
1999	0.841	7
2000	0.039	6

TABLE 2: Risk-neutral Weibull parameter estimates on extreme 2-month BKX call and put options.

Date	maturity	cdrn	adrn	curn	aurn
Oct. 2000	.1804	0.0563	0.6465	0.0821	1.1453
Nov. 2000	.1968	0.0627	0.7004	0.0176	0.4245
Dec. 2000	.1779	0.0636	0.6932	0.0272	0.5128
Jan. 2001	.1781	0.0665	0.6701	0.0540	0.6997
Feb. 2001	.1779	0.0527	0.6749	0.0659	0.9728
Mar. 2001	.1779	0.0342	0.4549	0.0914	1.0725
Apr. 2001	.1753	0.0736	0.6703	0.0867	0.9985
May 2001	.1945	0.0632	0.7057	0.0680	0.9862
Jun. 2001	.1753	0.0365	0.6122	0.0524	1.0211
Jul. 2001	.1945	0.0538	0.6794	0.0734	1.1855
Aug. 2001	.1917	0.0406	0.6049	0.0637	1.1206
Sep. 2001	.1563	0.1303	1.0758	0.0913	1.1556
Mean Level	.1814	0.0612	0.6824	0.0645	0.9413

Table 3. Exponential tilt coefficients for out-of-the-money put and call options on the BKX index

Date	Maturity	Put Options	Call Options
Oct. 2000	0.1804	-0.6403	0.1348
Nov. 2000	0.1968	-0.6183	0.2533
Dec. 2000	0.1779	-0.6392	0.2588
Jan. 2001	0.1781	-0.6498	0.2415
Feb. 2001	0.1779	-0.6298	0.1747
Mar. 2001	0.1779	-0.6846	0.1841
Apr. 2001	0.1753	-0.6597	0.2026
May 2001	0.1945	-0.6177	0.1596
Jun. 2001	0.1753	-0.6292	0.1135
Jul. 2001	0.1945	-0.6138	0.0732
Aug. 2001	0.1917	-0.6608	0.1152
Sep. 2001	0.1563	-0.6216	0.1763
Mean Level	0.1814	-0.6387	0.1739

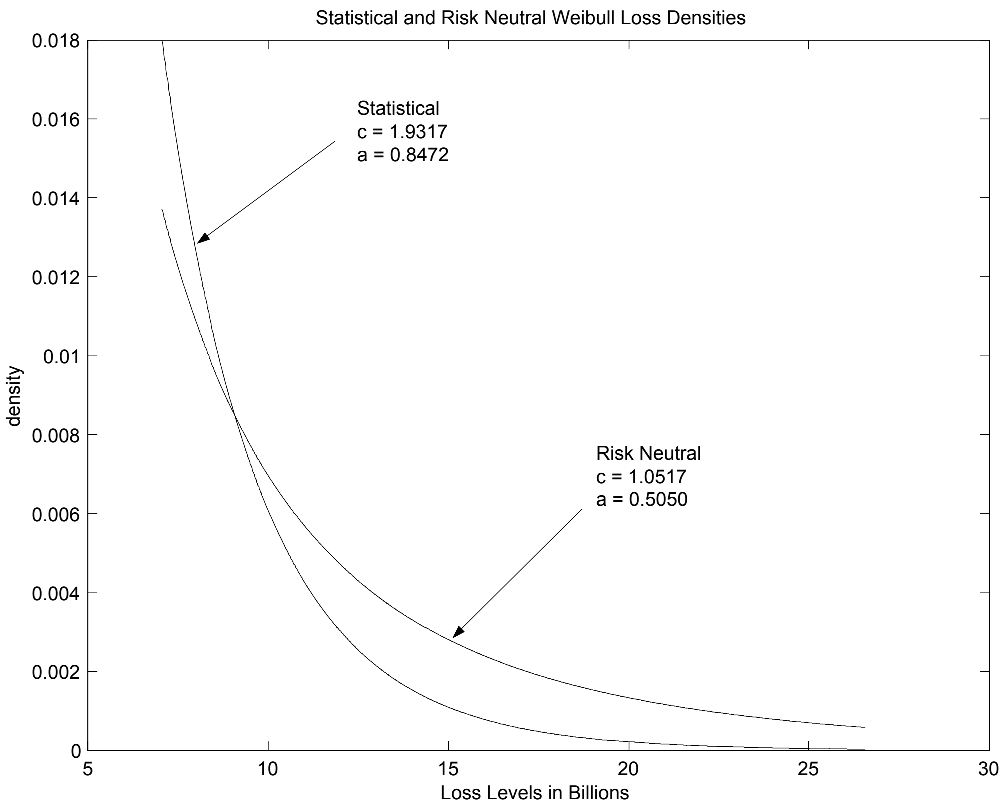
Statistical and Risk Neutral Weibull Loss Densities
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
density
Statistical
c = 1.9317
a = 0.8472
Risk Neutral
c = 1.0517
a = 0.5050
5
10
15
20
25
30
Loss Levels in Billions

Figure 1: