An Integrated Structural Model for Portfolio Market and Credit Risk

by

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ABSTRACT

A single factor migration-style credit risk model is extended to measure the market risks of the non-defaulting credits in an asymptotic portfolio. Correlations among credit and default risks are modeled using a common Gaussian factor. Idiosyncratic default and rating migration risks are fully diversified in an asymptotic portfolio, but market risks on performing credits are not diversifiable. A closed-form representation of an asymptotic portfolio's conditional future market value is derived and used to generate an estimate of the portfolio's unconditional future value distribution using Monte Carlo methods for a sample portfolio. This integrated exposure distribution is used to construct economic capital allocations, and these capital allocations are compared to piecemeal approaches for measuring risks and estimating economic capital. The capital comparisons show that the issues of diversification and capital benefits are complex, and it is impossible to make general statements about potential for capital savings. The results show that capital allocations derived from an integrated market and credit risk measure can be larger or smaller than capital allocations that are estimated from piecemeal risk measures—the sign and magnitude of the difference depends on the piecemeal approaches used to measure market and credit risks and the method used to construct economic capital allocations. Regarding specifically the Basel II AIRB approach, the results clearly show that no further diversification benefit should be considered for banking book positions as no market risk capital is required and Basel II AIRB capital requirements fall far short of the capital required by an integrated risk measure. For trading book positions subject to 10-day market risk requirements and incremental default capital requirements, the issues are unclear because of differences in measurement horizons and as well as allowed variation in bank market and credit risk measures in addition to shortcomings in the EL+UL capital allocation framework.

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I. INTRODUCTION

The Gaussian asymptotic single factor model (ASFM) of portfolio credit losses, developed by Vasicek (1987), Finger (1999), Schönbucher (2000), Gordy (2003) and others, provides an approximation for the default rate distribution for a credit portfolio in which default dependence is driven by a single common factor. In a large portfolio of credits, idiosyncratic risk is fully diversified and the only source of portfolio loss uncertainty is the default rate that is driven by the common latent Gaussian factor.[1]

By construction, the ASFM model is a default-mode model meaning that all credits are assumed to either perform or default within the model's risk measurement horizon. Defaulting credits' losses are measured by the model. Income earned on non-defaulting credits is not recognized in the Vasicek (and Basel II AIRB) loss distribution estimate.

CreditMetrics[2] popularized mark-to-market (MTM) credit-migration style risk measurement models. The credit-migration class of models generalizes the Vasicek (1987) default-mode model to include the potential for migration of non-defaulting credits among credit quality groups. When these migrating credits are re-priced at the end of the risk measurement horizon, they generate capital gains or losses. The

[1] The ASFM assumes the unconditional probability of default, exposure at default, and loss rates in default (*LGD*) are known non-stochastic quantities for all obligors.

[2] See, "CreditMeterics Technical Document," JP Morgan, New York, April 2, 1997.

CreditMeterics model uses estimates of credit quality transition matrices and a single latent common factor to drive changes in an obligor's credit quality and trigger default. The portfolio's future value distribution includes interest income, MTM gains and losses on non-defaulting credits, and default losses on the portfolio's positions.

The CreditMetrics approach incorporates correlation among credit quality transitions and defaults, but it does not incorporate market risk. While credits may transition among credit-quality grades, the MTM value of a future cash flow is known and non-stochastic. The end-of-horizon values of performing credits are calculated using credit quality specific implied forward rates that are bootstrapped from the spot yield curves for each credit quality grade used in the model.

When it was initially introduced, CreditMetrics estimated its MTM portfolio value distributions using Monte Carlo simulation. Subsequently, Finger (1999) and Gordy (2003) derived a closed-form expression for the conditional loss rate of an asymptotic portfolio of credits. These asymptotic portfolio models assume that the future market values of performing credits in each credit quality grade are known and deterministic.[3]

This paper develops a model that generalizes the ratings-migration framework to incorporate the valuation effects of market risk on non-defaulting credits. It derives the MTM value distribution for an asymptotic credit portfolio. This MTM value distribution

[33] In deriving his results, Gordy (2003) p. 211 notes, "In principal, however, we can allow (credit) spreads to be non-stochastic functions of X (the common factor)." In contrast , the model for the conditional asymptotic portfolio's future value developed in this paper (expression(33)) is a generalization that incorporates credit spreads that are stochastic functions conditional on the common factor.

can be used to estimate portfolio-invariant capital allocations.[4] Non-defaulting portfolio credits may migrate among credit quality grades. The end-of-horizon MTM value of each credit quality is, however, stochastic. The distributions of the future market values of performing credits are modeled, and the model distributions are calibrated using historical data on the market yields of alternative credit quality instruments. The model incorporates correlations between portfolio default rates, credit migration probabilities and credit-quality specific market yields using a single common latent factor. The correlations among the yields on the alternative credit-quality classes are calibrated using historical data.

Idiosyncratic default and rating migration risks are fully diversified in an asymptotic portfolio, but market risks on performing credits are not diversifiable in this model.[5] As a consequence, the model produces a closed-form expression for the conditional distribution of the asymptotic portfolio's MTM value that depends on the common latent factor and, in addition, on the idiosyncratic factors that drive the market-wide yields on each credit-quality class. The conditional asymptotic distribution has low dimensionality so standard Monte Carlo techniques can be used to derive an accurate

[4] A portfolio-invariant capital allocation is one in which the economic capital required on the marginal investment is independent of the composition of the portfolio to which the asset is added. In a so-called asymptotic portfolio, all idiosyncratic risks are diversified, and the required capitalization rate on the marginal asset equals the overall portfolio capitalization rate. See Gordy (2003) for further discussion.

[5] The model developed herein does not explicitly include instrument-specific yield risks as all instruments in a credit-quality grade are assumed to have the same required yield. The model can be extended to include instrument-specific yield risks and it can be formally demonstrated that these risks will be completely diversified in an asymptotic portfolio.

estimate of the portfolio's full MTM value distribution including correlated market, credit quality, and default sources of risk.

A specific stylized sample asymptotic portfolio of BBB-quality credits is used to illustrate model calibration and estimation of the integrated MTM portfolio value distribution. The estimated unconditional asymptotic portfolio loss distribution is used to construct estimates of the economic capital allocations needed to support a desired solvency standard and to measure the potential for realizing economic capital savings from integrated risk measurement.[6]

Within the banking industry, it is common practice to use separate models to measure market and credit risk, and to estimate capital needs using either only one measure of risk or in some cases, the sum of the market and credit risk measures. An important issue related to integrated risk measurement is the magnitude of capital savings that might be achieved by jointly modeling market and credit risks. It is axiomatic that a well-constructed and well-executed integrated risk measurement and capital allocation framework will produce more accurate risk and capital estimates than can be produced by piecemeal approaches.

The results of this study show that, relative to market and credit risk estimation methods commonly in use, the integrated modeling of market and credit risk will alter estimates of the minimum capital needed to achieve a given target solvency margin. Differences among capital estimates can be large. Whether or not the adoption of an integrated risk measurement framework will reduce an institution's estimated capital

[6] Solvency standards are typically expressed as target survival rates, i.e. a target minimum probability that the firm will be able to payoff its debts at the end of the risk measurement horizon using revenues generated exclusively from the sale of investment portfolio.

allocation depends on how the institution determines its capital needs prior to applying the integrated approach. The results show that piecemeal approaches for measuring market and credit risk can lead to under- or over-stated estimates of capital relative to the capital allocation estimated using an integrated model.

Focusing specifically on the Basel II AIRB capital requirement, the results do not support arguments that recommend capital relief in the current framework on the premise that an integrated capital framework would generate capital savings that are not realized in the current Basel approach. The current Basel II AIRB paradigm significantly understates capital needs relative to the capital estimates produced by an accurate integrated market and credit risk capital allocation methodology.

The following section explains the assumptions that underlie the integrated risk measurement model and formally derives a closed-form expression for the conditional value of an asymptotic portfolio of credits at the end of the selected risk measurement horizon. Section III illustrates the estimation of an asymptotic portfolio's unconditional future value distribution using Monte Carlo simulation. Section IV discusses capital allocation and measures the capital savings (or shortfalls) that are identified by the integrated risk measurement model. A final section concludes the paper.

II. THE MODEL

1. CHARACTERISTICS OF INDIVIDUAL CREDITS

The individual credits analyzed in this structural model are zero coupon credits that repay principal and accrued interest at maturity. We assume that the borrower receives \$1 at initiation of the loan contract and promises to repay $(1+Y_0)^m$ at loan maturity, where Y_0 is the initial contract yield-to-maturity.

2. Individual Credit Default Process

Define a latent unobserved factor, $\widetilde{V}_i$, for credit i whose value is realized at the end of the horizon of interest. The modeling time horizon is completely flexible. Time is not an independent factor in this structural model but is only implicitly recognized through the calibration of model's parameters (e.g. probability of default or downgrade). For expositional convenience we will develop the model using a one-year time horizon but shorter or longer periods can be substituted.[7] As a consequence, the latent random factor value $\widetilde{V}_i$ is realized at the end of a year and so the model abstracts away form defaults or downgrades that may occur prior to this date. $\widetilde{V}_i$ has the following properties,

$$\begin{aligned} &\widetilde{V}_i = \sqrt{\rho_V}\,\widetilde{e}_M + \sqrt{1-\rho_V}\,\widetilde{e}_{id} \\ &\widetilde{e}_M \sim \phi(e_M) \\ &e_{id} \sim \phi(e_{id}), \\ &E(\widetilde{e}_{id}\,\widetilde{e}_{jd}) = E(\widetilde{e}_M\,\widetilde{e}_{jd}) = 0, \ \ \forall i, j. \end{aligned} \tag{1}$$

$\widetilde{V}_i$ is distributed standard normal, $E\left(\widetilde{V}_i\right)=0,$ and $E\left(\widetilde{V}_i^2\right)=1.$ $\widetilde{e}_M$ is a factor common to all credits' associated latent factors, $\widetilde{V}_i$. The correlation between individual credits' latent factors is ρ_V. $\widetilde{V}_i$ is often interpreted as a proxy for the market value of the firm that issued credit i.

[7] The example in Section III uses a 6-month risk measurement horizon.

Credit i is assumed to default when its latent factor takes on a value less than a credit-specific threshold, $\widetilde{V}_i < D_i$. Since no cash payments are due until contract maturity, if the horizon of interest is less than maturity, default is interpreted as a violation of a contract covenant that causes the bank to call the loan balance. The unconditional probability that credit i defaults is, $PD = \Phi(D_i)$, where $\Phi(\cdot)$ represents the cumulative standard normal density function. Define a default indicator function for credit i,

$$1_{D_i}\left(\widetilde{V}_i\right) = \begin{cases} 1 & \text{if } \widetilde{V}_i < D_i \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In default, assume the bank recovers a fixed proportion of the initial loan balance, $(1 - LGD)$.

3. End-of-Period Credit Quality

We focus on the case when the portfolios' credits maturity exceeds the risk measurement horizon. At the end of the measurement horizon, the credit will either default or continue to accrue interest. If the credit continues to accrue interest, it is still possible that its credit quality has changed over the risk measurement horizon. We assume that credit quality, while continuous, is measured and priced according to discrete credit quality classes or "ratings".

A credit's rating at the end of the risk measurement horizon is assumed to be determined by the credit's realized "distance to default, or $\widetilde{V}_i - D_i$. In other word, the realized distance to default determines the credit's unconditional probability of default in the subsequent period. If the credit's distance to default is large, then the credit is said to be "high quality" and its unconditional probability of default in the subsequent period is assumed to be small. When the realized distance to default is small, the credit is said to be of low quality and the unconditional probability that it defaults in the subsequent period is assumed to be large.

For expositional purposes, we assume there are three distinct credit qualities: high, medium and low. Associated with each of these credit qualities, we assume that there is an observable term-structure of yields at which new credits could be issued. A credit will be of high quality (low probability of default) if at the end of the period, $\widetilde{V}_i > G_1$; the credit will be medium quality if, $G_2 < \widetilde{V}_i < G_1$; and the credit will be low quality (high probability of default) if, $D_i < \widetilde{V}_i < G_2$. The number of credit quality states can be expanded with no additional difficulty and indeed the example in Section III uses four credit quality grades.

Define three functions that indicate whether a credit is of high, medium, or low quality at the end of the period,

$$1_H\left(\widetilde{V}_i\right) = \begin{cases} 1 & \text{if } \widetilde{V}_i > G_1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$1_K\left(\widetilde{V}_i\right) = \begin{cases} 1 & \text{if } G_2 < \widetilde{V}_i < G_1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$1_L\left(\widetilde{V}_i\right) = \begin{cases} 1 & \text{if } D_i < \widetilde{V}_i < G_2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Under these assumptions, the probability that a credit becomes high quality is, $1 - \Phi^{-1}(G_1)$; the probability that a credit transitions to medium quality is, $\Phi^{-1}(G_1) - \Phi^{-1}(G_2)$; and $\Phi^{-1}(G_2) - \Phi^{-1}(D_i)$ is the probability that a credit becomes low quality.[8]

[8] The extension to include additional credit grades is accomplished by introducing additional thresholds and indicator function between D_i and G_1.

4. Individual Credit Capital Gain/Loss Process

At the end of the risk measurement horizon, individual credits are assumed to have remaining maturities of m years. We assume that there are historical time series or other techniques that can be used to estimate the distribution of the discount factors that may prevail at the end of the risk measurement horizon for high-, medium-, and low-quality cash flows with m- periods remaining until maturity. How these distributions are constructed is not important for purposes of deriving the structural model. The distributions could for example be determined from forwards rates and implied volatilities using a theoretical model or they could be constructed from historical time series data. The sample risk-measurement calculation in Section III uses historical time series data on yields to construct distribution for future discount factor values.

Let $\tilde{B}_m^H$ represent the market value of a promised cash payment of \$1 to be received after $m+l$-periods at the end of an l- period risk measurement horizon.[9] If $\tilde{Y}_{Hm}$ represents the distribution of the potential yield on high-quality m-period credits on date l,

$$\tilde{B}_m^H = \frac{\left(1+Y_0\right)^{m+l}}{\left(1+\tilde{Y}_{Hm}\right)^m} \tag{6}$$

Similarly, define $\tilde{B}_m^K$ to be the discount factor for a promised cash payment of \$1 to be received after $m+l$-periods if the credit is valued using the prevailing medium-quality yield on m- period credits (at time l), $\tilde{Y}_{Km}$,

[9] Consistent with Section II.2, $l=1$ year throughout Section II, but the model can incorporate any horizon.

$$\tilde{B}_m^K = \frac{(1+Y_0)^{m+l}}{\left(1+\tilde{Y}_{Km}\right)^m} \tag{7}$$

$\tilde{B}_m^L$ is the corresponding discount factor for a low-quality credit when the low-quality spot yield is $\tilde{Y}_{Lm}$,

$$\tilde{B}_m^L = \frac{(1+Y_0)^{m+l}}{\left(1+\tilde{Y}_{Lm}\right)^m} \tag{8}$$

Using these quality-specific discount factors and the indicator functions defined in expression (3)-(5), the end-of-period value of a performing credit can be written,

$$\tilde{B}_m = 1_H\left(\tilde{V}_i\right)\tilde{B}_m^H + 1_K\left(\tilde{V}_i\right)\tilde{B}_m^K + 1_L\left(\tilde{V}_i\right)\tilde{B}_m^L \tag{9}$$

The capital gain or loss on a performing credit over the first period is $\tilde{B}_m - 1$.

5. End-of-Period Conditional Valuation Distributions

The end-of-period value of a performing credit is drawn from a different probability distribution depending on whether the credit ends the period as a high-, medium- or low-quality asset. Each of these distributions are determined by the credit's promised cash flows at maturity and the probability distribution for the required spot market yields that will prevail at the end of the risk measurement horizon. As we may choose between modeling quality-dependent distributions for discount factors or yields, we elect to model the distributions for quality-dependent discount factors directly, $\Pi_H\left(\tilde{B}_m^H\right)$, $\Pi_K\left(\tilde{B}_m^K\right)$, and $\Pi_L\left(\tilde{B}_m^L\right)$.

Systematic dependence between defaults and ratings migrations are incorporated by assuming that the realizations from the discount factor distributions are driven by three latent Gaussian factors, $\tilde{Z}_H$, $\tilde{Z}_K$, and $\tilde{Z}_L$ with the following properties,

$$
\begin{aligned}
&\tilde{Z}_H = \sqrt{\rho_{ZH}}\,\tilde{e}_M + \sqrt{1-\rho_{ZH}}\,\tilde{e}_H \\
&\tilde{Z}_K = \sqrt{\rho_{ZK}}\,\tilde{e}_M + \sqrt{1-\rho_{ZK}}\,\tilde{e}_K \\
&\tilde{Z}_L = \sqrt{\rho_{ZL}}\,\tilde{e}_M + \sqrt{1-\rho_{ZL}}\,\tilde{e}_L \\
&\tilde{e}_M \sim \phi(e_M), e_H \sim \phi(e_H), \tilde{e}_K \sim \phi(e_K), \tilde{e}_L \sim \phi(e_L), \\
&E(\tilde{e}_M\,\tilde{e}_H) = E(\tilde{e}_M\,\tilde{e}_K) = E(\tilde{e}_M\,\tilde{e}_K) = E(\tilde{e}_H\,\tilde{e}_K) = E(\tilde{e}_H\,\tilde{e}_L) = E(\tilde{e}_K\,\tilde{e}_L) \\
&E(\tilde{e}_{id}\,\tilde{e}_K) = E(\tilde{e}_{id}\,\tilde{e}_L) = E(\tilde{e}_{id}\,\tilde{e}_L) \quad \forall\, i
\end{aligned}
\tag{10}
$$

We adopt the normalization convention that higher end-of-period contract valuations are associated with larger realizations of the latent variables, $\tilde{Z}_H, \tilde{Z}_K$, and $\tilde{Z}_L$. The correlations between the latent variables that drive the end-of-period discount factors for the high- and medium-quality, high- and-low quality, and medium- and low-quality are, respectively, $\sqrt{\rho_{ZH}\,\rho_{ZK}}$, $\sqrt{\rho_{ZH}\,\rho_{ZL}}$, and, $\sqrt{\rho_{ZK}\,\rho_{ZL}}$.

Correlations among realizations of the end-of-period discount factors for the alternative credit quality grades are introduced using the integral transformation,

$$
\begin{aligned}
&\tilde{B}_m^H \sim \Pi_H^{-1}\left(\Phi\left(\tilde{Z}_H\right)\right) \\
&\tilde{B}_m^K \sim \Pi_K^{-1}\left(\Phi\left(\tilde{Z}_K\right)\right). \\
&\tilde{B}_m^L \sim \Pi_L^{-1}\left(\Phi\left(\tilde{Z}_L\right)\right)
\end{aligned}
\tag{11}
$$

6. Holding Period Mark-to-Market Value for an Individual Credit

At the end of the capital allocation horizon, a credit either performs or it defaults. If it performs, the bank continues to accrue interest and must value the contract on market-value basis if the contract is in the trading- or available-for-sale accounts. The mark-to-market process may generate a capital gain or loss in the mark-to-market value of the credit as risk-free interest rates, alternative quality credit spreads, and the credit quality of the borrower may change over the risk-measurement horizon. If the credit defaults, in this model the investor is assumed to receive a fixed recovery, i.e., the loss given default rate (LGD) is a constant.[10] The MTM value of a credit with a remaining maturity of *m* at the end of the measurement horizon, $\tilde{b}_i$, can be written,

$$\tilde{b}_i \approx (1-LGD)\, 1_{D_i}\left(\tilde{V}_i\right) + 1_H\left(\tilde{V}_i\right)\tilde{B}_m^H + 1_K\left(\tilde{V}_i\right)\tilde{B}_m^K + 1_L\left(\tilde{V}_i\right)\tilde{B}_m^L \ . \tag{12}$$

7. Conditional Mark-to-Market Value of an Individual Credit

It is useful to evaluate the indicator functions that are included in an individual credit's mark-to-market value conditional on a specific realized value for the common Gaussian factor, e_M. Let $1_{D_i}\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right)$ represent the default indicator function conditional on a realized value of e_M,

$$1_{D_i}\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right) = \begin{cases} 1 & \text{if} \quad \tilde{e}_{id} < \dfrac{D_i - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}} \\ 0 & \text{otherwise} \end{cases} . \tag{13}$$

The conditional indicator function is a binomial random variable with an expected value,

$$E\left(1_{D_i}\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right)\right) = \Phi\left(\frac{D_i - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right). \tag{14}$$

Conditional default indicator functions are independent across credits.

Similarly, let $1_H\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right)$, $1_K\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right)$, and $1_L\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right)$ represent, respectively, the default indicator functions for high-, medium- and low-quality credit status at the end of the horizon conditional on a realized value of e_M,

[10] The results in Kupiec (2007) can be used to generalize the model to include correlated stochastic recovery rates.

$$1_H\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right) = \begin{cases} 1 & \text{if} \quad \tilde{e}_{id} > \dfrac{G_1 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}} \\ 0 & \text{otherwise} \end{cases}, \tag{15}$$

$$1_K\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right) = \begin{cases} 1 & \text{if} \quad \dfrac{G_2 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}} < \tilde{e}_{id} < \dfrac{G_1 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}} \\ 0 & \text{otherwise} \end{cases}, \tag{16}$$

$$1_L\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right) = \begin{cases} 1 & \text{if} \quad \dfrac{D_i - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}} < \tilde{e}_{id} < \dfrac{G_2 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}} \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

These conditional indictor functions are independent across credits for high-, medium-, and low-quality credits. For example, for the indicator function for high-quality credits,

$E\left(1_H\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right) \cdot 1_H\left(\tilde{V}_j \mid \tilde{e}_M = e_M\right)\right) = 0 \quad \forall\, i \neq j.$

A similar relationship holds true for medium- and low-quality credits. The conditional expected values of these indicator functions are,

$$E\left(1_H\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right)\right) = 1 - \Phi\left(\frac{G_1 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right), \tag{18}$$

$$E\left(1_K\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right)\right) = \Phi\left(\frac{G_1 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) - \Phi\left(\frac{G_2 - \sqrt{\rho_V}}{\sqrt{1-\rho_V}}\right) \text{ and,} \tag{19}$$

$$E\left(1_L\left(\tilde{V}_i \mid \tilde{e}_M = e_M\right)\right) = \Phi\left(\frac{G_2 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) - \Phi\left(\frac{D_i - \sqrt{\rho_V}}{\sqrt{1-\rho_V}}\right). \tag{20}$$

8. Mark-to-Market Value of an Asymptotic Portfolio of Credits

Consider a portfolio of N credits that are identical in all respects except for the idiosyncratic sources of risk in their latent variables $\tilde{V}_i$'s. These risks are modeled as standard normal deviates which are distributed independently across credits,

$E(\tilde{e}_{id}, \tilde{e}_{jd}) = 0 \quad \forall \quad i \neq j$. The credits are otherwise identical with par values of \$1, default thresholds of $D = D_i$, maturities of $m - l$ years, and identical initial yields, $Y_{i0} = Y_0$. The mark-to-market value of a portfolio of *N* credits, at time *l* is, $\tilde{b}_p$,

$$\tilde{b}_p = (1 - LGD)\left(\frac{\sum_{i=1}^{N} 1_D(\tilde{V}_i)}{N}\right) + \left(\frac{\sum_{i=1}^{N} 1_H(\tilde{V}_i)\tilde{B}_m^H}{N}\right) + \left(\frac{\sum_{i=1}^{N} 1_K(\tilde{V}_i)\tilde{B}_m^K}{N}\right) + \left(\frac{\sum_{i=1}^{N} 1_L(\tilde{V}_i)\tilde{B}_m^L}{N}\right). \tag{21}$$

The MTM value of an asymptotic portfolio of credits is the limit of expression (21) taken as $N \to \infty$.

9. The Conditional Distribution of the Market Value of Performing Credits

Before deriving the distribution function for an asymptotic portfolio of credits, it is necessary to consider the distribution function of the end-of-period discount factors for performing credits, conditional on a realized value of the common market factor, $\tilde{e}_M = e_M$. The market value of a credit depends on its end-of-period credit quality as well as the realized values of the latent factors that determine the discount factors for the alternative ratings-quality categories.

Consider the conditional cumulative probability distributions of the latent factors that drive the cash flow discount factors. Conditional on a realized value of the common market factor, $\tilde{e}_M = e_M$,

$$\Phi(\tilde{Z}_H < Z_H \mid \tilde{e}_M = e_M) = \Phi\left(\frac{Z_H - \sqrt{\rho_{ZH}}\, e_M}{\sqrt{1 - \rho_{ZH}}}\right), \tag{22}$$

$$\Phi(\tilde{Z}_K < Z_K \mid \tilde{e}_M = e_M) = \Phi\left(\frac{Z_K - \sqrt{\rho_{KH}}\, e_M}{\sqrt{1 - \rho_{KH}}}\right), \tag{23}$$

$$\Phi\left(\widetilde{Z}_L < Z_L \mid \widetilde{e}_M = e_M\right) = \Phi\left(\frac{Z_L - \sqrt{\rho_{LH}}\, e_M}{\sqrt{1-\rho_{LH}}}\right). \tag{24}$$

Using the integral transform, the conditional cumulative densities for the discount factors of the alternative credit qualities grades can be written as implicit functions,

$$\widetilde{B}_m^H \mid \widetilde{e}_M = e_M \sim \left\{\Pi_H^{-1}\left(\Phi\left(\frac{Z_H - \sqrt{\rho_{ZH}}\, e_M}{\sqrt{1-\rho_{ZH}}}\right)\right), \phi(Z_H)\right\}, Z_H \in (-\infty, \infty), \tag{25}$$

$$\widetilde{B}_m^K \mid \widetilde{e}_M = e_M \sim \left\{\Pi_K^{-1}\left(\Phi\left(\frac{Z_K - \sqrt{\rho_{ZK}}\, e_M}{\sqrt{1-\rho_{ZK}}}\right)\right), \phi(Z_K)\right\}, Z_K \in (-\infty, \infty), \tag{26}$$

$$\widetilde{B}_m^L \mid \widetilde{e}_M = e_M \sim \left\{\Pi_L^{-1}\left(\Phi\left(\frac{Z_H - \sqrt{\rho_{ZL}}\, e_M}{\sqrt{1-\rho_{ZL}}}\right)\right), \phi(Z_L)\right\}, Z_L \in (-\infty, \infty). \tag{27}$$

Notice that the conditional discount factor distributions are independent of all individual credit idiosyncratic risk characteristics (i.e., they are independent of $e_{id}\ \forall\ i.$).

10. The Market-to-Market Value Distribution on an Asymptotic Portfolio of Credits

Consider the MTM value distribution for an asymptotic portfolio of credits conditional on a realization of the common factor, $\widetilde{e}_M = e_M$,

$$\begin{aligned}\widetilde{b}_p = (1-LGD)\lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N} 1_D\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)}{N}\right) \\ + \left(\widetilde{B}_m^H \mid \widetilde{e}_M = e_M\right)\lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N} 1_H\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)}{N}\right) + \left(\widetilde{B}_m^K \mid \widetilde{e}_M = e_M\right)\lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N} 1_K\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)}{N}\right) \\ + \left(\widetilde{B}_m^L \mid \widetilde{e}_M = e_M\right)\lim_{N\to\infty}\left(\frac{\sum_{i=1}^{N} 1_L\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)}{N}\right)\end{aligned} \tag{28}$$

Because the conditional indicator functions are independent and identically distributed across individual credits, the strong law of large numbers requires that the limiting terms converge almost surely to their expected values,

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} 1_D\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)}{N} \xrightarrow[a.s]{} E\left(1_D\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)\right) = \Phi\left(\frac{D - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right), \tag{29}$$

$$\lim_{N\to\infty} \left(\frac{\sum_{i=1}^{N} 1_H\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)}{N}\right) \xrightarrow[a.s.]{} E\left(1_H\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)\right) = 1 - \Phi\left(\frac{G_1 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right), \tag{30}$$

$$\lim_{N\to\infty} \left(\frac{\sum_{i=1}^{N} 1_K\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)}{N}\right) \xrightarrow[a.s]{} E\left(1_K\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)\right) = \Phi\left(\frac{G_1 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) - \Phi\left(\frac{G_2 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right), \tag{31}$$

$$\lim_{N\to\infty} \left(\frac{\sum_{i=1}^{N} 1_L\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)}{N}\right) \xrightarrow[a.s]{} E\left(1_L\left(\widetilde{V}_i \mid \widetilde{e}_M = e_M\right)\right) = \Phi\left(\frac{G_2 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) - \Phi\left(\frac{D_i - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right). \tag{32}$$

Substitution produces the following expression for the asymptotic portfolio's future value per dollar invested,

$$\begin{aligned}
\widetilde{b}_p = {} & (1 - LGD)\,\Phi\left(\frac{D - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) \\
& + \left(\widetilde{B}_m^H \mid \widetilde{e}_M = e_M\right)\left(1 - \Phi\left(\frac{G_1 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right)\right) \\
& + \left(\widetilde{B}_m^K \mid \widetilde{e}_M = e_M\right)\left(\Phi\left(\frac{G_1 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) - \Phi\left(\frac{G_2 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right)\right) \\
& + \left(\widetilde{B}_m^L \mid \widetilde{e}_M = e_M\right)\left(\Phi\left(\frac{G_2 - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) - \Phi\left(\frac{D - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right)\right)
\end{aligned} \tag{33}$$

The terms $\left\{\left(\tilde{B}_m^H \mid \tilde{e}_M = e_M\right), \left(\tilde{B}_m^K \mid \tilde{e}_M = e_M\right), \left(\tilde{B}_m^L \mid \tilde{e}_M = e_M\right)\right\}$ in expression (33), are stochastic, but they depend only on the idiosyncratic risk components that drive the discount factors used to estimate the mark-to-market valuations of the high-, medium-, and low-quality credits. The larger are the realizations of $\tilde{e}_H, \tilde{e}_K$, and $\tilde{e}_L$, the higher are the market values of each credit quality group. The factors $\tilde{e}_H, \tilde{e}_K$, and $\tilde{e}_L$ are independent of one another, but they are identical for all credits of a given quality at any given time, and so their influence on the portfolio's value is not diversified in an asymptotic portfolio.

The unconditional distribution of the end-of-horizon MTM value of the asymptotic portfolio can be determined using Monte Carlo simulation, sampling over the distributions of the 4 latent factors $\{\tilde{e}_M, \tilde{e}_H, \tilde{e}_K$, and $\tilde{e}_L\}$ that determine the asymptotic portfolio's MTM value. The following section provides an explicit example.

III. MONTE CARLO SIMULATION OF A PORTFOLIO'S FUTURE MTM VALUE DISTRIBUTION

1. Overview

In this section, the distribution of an asymptotic portfolio's future MTM value is calculated using standard Monte Carlo techniques. The example uses a hypothetical portfolio of credits that have approximately BBB credit quality. The credits are assumed to have an original maturity of 18-months and the example will derive the distribution of the portfolio's MTM value after 6 months, when the portfolio credits have 12 months remaining to maturity.

To simplify the analysis, it is assumed that the portfolio is composed of individual credits that are zero-coupon instruments with an initial market value of $1. The example uses four possible fully performing credit-quality states (instead of three) in addition to the default state.

2. Model Calibration

Distributions for the discount factors and their latent factor correlations are calibrated using historical yield data on Standard & Poor's credit-quality indices for AAA, A, BBB, and B credits as reported by Bloomberg. Associated with each credit quality is a latent Gaussian factor, $\tilde{Z}_{AAA}, \tilde{Z}_{A}, \tilde{Z}_{BBB}$, and, $\tilde{Z}_{B}$, which in turn have correlation parameters (respectively), $\rho_{AAA}, \rho_{A}, \rho_{BBB}$, and, ρ_{B}. Following Basel II conventions for wholesale credits, the example uses 20 percent as the correlation for the latent default factors, $\rho_V = 0.20$.

The yields on 1-year AAA, A, BBB, and B bonds are transformed into discount factor series that represent the current value of $1 to be received after 1-year on contracts with credit risks equivalent to those of AAA-, A-, BBB-, and B-quality instruments. Figure 1 plots data on the discount factors for 1-year maturity cash flows. The discount factors are sampled at 6-months intervals beginning January 1, 1997 and ending January 1, 2007. The fist-differences of these respective series are used to estimate the probability distribution for 6-month changes in each quality-dependent discount factor series. Figure 2 plots histograms of the 6-month changes in the AAA-, A-, BBB- and B-discount factors series.

Figure 1: 1-Year Credit Quality-Specific Discount Factors

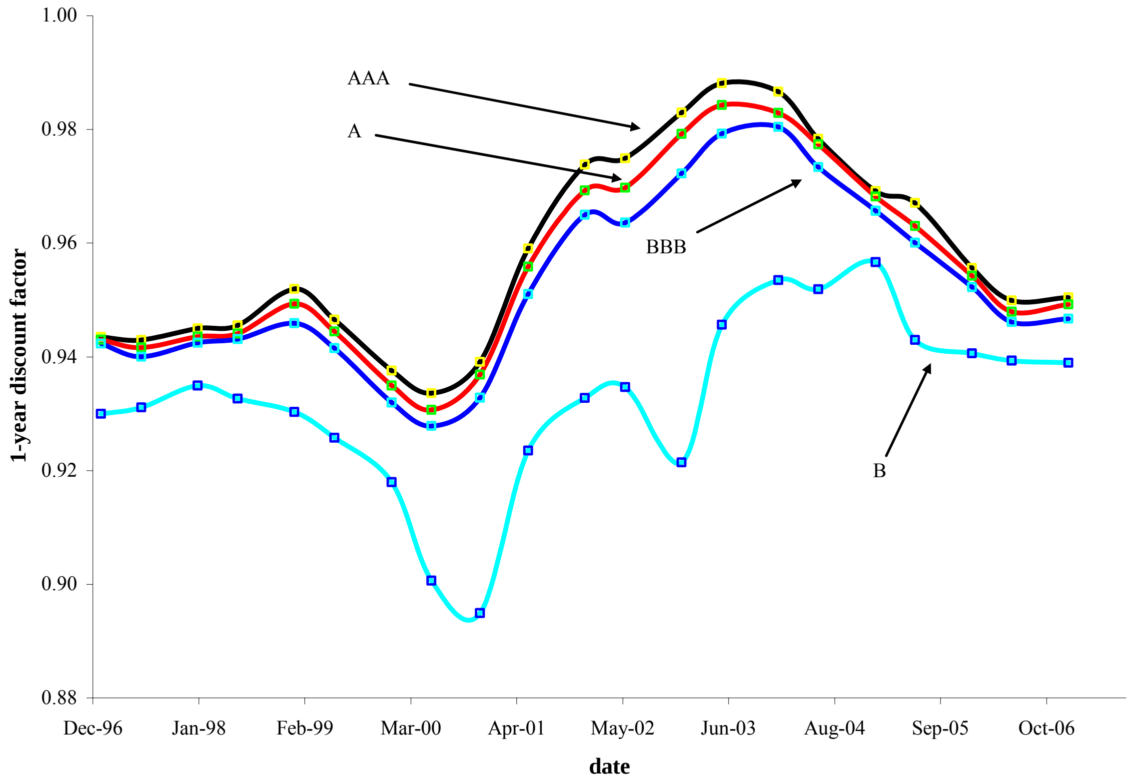
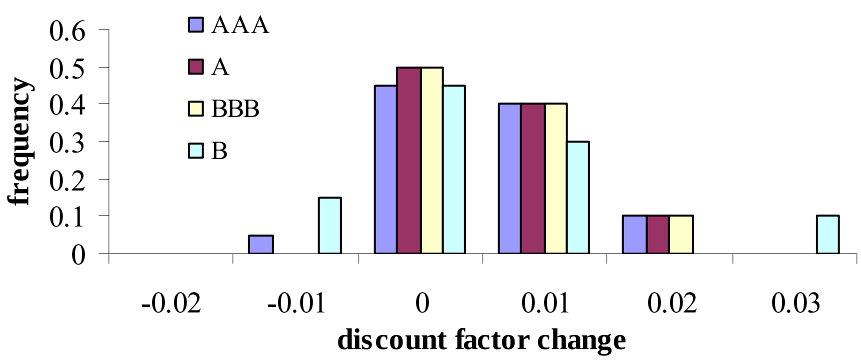


Figure 2: Histograms of 6-Month Changes in AAA, A, BBB, and B Discount Factors (1997-2007)



The histograms in Figure 2 have characteristics that are similar to random variables drawn from individual Beta distributions. End-of-horizon contract price uncertainty on performing credits will be modeled using individual quality-specific beta distributions. There are 4 parameters that determine the shape and location of a beta distribution; two shape parameters: p, and q; and the minimum (a) and maximum (b) values of the distribution. The probability density associated with an individual observation, y_i, from a beta distribution is given by,

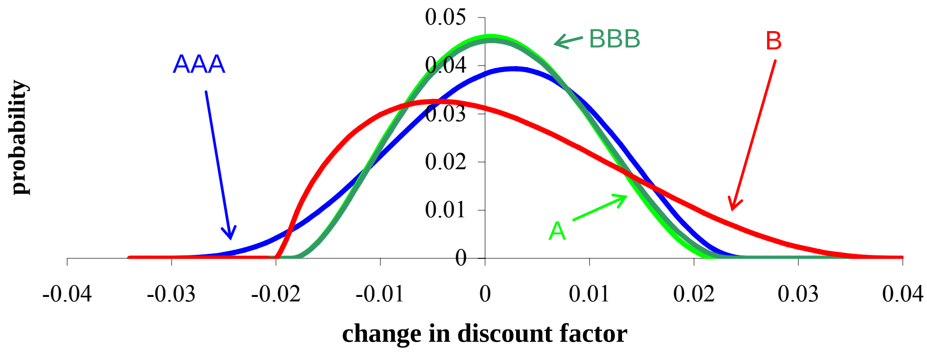
$$p(y_i) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{(y_i - a)^{p-1}(b - y_i)^{q-1}}{(b-a)^{p+q-1}}, \tag{34}$$

The parameters of the individual beta distributions are estimated using maximum likelihood. The log likelihood associated with a sample of Q independent observations from the same beta distribution is, $\ln L(y_1, y_2, \ldots, y_Q) = \sum_{i=1}^{Q} \ln(p(y_i))$. The maximum likelihood estimates for the parameters of individual discount factor beta distributions are reported in Table 1. The parameter estimates are nearly identical for the A- and BBB-quality discount factors. Compared to the other distributions, the AAA-quality distribution shows more downside risk whereas the B-quality distribution shows a higher potential for significant price appreciation. Plots of the estimated factor-change distributions appear in Figure 3.

Table 1: Beta Distribution ML Parameter Estimates

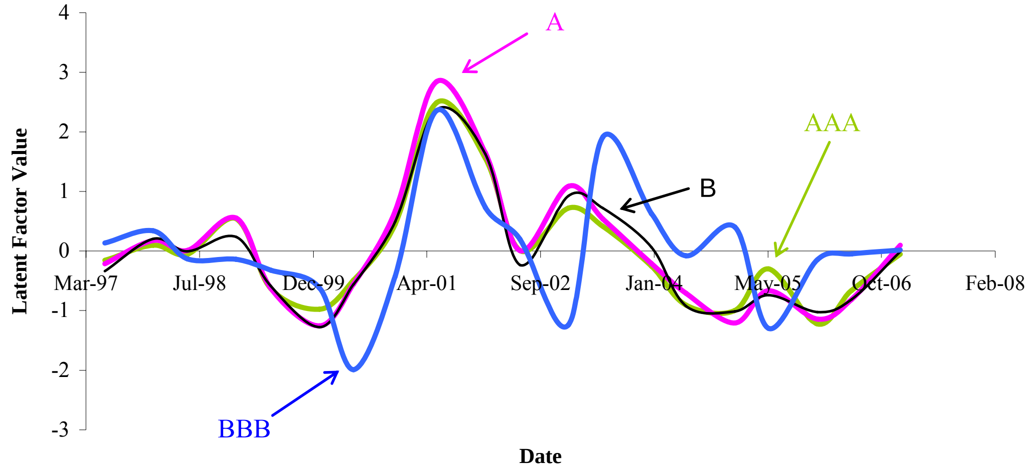
	AAA	A	BBB	B
sample min	-0.012	-0.010	-0.010	-0.018
sample max	0.020	0.019	0.019	0.030
ML "a" estimate	-0.033	-0.019	-0.019	-0.020
ML "b" estimate	0.025	0.022	0.024	0.039
ML "p" estimate	4.809	2.888	2.917	1.803
ML "q" estimate	3.427	3.175	3.353	3.377

Figure 3: Maximum Liklihood Estimates of Discount Factor Change Densities



Each discount factor change is converted into a cumulative probability using its associated estimated Beta distribution. The estimated realization of the latent Gaussian driving factor is recovered from these cumulative densities by inverting the integral transformation. For example, the first-difference of the discount factor on the AAA-quality on date t, ΔB_{1t}^{AAA}, has an associated latent Gaussian factor value, $Z_{AAA,t}$, given by $Z_{AAA,t} = \Phi^{-1}\left(Beta\left(\Delta B_{1t}^{AAA}, 4.809, 3.427, -.033, .025\right)\right)$, where $Beta(x, 4.809, 3.427, -.033, .025)$ represents the cumulative distribution function for the AAA-Beta distribution evaluated at x using the maximum likelihood parameter estimates, $p = 4.809$, $q = 3.427$, $a = -0.033$, and $b = 0.025$. Analogous transformations are used to recover the latent factor series realizations for $\widetilde{Z}_A, \widetilde{Z}_{BBB}$, and, $\widetilde{Z}_B$, and the resulting Gaussian latent factor estimates are plotted in Figure 4.

Figure 4: ML Estimates of Implied Discount Function Latent Factor Realizations



The sample correlations between the latent Gaussian factors are calculated and the estimates appear in Table 2. As the correlation estimates indicate, changes in the discount factors for investment quality credits are highly correlated, while changes in the discount factors for the sub-investment grade credits do not move in lock-step with the investment grade market. The correlation estimates in Table 2 are used to solve for the implied values of the correlation parameters, $\rho_V, \rho_{AAA}, \rho_A, \rho_{BBB}$; these parameter estimates appear in Table 3.

Table 2: Correlations Among 6-Month Changes in Discount Factors

	AAA	**A**	**BBB**	**B**
AAA	1.000	0.939	0.927	0.483
A		1.000	0.938	0.489
BBB			1.000	0.528
B				1.000

Table 3: Common Latent Factor Coefficient Estimates

AAA	0.792
A	0.811
BBB	0.944
B	0.295

The individual portfolio credits will be modeled as if they are initially underwritten as BBB-quality credits. The assumed transition probabilities and corresponding boundaries for the individual credit latent Gaussian factor $\tilde{V}_i$ are reported in Table 4. Note that these transition probabilities are assumed to be transitions over the 6-month capital measurement horizon. Should a credit default, the assumed recovery value is nonrandom and equal to 80 percent of the initial credit extended.

Table 4: Transition Probabilities and Associated Latent Factor Threshold Values

credit quality	probability	lower threshold	upper threshold
AAA	0.005	2.57583	
A	0.015	2.05375	2.57583
BBB	0.96	-2.05375	2.05375
B	0.015	-2.57583	-2.05375
D	0.005	-2.57583	-2.57583

The remaining set of information that underlies the example are the current yields associated with the alternative credit qualities at the time the credits were underwritten. For these data, we assume that the term structure is flat between 12 and 18 months, with yields to maturity and discount factors as reported in Table 5. The yields used in the example correspond to those that prevailed in early January 2007.

Table 5: Term Structure of Alternative Credit Quality Grades at Issuance

credit quality	yield	1-year discount factor	18-month discount factor
AAA	0.0526	0.95	0.926
A	0.0537	0.949	0.924
BBB	0.0560	0.947	0.922
B	0.0650	0.939	0.910

3. Monte Carlo Density Estimation

The portfolio's future value density can be estimated using standard Monte Carlo techniques. The distribution of the end-of-horizon MTM value of the portfolio can be estimated by repeatedly sampling over the distributions of the 5 independent latent factors that determine the asymptotic portfolio's MTM value, $(\tilde{e}_M, \tilde{e}_{AAA}, \tilde{e}_A, \tilde{e}_{BBB}, \tilde{e}_B)$, and calculating the implied MTM value of the asymptotic portfolio.

Given a single random draw from the four independent standard Gaussian distributions, $(e_{M\,i}, e_{AAA\,i}, e_{A\,i}, e_{BBB\,i}, e_{B\,i})$, calculate the proportion of the asymptotic portfolio that migrates to each of the credit quality grades or the default state. The calculations needed to define these proportions are reported in Table 6.

Next calculate the latent factor realizations that determine each credit quality discount factor,

$$\begin{aligned}
Z_{AAA\,i} &= \sqrt{.792}\, e_{M\,i} + \sqrt{1-.792}\, e_{AAAi} \\
Z_{A\,i} &= \sqrt{.811}\, e_{M\,i} + \sqrt{1-.811}\, e_{Ai} \\
Z_{BBB\,i} &= \sqrt{.944}\, e_{M\,i} + \sqrt{1-.944}\, e_{BBBi} \\
Z_{B\,i} &= \sqrt{.295}\, e_{M\,i} + \sqrt{1-.295}\, e_{Bi}
\end{aligned} \tag{35}$$

Use the latent factor realizations to calculate the realizations of the discount factor change distributions,

$$
\begin{aligned}
\Delta B_i^{AAA} &= Beta^{-1}\left(\Phi\left(Z_{AAA\,i}\right), 4.809, 3.427, -0.033, 0.025\right),\\
\Delta B_i^{A} &= Beta^{-1}\left(\Phi\left(Z_{A\,i}\right), 2.888, 3.173, -0.019, 0.022\right),\\
\Delta B_i^{BBB} &= Beta^{-1}\left(\Phi\left(Z_{BBB\,i}\right), 2.917, 3.353, -0.019, 0.024\right),\\
\Delta B_i^{B} &= Beta^{-1}\left(\Phi\left(Z_{B\,i}\right), 1.803, 3.377, -0.02, 0.039\right),
\end{aligned}
\tag{36}
$$

where $Beta^{-1}(y, 1.8, 3.3, -0.02, 0.03)$ represents the inverse of the Beta distribution with parameter values, $p = 1.8, q = 3.3, a = -0.02$, and $b = 0.03$ evaluated at $y \in [0,1]$.

Table 6: Portfolio Composition Conditional on a Realization of the Common Market Factor, $\tilde{e}_M$

Credit Quality	Proportion of an Asymptotic Portfolio in State
AAA	$\left(1-\Phi\left(\frac{(\Phi^{-1}(0.995)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.2}}\right)\right)$
A	$\Phi\left(\frac{\Phi^{-1}(0.995)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-0.2}}\right)-\Phi\left(\frac{\Phi^{-1}(0.98)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-0.2}}\right)$
BBB	$\Phi\left(\frac{\Phi^{-1}(0.98)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right)-\Phi\left(\frac{\Phi^{-1}(0.02)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right)$
B	$\Phi\left(\frac{\Phi^{-1}(.002)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right)-\Phi\left(\frac{\Phi^{-1}(.0005)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right)$
Default	$\Phi\left(\frac{\Phi^{-1}(0.005)-\sqrt{.02}\,e_{M\,i}}{\sqrt{1-0.2}}\right)$

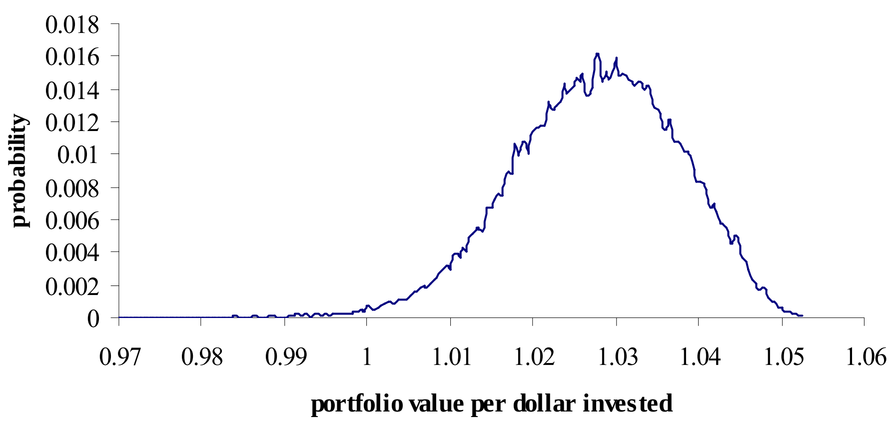
For a single realization of the latent factors, $(e_{M\,i}, e_{AAA\,i}, e_{A\,i}, e_{BBB\,i}, e_{B\,i})$, the end-of-period MTM value of the portfolio is,

$$\tilde{b}_{p\,i} = .80\,\Phi\left(\frac{\Phi^{-1}(.005)-\sqrt{.2}\,e_{M\,i}}{\sqrt{1-.2}}\right)+(1.056)^{\frac{3}{2}}\times$$

$$\left(\begin{array}{l}
\left(0.95+Beta^{-1}\left(\Phi\left(Z_{AAA\,i}\right), 4.809, 3.427, -0.033, 0.025\right)\right)\left(1-\Phi\left(\frac{(\Phi^{-1}(0.995)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right)\right) \\
+\left(0.949+Beta^{-1}\left(\Phi\left(Z_{A\,i}\right), 2.888, 3.173, -0.019, 0.022\right)\right)\Phi\left(\frac{\Phi^{-1}(0.995)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-0.2}}\right)-\Phi\left(\frac{\Phi^{-1}(0.98)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right) \\
+\left(0.947+Beta^{-1}\left(\Phi\left(Z_{BBB\,i}\right), 2.917, 3.353, -0.019, 0.024\right)\right)\Phi\left(\frac{\Phi^{-1}(0.98)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right)-\Phi\left(\frac{\Phi^{-1}(0.02)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right) \\
+\left(0.939+Beta^{-1}\left(\Phi\left(Z_{B\,i}\right), 1.803, 3.377, -0.02, 0.039\right)\right)\Phi\left(\frac{\Phi^{-1}(.002)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right)-\Phi\left(\frac{\Phi^{-1}(.0005)-\sqrt{0.2}\,e_{M\,i}}{\sqrt{1-.02}}\right)
\end{array}\right) \tag{37}$$

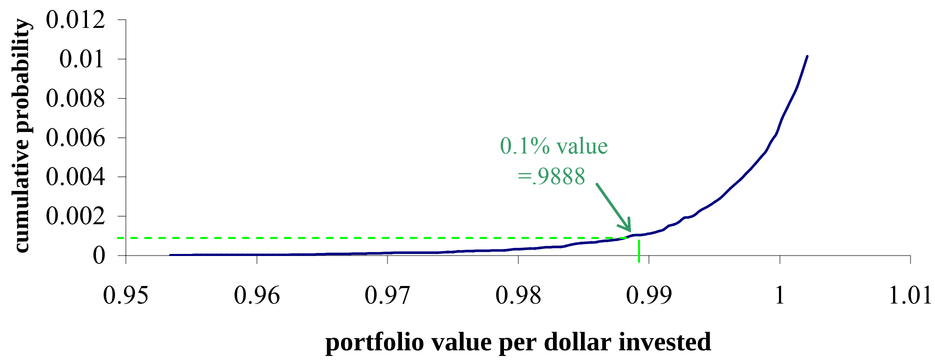
The only element in expression (37) that remains to be explained is the multiplication factor, $(1.056)^{\frac{3}{2}} \approx 1.085$, which is the payoff on these zero-coupon instruments should they perform at maturity. This term must be included because the fitted discount function distributions were specified for the receipt of a single future dollar.

Figure 5: Estimated Portfolio Value Probability Density



The end-of-period MTM portfolio value was estimated using a random sample of sixty-five thousand latent factor realizations, $\{(e_{M\,i}, e_{AAA\,i}, e_{A\,i}, e_{BBB\,i}, e_{B\,i}), i = 1,....,65000\}$, and for each one we repeat the calculations in expression (37).[11] This procedure simulates a random sample of size sixty-five thousand drawn from the asymptotic portfolio's end-of-period MTM value distribution. A histogram of the Monte Carlo generated sample is plotted in Figure 5.

Figure 6: Estimated Left Tail of the Portfolio Value Distribution



IV. CAPITAL ALLOCATION AND DIVERSIFICATION BENEFITS

1. Asymptotic Portfolio Loss Distribution Function

Figure 6 plots the left-hand tail of the portfolio's cumulative MTM future value distribution. Highlighted in green is the sample estimate (0.9888) of the probability density's 0.1 percent critical value. If potential portfolio losses are calculated relative to the portfolio's initial value (taken as $1 or 100 percent), the 99.9 percent quantile of the portfolio's loss distribution is 1-0.9888, or 1.12 percent. This value represents the sum of expected and unexpected loss (EL+UL).

Before considering capital allocation and measuring the potential capital savings that might be derived from the diversification benefits recognized in an integrated model,

[11] The Monte Carlo sampling used antithetic variates to improve estimation efficiency

it is important to consider alternative approaches for measuring credit and market risk that might serve as basis of comparison.

2. Alternative Models of an Asymptotic Portfolio's Credit Loss Distribution

The are at least three alternative models of an as asymptotic portfolio's credit loss distribution that should be considered when analyzing the issue of potential capital savings that might be gained using and integrated risk measure. The first model is the well-known Vasicek model. This model is used to measure risk in the Basel II AIRB approach for setting regulatory capital for credit risk. The expression for an asymptotic portfolio's loss rate conditional on a realization of the common factor (e_M) is $LR^V(e_M)$,

$$LR^V(e_M) = LGD \times \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho_V}\, e_M}{\sqrt{1-\rho_V}}\right) = 0.20 \times \Phi\left(\frac{\Phi^{-1}(.005) - \sqrt{0.2}\, e_M}{\sqrt{1-0.02}}\right). \tag{38}$$

Because low value realizations of e_M are associated with a high probability of default, the 99.9 percent quantile of the Vasicek portfolio loss distribution is given by, $LR^V\left(\Phi^{-1}(0.001)\right)$.

The Vasicek model does not include recognition of any interest earning generated by the non-defaulting credits in the portfolio. Kupiec (2004b) modifies the Vasiciek model so that it includes interest earnings on the non-defaulting credits, and these earning offset losses on defaulting credits. The Kupiec model is an accounting model and measures only accrued interest earnings on non-defaulting credits—it does not consider the capital gains and losses that might be generated by credit quality migration. If y_0 is the promised annual yield to maturity on the portfolio's individual credits, the future value of an asymptotic portfolio (per dollar invested) conditional on a realization of the common market factor $b_p^K(e_M)$ is,

$$
\begin{aligned}
b_p^K(e_M) &= (1+y_0)\left(1-\Phi\left(\frac{\Phi^{-1}(PD)-\sqrt{\rho_V}\ e_M}{\sqrt{1-\rho_V}}\right)\right)+(1-LGD)\Phi\left(\frac{\Phi^{-1}(PD)-\sqrt{\rho_V}\ e_M}{\sqrt{1-\rho_V}}\right) \\
&= \sqrt{1.056}\left(1-\Phi\left(\frac{\Phi^{-1}(.005)-\sqrt{0.2}\ e_M}{\sqrt{1-0.02}}\right)\right)+(0.8)\Phi\left(\frac{\Phi^{-1}(.005)-\sqrt{0.2}\ e_M}{\sqrt{1-0.02}}\right)
\end{aligned}
\quad . \qquad (39)
$$

Low value realizations of e_M are associated with a high probability of default, and so the 99.9 percent quantile of the loss rate distribution is given by $1-b_p^K\left(\Phi^{-1}(0.001)\right)$.

The final credit risk model considered is the asymptotic portfolio formulation of a credit migration model similar to CreditMetrics. This framework develops a mark-to-market model without market risk. Interest income is recognized and non-defaulting portfolio credits may migrate among credit quality classes and generate capital gains or losses as they are market-to-market using non-stochastic discount functions. Under this approach, for the portfolio parameterization in Section III, an asymptotic portfolio's future value, conditional on the common market factor e_M, is identical to expression (37) after substituting non-stochastic discount factors for the stochastic discount factor terms in the expression,

$$
\begin{aligned}
b_p^{CM}(e_M) &= 0.8\,\Phi\left(\frac{\Phi^{-1}(0.005)-\sqrt{.02}\ e_M}{\sqrt{1-0.2}}\right) \\
&+1.056^{\frac{3}{2}}\left(\begin{aligned}
&\frac{1}{1.0526}\left(1-\Phi\left(\frac{(\Phi^{-1}(0.995)-\sqrt{0.2}\ e_{Mi}}{\sqrt{1-.2}}\right)\right) \\
&\quad+\frac{1}{1.0537}\left(\Phi\left(\frac{\Phi^{-1}(0.995)-\sqrt{0.2}\ e_M}{\sqrt{1-0.2}}\right)-\Phi\left(\frac{\Phi^{-1}(0.98)-\sqrt{0.2}\ e_M}{\sqrt{1-0.2}}\right)\right) \\
&\quad+\frac{1}{1.056}\left(\Phi\left(\frac{\Phi^{-1}(0.98)-\sqrt{0.2}\ e_M}{\sqrt{1-.02}}\right)-\Phi\left(\frac{\Phi^{-1}(0.02)-\sqrt{0.2}\ e_M}{\sqrt{1-.02}}\right)\right) \\
&\quad+\frac{1}{1.065}\left(\Phi\left(\frac{\Phi^{-1}(.002)-\sqrt{0.2}\ e_M}{\sqrt{1-.02}}\right)-\Phi\left(\frac{\Phi^{-1}(.0005)-\sqrt{0.2}\ e_M}{\sqrt{1-.02}}\right)\right)
\end{aligned}\right).
\end{aligned}
\qquad (40)
$$

The 99.9 percent critical value of the migration model asymptotic portfolio loss distribution is $1-b_p^{CM}\left(\Phi^{-1}(0.001)\right)$.

3. Alternative Approaches for Measuring Portfolio Market Risk

Typically, banks measure the market risk of credit instruments using a value-at-risk (VaR) model to estimate the distribution of potential mark-to-market value changes that might be experienced by a credit instrument over the risk-measurement horizon. Most VaR models in use are short-horizon (1-day to 10 days) measures, and most assume that the instrument's credit quality is unchanged over the horizon. In addition to excluding credit-quality migration, these VaR models also often ignore valuation changes that are associated with accruing interest in part because, on many fixed income instruments, the market convention is for dealer quotes (and trade prices) to be reported net of the accrued interest because the buyer pays the seller accrued interest at settlement.

If a bank were to extend their market risk VaR models to a longer horizons, it is unclear how they might parameterize and calibrate these models. In some extended-horizon market risk applications, banks might simply scale up 1-day value at risk exposure estimates.[12]

In this analysis, we will consider two alternative market risk measures. Both are VaR measures estimated for 6-month horizons. Both measures will assume that the credit quality is unchanged. One measure will include accrued interest and be based on full revaluation of the position, while the other method will rely on the Taylor series expansion or so-called delta-gamma method. The delta-gamma method ignores

[12] The support for the statements in this and the prior paragraph is the author's bank supervision model review experience.

accumulating interest and so, on instruments that accrue interest, it understates potential market values over longer horizons when accrued interest value can be important.[13]

Both VaR measures considered will be calibrated using data 6-month changes in market discount factors—they will not be time-scaled short horizon measures. The biases that might be introduced by the scaling of 1-day market risk measures are not considered in this draft of the paper (but they may well be important). The two portfolio market risk measures are, $FR\text{-}VaR$ for-full revaluation VaR and, $DG\text{-}VaR$ which is based on a Taylor series approximation for changes in the asset's value. The 99.9 percent coverage rates for these VaR measures are,

$$FR-VaR(.999)=(1.056)^{\frac{3}{2}}\times\left(0.947+Beta^{-1}(.001,\,2.917,3.353,-0.019,\,0.024)\right)-1\,, \tag{41}$$

and,

$$DG-VaR(.999)=(1.056)^{\frac{3}{2}}\times Beta^{-1}(.001,\,2.917,3.353,-0.019,0.024). \tag{42}$$

In expression (42), $1.056^{\frac{3}{2}}$ is the weight on the linear term on the Taylor series expansion with respected to changes in the BBB-quality discount factor. The credit's MTM value is a linear function of the discount factor so there are no higher order terms in the expansion.

4. Risk Measurement and the Potential for Economic Capital Savings

The top panel in Table 7 reports the 99.9 percent loss coverage EL+UL estimates produced by the three alternative asymptotic credit risk models. The Kupiec model produces the smallest loss estimate (a gain) because it recognizes the full accrued interest income on non-defaulting credits without any MTM losses from credit quality migration. The 0.001 critical value of the portfolio future value distribution estimate also produces a

[13] This effect is often called the pull-to-par effect on discount instruments. This positive underlying time drift is ignored in the delta-gamma approach.

gain for the credit migration model, but the gain is slightly smaller than the Kupiec model estimate because accrued interest earnings are (on balance) offset by MTM losses on downgraded credits. Of these estimates, the only the Vasicek estimate overstates the integrated model 99.9 percent loss coverage estimate of 1.12 percent of the portfolio's initial value.

If the object of capital allocation is to ensure a 99.9 percent solvency standard[14], it has been shown elsewhere (Kupiec 2004a, 2004b, 2006) that capital must be set equal to the bank's estimate of EL + UL, plus in addition, the bank must allocate additional capital to ensure that it can service its own debt should the 99.9 percent loss be realized.[15] A simple EL+UL rule understates capital significantly on a high quality credit portfolio. On a high quality credit portfolio, estimates of EL+UL are small, in some cases 1 percent or less, and so a very high share of the credit portfolio is funded with debt. In such an instance, the interest component of the capital allocation is by far the largest component.

[14] The Basel II solvency standard is a 99.9 percent probability that the bank survives over a one-year the risk measurement horizon.

[15] Reserves would be included in this equity capital measure.

Table 7: Economic Capital Estimates Under Alternative Credit Risk Models

No Supplemental Market Risk Measure	Credit Risk Models: Vasicek	Kupiec	Credit Migration
Portfolio Distribution 0.001 Critical Value MTM Value Estimate	0.9818	1.0069	1.0057
Estimated Portfolio Credit Model 99.9 Percent Loss Rate (EL+UL)	0.0182	-0.0069	-0.0057
Economic Capital Calculation @ 5.26% (AAA) Funding Cost			
Amount Financed Under Piecemeal Risk Estimates	0.9818	1	1
Interest Capital at AAA rate (5.26%)	0.0255	0.0260	0.0260
Estimated Economic Capital Required	0.0437	0.0191	0.0202
Economic Capital Required Under Integrated Approach @ 5.26%	0.0369	0.0369	0.0369
Estimated Capital Surplus (Shortall)	0.0068	(0.0178)	(0.0166)
Economic Capital Calculation @ 5.37% (A) Funding Cost			
Amount Financed Under Piecemeal Risk Estimates	0.9932	1	1
Interest Capital at A rate (5.37%)	0.0263	0.0265	0.0265
Estimated Economic Capital Required	0.0445	0.0196	0.0208
Economic Capital Required Under Integrated Approach @ 5.37%	0.0374	0.0374	0.0374
Estimated Capital Surplus (Shortall)	0.0071	(0.0178)	(0.0166)
Economic Capital Calculation @ 5.60% (BBB) Funding Cost			
Amount Financed Under Piecemeal Risk Estimates	0.9929	1	1
Interest Capital at BBB rate (5.60%)	0.0274	0.0276	0.0276
Estimated Economic Capital Required	0.0456	0.0207	0.0219
Economic Capital Required Under Integrated Approach @ 5.60%	0.0385	0.0385	0.0385
Estimated Capital Surplus (Shortall)	0.0071	(0.0178)	(0.0166)

The lower panels of Table 7 show alternative capital allocation estimates for different funding cost estimates when the capital allocation is based on measured credit risk alone. If the portfolio is 100 percent financed with debt, then the via Modigliani-Miller arguments, the cost of bank debt would be the yield on the portfolio (5.60 percent).[16] The greater the share of equity used to fund the portfolio, the lower the interest rate the bank should receive on its funding debt. While formal pricing models can be derived to price the bank's debt (see Kupiec 2006), in this context we simply consider three alternative funding costs. In all cases, use of the Vasicek model credit loss estimates and a capital allocation equal to 'EL+UL+bank interest expense' leads to an

[16] If there were explicit or implicit guarantees on the bank's funding debt, the banks funding rate would be reduced toward the risk free rate.

overstatement of economic capital needs relative to estimates derived using the integrated risk measurement model.

Table 8: Comparison of Capital Estimates for Alternative Credit Risk Models when Market Risk is Measured Using the Delta-Gama Approach

	Credit Risk Models		
Delta-Gamma Market Risk VaR Estimate	Vasicek	Kupiec	Credit Migration
Portfolio Distribution 0.001 Critical Value MTM Value Estimate	0.9818	1.0069	1.0057
Estimated Portfolio Credit Loss Rate	0.0182	-0.0069	-0.0057
Estimated DG VaR Market Risk Loss Rate	0.0187	0.0187	0.0187
Estimate Market + Credit Loss @ 99.9% Coverage Rate (EL+UL)	0.0369	0.0118	0.0130
Integrated Model Estimated Total Loss Rate EL+UL	0.0112	0.0112	0.0112
Economic Capital Calculation @ 5.26% (AAA) Funding Rate			
Amount Financed Under Piecemeal Risk Estimates	0.9631	0.9882	0.9870
Interest Capital at AAA rate (5.26%)	0.0250	0.0257	0.0256
Estimated Economic Capital Required	0.0619	0.0375	0.0386
Economic Capital Required Under Integrated Approach (5.26%)	0.0369	0.0369	0.0369
Estimated Capital Surplus (Shortall)	0.0251	0.0006	0.0018
Economic Capital Calculation @ 5.60% (BBB) Funding Rate			
Amount Financed Under Piecemeal Risk Estimates	0.9631	0.9882	0.9870
Interest Capital at BBB rate (5.60%)	0.0266	0.0273	0.0273
Estimated Economic Capital Required	0.0635	0.0391	0.0403
Economic Capital Required Under Integrated Approach (5.60%)	0.0385	0.0385	0.0385
Estimated Capital Surplus (Shortall)	0.0250	0.0006	0.0017

Table 8 compares the integrated model capital estimates to estimate produced by a piecemeal approach, where a credit risk model's EL+UL estimate is added to a Delta-Gamma VaR market risk (99.9 percent coverage) estimate (0.187). Capital allocation estimates are presented for two alternative rates on the bank's funding debt, 5.26 percent (AAA rate), and 5.60 percent (BBB rate). For the three asymptotic credit risk models considered, economic capital is overstated when capital is estimated by the sum of estimated EL+UL and estimates of market risk VaR from a delta-gamma approach. The overstatement is largest for the Vasicek model as both the Vasicek model and the delta-gamma VaR model exclude recognition of all interest earnings on the portfolio.

Table 9: Comparison of Capital Estimates for Alternative Credit Risk Models when Market Risk is Measured Using the Full Revaluation Approach

	Credit Risk Models		
Future Value Market Risk VaR	Vasicek	Kupiec	Credit Migration
Portfolio Distribution 0.001 Critical Value MTM Value Estimate	0.9818	1.0069	1.0057
Estimated Portfolio Credit Loss Rate	0.0182	-0.0069	-0.0057
Estimated FV VaR Market Risk Loss Rate	-0.0089	-0.0089	-0.0089
Estimate Market + Credit Loss @ 99.9% Coverage Rate (EL+UL)	0.0093	-0.0158	-0.0147
Integrated Model Estimated Total Loss Rate EL+UL	0.0112	0.0112	0.0112
Economic Capital Calculation @ 5.26% Fundingg Rate			
Amount Financed Under Piecemeal Risk Estimates	0.9907	1.0000	1.0000
Interest Capital at AAA rate (5.26%)	0.0257	0.0260	0.0260
Estimated Economic Capital Required	0.0350	0.0101	0.0113
Economic Capital Required Under Integrated Approach (5.26%)	0.0369	0.0369	0.0369
Estimated Capital Surplus (Shortfall)	(0.0019)	(0.0267)	(0.0256)
Economic Capital Calculation @ 5.60% Funding Rate			
Amount Financed Under Piecemeal Risk Estimates	0.9907	1.0000	1.0000
Interest Capital at AAA rate (5.26%)	0.0257	0.0260	0.0260
Estimated Economic Capital Required	0.0350	0.0101	0.0113
Economic Capital Required Under Integrated Approach (5.26%)	0.0385	0.0385	0.0385
Estimated Capital Surplus (Shortfall)	(0.0035)	(0.0284)	(0.0272)

Table 9 reports the results of a comparison of an alternative method for capitalizing portfolio market and credit risk. In this comparison, market risks are measured using a full-revaluation VaR model with 99.9 percent VaR coverage rates. This loss rate estimate is added to the 99.9 percent coverage estimates of EL+UL produced by the three alternative credit portfolio models. When full revaluation is used to measure market risk, 6-months of accrued interest in this example are large enough to ensure that the 0.001 cumulative value of the market risk future value distribution is positive. Thus, the market risk VaR estimate provides an offset for credit risk estimates of EL+UL.

Table 9 provides economic capital estimates for two different funding rate assumptions (5.26 percent (AAA) and 5.60 percent (BBB)). In all cases, economic capital allocations constructed from piecemeal market and credit risk estimates understate the economic capital allocation that is estimated using the integrated risk measure of portfolio losses.

4. Potential Capital Savings the Under Basel II AIRB Model

The Basel II capital allocation framework sets capital equal to a 99.9 percent coverage estimate of EL+UL. The Basel II AIRB estimates EL+UL using a modified version of the Vasicek credit risk model. For positions held in the banking book, there is no requirement to hold market risk capital. In this instance, the results in Table 7 show that the Vasicek estimate of EL+UL at the 99.9 percent coverage rate is 1.82 percent. This estimate of economic capital is only about half of the capital required by the integrated risk measurement approach.[17]

[17] Integrated capital estimates range between 3.69 and 3.85 percent depending on the funding cost used in the analysis.

For positions held in a bank's trading book, the market risk amendment in the US implementation of Basel II propose an incremental default risk capital charge in addition to the 10-day capital requirement market risk. Complicating any analysis of trading book capital levels are differences in risk measurement horizons between Basel measures of market and credit risk. In contrast with the market risk measurement horizon, the default risk add-on may be measured over a different horizon that may be as long as a year.[18]

If market and credit risks were measured on the same horizon, the results in Tables 8 and 9 show that a piecemeal approach for estimating market and credit risk may lead to capital allocation estimates that overstate or understate actual integrated capital needs depending on the method used to measure market risk. The picture would undoubtedly become even more clouded if market and credit risk measures were based on different horizons.

V. CONCLUSIONS

This paper has developed a single common factor migration-style credit risk model that includes market risks on the non-defaulting credits in an asymptotic portfolio. A closed-form representation of an asymptotic portfolio's conditional future market value was derived and used to generate an estimate of the portfolio's unconditional future value distribution using Monte Carlo methods for a sample portfolio. This integrated exposure distribution was used to construct economic capital allocations, and these capital allocations were compared to piecemeal approaches for measuring risks and estimating

[18] It has been proposed that the default risk measurement horizon may be shortened based on the market liquidity of an instrument, but modalities of how this might work in practice have yet to be determined.

economic capital. The capital comparisons show that the issues of diversification and capital benefits are complex, and it is impossible to make general statements about potential for capital savings. Capital allocations derived from an integrated market and credit risk measure can be larger or smaller than capital allocations that are estimated from piecemeal risk measures. The results show that no further diversification benefit should be considered in the Basel II AIRB approach as no market risk capital is required for banking book positions and Basel II AIRB capital requirements fall far short of the capital required by an integrated risk measure. For trading book positions subject to 10-day market risk requirements and incremental default capital requirements, the issues are unclear because of differences in measurement horizons and as well as allowed variation in bank market and credit risk measures and capital allocation framework.

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