

A Generalized Single Common Factor Model of Portfolio Credit Risk

by

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March 2007

ABSTRACT

The Vasicek single factor model of portfolio credit loss is generalized to include correlated stochastic exposures and loss rates. The new model can accommodate any distribution and correlation assumptions for the loss and exposure rates of individual credits and will produce a closed-form approximation for an asymptotic portfolio's conditional loss rate. Revolving exposures are modeled as draws against committed lines of credit. Draw rates and loss rates on defaulted credits are random variables with known probability distributions. Dependence among defaults, individual exposures, and loss rates are modeled using a single common Gaussian factor. A closed-form expression for an asymptotic portfolio's inverse cumulative conditional loss rate is used to calculate a portfolio's unconditional loss rate distribution, estimate economic capital allocations, and analyze portfolio loss rate characteristics. Positive correlation in individual credit exposures and loss rates increases systematic risk. As a consequence, portfolio loss rate distributions exhibit wider ranges and greater skewness.

* Division of Insurance and Research, Federal Deposit Insurance Corporation. The views expressed are those of the author and do not reflect the views of the FDIC. I am grateful to Rosalind Bennett, Steve Burton, Sanjiv Das, and Robert Jarrow for comments on an earlier draft of this paper. Email: pkupiec@fdic.gov

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1. INTRODUCTION

The Gaussian asymptotic single factor model of portfolio credit losses (ASFM), developed by Vasicek (1987), Finger (1999), Schönbucher (2000), Gordy (2003) and others, provides an approximation for the loss rate distribution for a credit portfolio in which the dependence among individual defaults is driven by a single common latent factor. The ASFM assumes the unconditional probability of default on an individual credit (PD) is fixed and known. In addition, all obligors' exposures at default (EAD), and loss rates in default (LGD) are assumed to be known non-stochastic quantities. In a large portfolio of credits, idiosyncratic risk is fully diversified and the only source of portfolio loss uncertainty is the uncertainty in the portfolio default rate that is driven by the common latent Gaussian factor.

The ASFM has been widely applied in the financial industry. It has been used to estimate economic capital allocations [e.g., Finger (1999), Schönbucher (2000), Gordy (2003), and others]. It is the model that underlies the Basel II Advanced Internal Ratings-based approach (AIRB) for setting banks' minimum regulatory capital requirements. In addition to capital allocation applications, the ASFM has been adapted to estimate potential loss distributions for tranches of portfolio credit products and to price basket credit risk transfer products.[e.g., Li (2000), Andersen, Sidenius, and Basu (2003), Gibson (2004), Gordy and Jones (2002) and others]. Notwithstanding widespread use of the ASFM, empirical findings suggest that the model omits important systematic factors that in part determine the characteristics of a portfolio's true underlying credit loss distribution.

In analysis of historical data, many studies find significant time variability among the realized *LGDs* for a given credit facility or ratings class and negative correlation between observed default frequencies and contemporaneous recovery rates on defaulted credits. Default losses increase in periods when default rates are elevated. Studies by Frye (2000b), Hu and Perraudin (2002), Schuermann (2004), Araten, Jacobs, and Varshney (2004), Altman, Brady, Resti and Sironi (2004), Hamilton, Varma, Ou and Cantor (2004), Carey and Gordy (2004), Emery, Cantor and Arnet (2004) and others show pronounced decreases in the recovery rates during recessions and other periods with elevated default rates. These results suggest the existence of a systematic relationship between default frequencies and default recovery rates that is not captured in the Vasicek ASRM framework.¹

In addition to issues related to stochastic *LGD*, the ASFM often is employed to estimate capital needs for portfolios of revolving credits even though the model is based on the assumption that individual credit *EADs* are fixed. The available evidence, including studies by Allen and Saunders (2003), Asarnow and Marker (1995), Araten and Jacobs (2001), and Jiménez, Lopez, and Saurina (2006) suggests that obligors draw down committed lines of credit as their credit quality deteriorates. Analysis of creditors' draw rate behavior shows that *EADs* on revolving exposures are positively correlated with default rates suggesting that there is at least one common factor that simultaneously determines portfolio *EAD* and default rate realizations.

¹ Cary and Gordy (2004) question the strength of systematic *LGD*-default rate correlations. Their estimates of correlations between default rates and total firm recovery rates—total dollars recovered to total dollar claims outstanding- show less pronounced correlations compared studies that estimate these correlations for specific liability classes.

The assumption of nonstochastic *LGD* and *EAD* precludes the ASFM model from incorporating important sources of systematic credit risk that are present in historical loss rate data. A number of existing models, including models by Frye (2000), Pykhtin (2003), Tasche (2004), and Andersen and Sidenius (2005), have extended the Vasicek ASFM framework to include stochastic *LGD* rates. These extensions all require complex numerical techniques or restrictive assumptions for the *LGD* distribution to produce tractable expressions for an asymptotic portfolios' loss rate distribution. No existing study (of which I am aware) extends the ASFM framework to include stochastic *EAD* and *LGD* and produce a closed-form expression for a portfolio loss rate distribution.

In the remainder of this paper, the Gaussian ASFM is extended to incorporate obligors with *EADs* and *LGDs* that are correlated random variables. In this extension, default is a random event driven by a compound latent factor as in the standard ASFM. Two additional compound latent factors are introduced to drive correlations among individual credits' *EADs* and *LGDs*. A closed-form expression for the inverse of the portfolio's credit loss distribution is constructed using a step function to approximate the underlying *LGD* and *EAD* distributions. The characteristics of the *LGD* and *EAD* distributions that can be modeled using this new approach are unrestricted. The approximation can be taken to any desired level of precision by adjusting the step function increment size. Portfolio loss rate distributions are analyzed under alternative *EAD* and *LGD* distribution and correlation assumptions that are consistent with stylized representations of alternative corporate and retail portfolios. Basel II-style capital allocation rules are constructed using the step-function approximation and some selected examples are used to illustrate the approach.

2. A GRID APPROXIMATION FOR A CUMULATIVE DISTRIBUTION FUNCTION

In this section we introduce an approximation method that can be used to represent any cumulative distribution function. A random variable's range of support is divided into a mesh of equal increments and the mesh representation is used in conjunction with compound latent Gaussian factors to construct a step-function approximation for individual credit *EAD* and *LGD* distributions. The step function approximation facilitates the derivation of a closed-form expression for the conditional loss rate and exposure distributions of an asymptotic portfolio of credits.

Let $\Xi(\tilde{a})$ represent the cumulative density function for $\tilde{a} \in [0,1]$. Because $\Xi(\tilde{a})$ is a cumulative density function, it is monotonic and non-decreasing in \tilde{a} . Over the range of support for \tilde{a} , define a mesh of n equal increments of size $\frac{1}{n}$, and use these to define a set

of overlapping events that span the support. Define $E(\tilde{a}, j, n)$, $j = 0, 1, 2, \dots, n$, such that: $E(\tilde{a}, 0, n)$ is the event $\tilde{a} = 0$; $E(\tilde{a}, 1, n)$ is the event $\tilde{a} \in \left[0, \frac{1}{n}\right]$; $E(\tilde{a}, j, n)$ is the event $\tilde{a} \in \left[0, \frac{j}{n}\right]$; $E(\tilde{a}, n-1, n)$ is the event $\tilde{a} \in \left[0, \frac{n-1}{n}\right]$; and, $E(\tilde{a}, n, n)$ is the event $\tilde{a} \in [0, 1]$. The probability that event $E(\tilde{a}, j, n)$ occurs is $\Xi\left(\frac{j}{n}\right)$.

Let $1_{E(\tilde{a}, j, n)}$ be the indicator function for the event $E(\tilde{a}, j, n)$,

$$1_{E(\tilde{a}, j, n)} = \begin{cases} 1 & \text{if } a_i \in E(\tilde{a}, j, n) \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of the indicator function is the probability of occurrence of the indicated event. It follows that $\Xi\left(\frac{j}{n}\right) = E\left(1_{E(\tilde{a}, j, n)}\right)$ for $j = 0, 1, 2, 3, \dots, n$, where the correspondence is exact for integer values of j , but is undefined for intermediate values.

To construct an approximation for $\Xi(\tilde{a})$ that spans the support of \tilde{a} for fixed n , define $x_i = \frac{a_i}{n}$ for any $a_i \in [0, 1]$. Using x_i , we approximate the cumulative density function for \tilde{a} as follows,

$$\Xi\left(a_i = \frac{x_i}{n}\right) \approx E\left(1_{E(\tilde{a},]x_i[, n)}\right), \quad (2)$$

where $]x_i[$ is the so-called ‘‘ceiling function’’ that returns x_i if x_i is an integer and returns x_i rounded up to the next largest integer value if x_i is not an integer. For non-integer x_i this approximation overstates the true cumulative probability, $\Xi\left(\frac{x_i}{n}\right)$, but the magnitude of the approximation error is decreasing in n and can be reduced to any desired degree of precision by choosing n sufficiently large.²

To gain additional understanding about the precision of the approximation, consider the compound event, $E(\tilde{a}, j-1, n) \cap E(\tilde{a}, j, n)$, as $n \rightarrow \infty$. In the limit as

² If $a_i \in [0, 1]$ is rational then $a_i = \frac{j}{n}$ for some integers, j and n . If $a_i \in [0, 1]$ is irrational, then Lagrange has shown, $\left|a_i - \frac{j}{n}\right| < \frac{1}{\sqrt{5} n^2}$. Thus $a_i \in [0, 1]$ can be approximated to any desired degree of accuracy by $a_i \approx \frac{j}{n}$ for some integers, j and n , where the precision of the approximation is increasing in n . See Conway and Guy (1996), pp. 187-189.

$n \rightarrow \infty$, $j \rightarrow \infty$, and $\frac{1}{n} \rightarrow 0$, but the ratio $\frac{j}{n}$ remains unchanged. In the continuous case, the

event $E(\tilde{a}, j-1, n) \cap E(\tilde{a}, j, n)$ converges to the point $\frac{j}{n} \in [0,1]$ as $n \rightarrow \infty$. Consequently,

$$\lim_{n \rightarrow \infty} E\left(\mathbb{1}_{E(\tilde{a}, j, n)}\right) - E\left(\mathbb{1}_{E(\tilde{a}, j-1, n)}\right) = \Xi'\left(\frac{j}{n}\right), \text{ the probability density of } \tilde{a} \text{ at the point } \frac{j}{n} \in [0,1].$$

In the discrete distribution case, as $n \rightarrow \infty$ each point in the support of \tilde{a} can be associated with a unique set of compound events, $E(\tilde{a}, i, n) \cap E(\tilde{a}, j, n)$, for some integers i and j , if n is set sufficiently large. Consequently, a discrete distribution $\Xi(\tilde{a})$ can be approximated exactly using this event-space representation.

It is useful to define the mathematical expectation of \tilde{a} in terms of the indicator functions defined in expression (1). Proposition 1 in the appendix shows,

$$E(\tilde{a}) = 1 - \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \sum_{j=0}^{n-1} E\left(\mathbb{1}_{E(\tilde{a}, j, n)}\right). \quad (3)$$

3. THE GAUSSIAN ASFM MODEL

The Vasicek single common factor model of portfolio credit risk assumes that uncertainty on credit i is driven by a latent unobserved factor, \tilde{V}_i with the following properties,

$$\begin{aligned} \tilde{V}_i &= \sqrt{\rho_V} \tilde{e}_M + \sqrt{1 - \rho_V} \tilde{e}_{id} \\ \tilde{e}_M &\sim \phi(e_M) \\ e_{id} &\sim \phi(e_{id}), \\ E(\tilde{e}_{id} \tilde{e}_{jd}) &= E(\tilde{e}_M \tilde{e}_{jd}) = 0, \quad \forall i, j. \end{aligned} \quad (4)$$

$\phi(\cdot)$ represents the standard normal density function. \tilde{V}_i is distributed standard normal, $E(\tilde{V}_i) = 0$, and $E(\tilde{V}_i^2) = 1$. \tilde{e}_M is a factor common to all credits' latent factors, \tilde{V}_i . The correlation between individual credits' latent factors is ρ_V . \tilde{V}_i is often interpreted as a proxy for the market value of the firm that issued credit i .

Credit i is assumed to default when its latent factor takes on a value less than a credit-specific threshold, $\tilde{V}_i < D_i$. The unconditional probability that credit i defaults is, $PD = \Phi(D_i)$, where $\Phi(\cdot)$ represents the cumulative standard normal density function. Time is not an independent factor in the ASFM but is implicitly recognized through the calibration of input values for PD.

4. A SINGLE FACTOR MODEL OF LOSSES ON A PORTFOLIO OF REVOLVING CREDITS WITH CORRELATED EXPOSURES AND LOSS RATES

A Model of Stochastic EAD

Assume that a generic revolving credit account, i , has a maximum line of credit, M_i , upon which it may draw. For any individual credit, the exposure at the end of the period, the facility utilization rate $\tilde{X}_i \in [0,1]$, is a random variable that determines the end-of-period account exposure, $\tilde{X}_i M_i$. Basel II conventions require that *EAD* be at least as large as initial exposure and so we model *EAD* by modeling an account's initial exposure and its draw rate $\tilde{\delta}_i$ on its remaining line of credit instead of directly modeling the account's utilization rate.

Assume an individual account begins the period with a drawn exposure of $Dr_{i0} M_i$, where Dr_{i0} is the initial share of the account line of credit that is utilized. The line of credit

that can be drawn by the creditor over the subsequent period is, $(1 - Dr_{i0})M_i$. Let $\tilde{\delta}_i \in [0,1]$ represent the share of the remaining line of credit that is borrowed over the period and let $\Omega(\tilde{\delta}_i)$ represent the cumulative density function for $\tilde{\delta}_i$. This representation accommodates the Basel II convention that requires that the exposure at the end of the one-year horizon is at least as large as the initial level of extended credit, $Dr_{i0} M_i$. This assumption can be relaxed and the model can recognize creditors' ability to reduce or eliminate their outstanding balances by setting $Dr_{i0} = 0$ and modeling an account's end-of-period utilization rate $\tilde{X}_i \in [0,1]$ directly instead of modeling its draw rate $\tilde{\delta}_i$. Under the draw rate specification, the account's end-of-period exposure is,

$$M_i \tilde{X}_i = M_i (Dr_{i0} + (1 - Dr_{i0}) \tilde{\delta}_i), \quad \tilde{\delta}_i \sim \Omega(\delta_i), \quad \delta_i \in [0,1]. \quad (5)$$

To construct the step function approximation, divide the $[0,1]$ range of support for the draw rate into $n+1$ overlapping regions and define $n+1$ corresponding events: $E(\tilde{\delta}, 0, n)$ is the event $\tilde{\delta}_i = 0$; $E(\tilde{\delta}, 1, n)$ is the event $\tilde{\delta}_i \in [0, \frac{1}{n}]$; $E(\tilde{\delta}, j, n)$ is the event $\tilde{\delta}_i \in [0, \frac{j}{n}]$; and, $E(\tilde{\delta}, n, n)$ is the event $\tilde{\delta}_i \in [0, 1]$.

Systematic dependence among individual account's draw rate behaviors can be incorporated by assuming that account draw rates are driven by a latent Gaussian factor, \tilde{Z}_i with the following properties,

$$\begin{aligned} \tilde{Z}_i &= \sqrt{\rho_Z} \tilde{e}_M + \sqrt{1 - \rho_Z} \tilde{e}_{iZ} \\ \tilde{e}_M &\sim \phi(e_M) \\ \tilde{e}_{iZ} &\sim \phi(e_{iZ}), \\ E(\tilde{e}_{iZ} \tilde{e}_{jZ}) &= E(\tilde{e}_M \tilde{e}_{jZ}) = E(\tilde{e}_{iZ} \tilde{e}_{jd}) = 0 \quad \forall i, j. \end{aligned} \quad (6)$$

We adopt the normalization convention that higher account draw rates are associated with smaller realizations of the latent variable, \tilde{Z}_i . The correlation between the latent variables that drive each account's draw rate is ρ_Z , and the correlation between the latent factors that drive account exposures and defaults is $\sqrt{\rho_Z \rho_V}$.

The probability distribution for an account's draw rate is approximated by a uniform-size step function defined on Z_i using the overlapping set of events defined above,

$$\tilde{\delta}_i = \begin{cases} 0 & \text{for } \tilde{Z}_i \geq A_{i1} \\ \left(\frac{1}{n}\right) & \text{for } A_{i2} \leq \tilde{Z}_i < A_{i1} \\ \left(\frac{2}{n}\right) & \text{for } A_{i3} \leq \tilde{Z}_i < A_{i2} \\ \vdots & \vdots \\ \left(\frac{j}{n}\right) & \text{for } A_{ij+1} \leq \tilde{Z}_i < A_{ij} \\ \vdots & \vdots \\ \left(\frac{n-1}{n}\right) & \text{or } A_{in} \leq \tilde{Z}_i < A_{in-1} \\ 1 & \text{for } \tilde{Z}_i < A_{in} \end{cases} \quad (7)$$

where $A_{in} < A_{in-1} < \dots < A_{i2} < A_{i1}$. Expression (7) models the draw rate as a monotonically decreasing function of \tilde{Z}_i with $n+1$ distinct draw rates with uniform increments of size $\frac{1}{n}$ beginning at $\delta_i = 1$. This partition is used to define the latent variable thresholds $\{A_{i1}, A_{i2}, \dots, A_{in}\}$ by equating the Gaussian probabilities for the latent variable thresholds to the probability that the corresponding events occur under the true draw rate distribution $\Omega(\tilde{\delta}_i)$. For example, the equality, $1 - \Phi(A_{i1}) = \Omega(0)$ defines $A_{i1} = \Phi^{-1}(1 - \Omega(0))$; Similarly,

$1 - \Phi(A_{i2}) = \Omega\left(\frac{1}{n}\right)$ defines $A_{i2} = \Phi^{-1}\left(1 - \Omega\left(\frac{1}{n}\right)\right)$, and so. The step function approximation for the unconditional draw rate distribution is given in Table 1.

Table 1: Step Function Approximation for an Individual Credit's Draw Rate Distribution

| Draw Rate | Event | Cumulative Probability of Draw Rate | Threshold Value for Latent Variable \tilde{Z}_i |
|-----------------|-----------------------------|-------------------------------------|---|
| 0 | $E(\tilde{d}, 0, n)$ | $\Omega(0)$ | $A_{i1} = \Phi^{-1}(1 - \Omega(0))$ |
| $\frac{1}{n}$ | $E(\tilde{\delta}, 1, n)$ | $\Omega\left(\frac{1}{n}\right)$ | $A_{i2} = \Phi^{-1}\left(1 - \Omega\left(\frac{1}{n}\right)\right)$ |
| $\frac{2}{n}$ | $E(\tilde{d}, 2, n)$ | $\Omega\left(\frac{2}{n}\right)$ | $A_{i3} = \Phi^{-1}\left(1 - \Omega\left(\frac{2}{n}\right)\right)$ |
| \vdots | \vdots | \vdots | \vdots |
| $\frac{n-1}{n}$ | $E(\tilde{\delta}, n-1, n)$ | $\Omega\left(\frac{n-1}{n}\right)$ | $A_{in} = \Phi^{-1}\left(1 - \Omega\left(\frac{n-1}{n}\right)\right)$ |
| 1 | $E(\tilde{\delta}, n, n)$ | 1 | |

A Model of Stochastic LGD

Let $\tilde{\lambda}_i \in [0,1]$ represent the loss rate that that will be experienced on credit i 's outstanding balance should the borrower default. Let $\Theta(\tilde{\lambda}_i)$ represent the cumulative density function for $\tilde{\lambda}_i$. Divide the interval $[0,1]$ into $n+1$ overlapping regions and define a

corresponding set of events: $E(\tilde{\lambda}, 0, n)$ is the event $\tilde{\lambda}_i = 0$; $E(\tilde{\lambda}, 1, n)$ is the event

$\tilde{\lambda}_i \in \left[0, \frac{1}{n}\right]$; $E(\tilde{\lambda}, j, n)$ is the event $\tilde{\lambda}_i \in \left[0, \frac{j}{n}\right]$; and $E(\tilde{\lambda}, n, n)$ is the event $\tilde{\lambda}_i \in [0,1]$.

Systematic dependence in individual credit's loss rates is introduced by assuming that

$\tilde{\lambda}_i$ is driven by a latent Gaussian factor, \tilde{Y}_i with the following properties,

$$\begin{aligned}
\tilde{Y}_i &= \sqrt{\rho_Y} \tilde{e}_M + \sqrt{1 - \rho_Y} \tilde{e}_{iY} \\
\tilde{e}_M &\sim \phi(e_M) \\
e_{iY} &\sim \phi(e_{iY}), \\
E(\tilde{e}_{iY} \tilde{e}_{jY}) &= E(\tilde{e}_M \tilde{e}_{jY}) = E(\tilde{e}_{iY} \tilde{e}_{jZ}) = E(\tilde{e}_{iY} \tilde{e}_{id}) = 0 \quad \forall i, j.
\end{aligned} \tag{8}$$

The common Gaussian factor, \tilde{e}_M , is identical in all latent factors, \tilde{V}_i , \tilde{Z}_i , and \tilde{Y}_i and so the three latent factor are positively correlated for a single credit and across portfolio credits provided $\sqrt{\rho_V} > 0$, $\sqrt{\rho_Z} > 0$, and $\sqrt{\rho_Y} > 0$. The correlation between the latent factors that determine default and loss given default is $\sqrt{\rho_V \rho_Y} > 0$, and the correlation between the Gaussian drivers of default and exposure at default is $\sqrt{\rho_V \rho_Z} > 0$.

$\Theta(\tilde{\lambda}_i)$ is approximated using the latent factor \tilde{Y}_i and the step function methodology outlined earlier. The model is normalized so that higher realized loss rates are associated with smaller realized values of \tilde{Y}_i . Using a uniform-size grid over the interval $[0,1]$ to define $n+1$ events, $\{E(\tilde{\lambda}, 0, n), E(\tilde{\lambda}, 1, n), \dots, E(\tilde{\lambda}, n, n)\}$, we approximate $\Theta(\tilde{\lambda}_i)$ as,

$$\tilde{\lambda}_i = \begin{cases} 0 & \text{for } \tilde{Y}_i \geq B_{i1} \\ \left(\frac{1}{n}\right) & \text{for } B_{i2} \leq \tilde{Y}_i < B_{i1} \\ \left(\frac{2}{n}\right) & \text{for } B_{i3} \leq \tilde{Y}_i < B_{i2} \\ \vdots & \vdots \\ \vdots & \vdots \\ \left(\frac{n-1}{n}\right) & \text{for } B_{in} \leq \tilde{Y}_i < B_{in-1} \\ 1 & \text{for } \tilde{Y}_i < B_{in} \end{cases} \tag{9}$$

for $B_{in} < B_{in-1} < \dots < B_{i2} < B_{i1}$. The latent variable thresholds are defined by equating the Gaussian threshold probabilities with the cumulative probability of the corresponding events under the true draw rate distribution. The threshold calibrations are illustrated in Table 2.

Table 2: Step Function Approximation for an Individual Credit's Loss Rate Distribution

| Loss Rate | Event | Cumulative Probability of Loss Rate | Threshold Value for Latent Variable \tilde{Y}_i |
|-----------------|------------------------------|-------------------------------------|---|
| 0 | $E(\tilde{\lambda}, 0, n)$ | $\Theta(0)$ | $B_{i1} = \Phi^{-1}(1 - \Theta(0))$ |
| $\frac{1}{n}$ | $E(\tilde{\lambda}, 1, n)$ | $\Theta\left(\frac{1}{n}\right)$ | $B_{i2} = \Phi^{-1}\left(1 - \Theta\left(\frac{1}{n}\right)\right)$ |
| $\frac{2}{n}$ | $E(\tilde{\lambda}, 2, n)$ | $\Theta\left(\frac{2}{n}\right)$ | $B_{i3} = \Phi^{-1}\left(1 - \Theta\left(\frac{2}{n}\right)\right)$ |
| \vdots | \vdots | \vdots | \vdots |
| $\frac{n-1}{n}$ | $E(\tilde{\lambda}, n-1, n)$ | $\Theta\left(\frac{n-1}{n}\right)$ | $B_{in-1} = \Phi^{-1}\left(1 - \Theta\left(\frac{n-2}{n}\right)\right)$ |
| 1 | $E(\tilde{\lambda}, n, n)$ | 1 | $B_{in} = \Phi^{-1}\left(1 - \Theta\left(\frac{n-1}{n}\right)\right)$ |

The Loss Rate for an Individual Credit Facility

The loss rate distribution for an individual account can be modeled using $2n + 1$ indicator functions defined over the latent variables \tilde{V}_i , \tilde{Z}_i , and \tilde{Y}_i . One indicator function indicates default status; n indicator functions are used to approximate the cumulative *EAD*

distribution, $\Omega(\tilde{\delta})$; and n indicator functions are used to approximate the cumulative *LGD* distribution, $\Theta(\tilde{\lambda})$,

$$\begin{aligned} 1_{D_i}(\tilde{V}_i) &= \begin{cases} 1 & \text{if } \tilde{V}_i < D_i \\ 0 & \text{otherwise} \end{cases}, \\ 1_{A_{ij}}(\tilde{Z}_i) &= \begin{cases} 1 & \text{if } \tilde{Z}_i < A_{ij} \\ 0 & \text{otherwise} \end{cases}, \quad \text{for } j = 0, 1, 2, 3, \dots, n \\ 1_{B_{ik}}(\tilde{Y}_i) &= \begin{cases} 1 & \text{if } \tilde{Y}_i < B_{ik} \\ 0 & \text{otherwise} \end{cases}, \quad \text{for } k = 0, 1, 2, 3, \dots, n. \end{aligned} \quad (10)$$

Each indicator function defines a binomial random variable with a mean equal to the cumulative standard normal distribution evaluated at its associated threshold value. For example, $1_{D_i}(\tilde{V}_i)$ has a binomial distribution with an expected value of $\Phi(D_i)$; similarly, $1_{A_{ij}}(\tilde{Z}_i)$ is distributed binomial with an expected value of $\Phi(A_{ij})$, and so on for the remaining indicator functions.

Let $\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n)$ represent the approximate loss rate for account i measured relative to the account's maximum credit limit, M_i . $\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n)$ is defined as,

$$\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n) = 1_{D_i}(\tilde{V}_i) \left(\left(Dr_{i0} + (1 - Dr_{i0}) \left(\frac{1}{n} \sum_{k=1}^n 1_{A_{ik}}(\tilde{Z}_i) \right) \right) \left(\frac{1}{n} \sum_{j=1}^n 1_{B_{ij}}(\tilde{Y}_i) \right) \right). \quad (11)$$

The notation indicates that the approximation depends on n , the number of step function increments used to approximate the account's *LGD* and *EAD* cumulative distribution functions.

The Conditional Loss Rate for an Individual Credit

Let $1_{D_i}(\tilde{V}_i | e_M)$ represent the value of the default indicator function conditional on a realized value for e_M , the common latent factor. Similarly, let $1_{A_{ij}}(\tilde{Z}_i | e_M)$ and $1_{B_{ij}}(\tilde{Y}_i | e_M)$ represent the values of the remaining indicator functions ($j=1, 2, 3, \dots, n$) conditional on a realized value for e_M . The conditional indicator functions define independent binomial random variables with properties,

$$E\left(1_{D_i}(\tilde{V}_i | e_M)\right) = \Phi\left(\frac{D_i - \sqrt{\rho_V} e_M}{\sqrt{1 - \rho_V}}\right),$$

$$E\left(1_{D_i}(\tilde{V}_i | e_M) - E\left(1_{D_i}(\tilde{V}_i | e_M)\right)\right)^2 = \Phi\left(\frac{D_i - \sqrt{\rho_V} e_M}{\sqrt{1 - \rho_V}}\right) \left(1 - \Phi\left(\frac{D_i - \sqrt{\rho_V} e_M}{\sqrt{1 - \rho_V}}\right)\right) \quad (12)$$

$$E\left(1_{A_{ij}}(\tilde{Z}_i | e_M)\right) = \Phi\left(\frac{A_{ij} - \sqrt{\rho_Z} e_M}{\sqrt{1 - \rho_Z}}\right),$$

$$E\left(1_{A_{ij}}(\tilde{Z}_i | e_M) - E\left(1_{A_{ij}}(\tilde{Z}_i | e_M)\right)\right)^2 = \Phi\left(\frac{A_{ij} - \sqrt{\rho_Z} e_M}{\sqrt{1 - \rho_Z}}\right) \left(1 - \Phi\left(\frac{A_{ij} - \sqrt{\rho_Z} e_M}{\sqrt{1 - \rho_Z}}\right)\right), \quad (13)$$

$$E\left(1_{B_{ij}}(\tilde{Y}_i | e_M)\right) = \Phi\left(\frac{B_{ij} - \sqrt{\rho_Y} e_M}{\sqrt{1 - \rho_Y}}\right),$$

$$E\left(1_{B_{ij}}(\tilde{Y}_i | e_M) - E\left(1_{B_{ij}}(\tilde{Y}_i | e_M)\right)\right)^2 = \Phi\left(\frac{B_{ij} - \sqrt{\rho_Y} e_M}{\sqrt{1 - \rho_Y}}\right) \left(1 - \Phi\left(\frac{B_{ij} - \sqrt{\rho_Y} e_M}{\sqrt{1 - \rho_Y}}\right)\right), \quad (14)$$

for $j=1, 2, 3, \dots, n$. An individual account's conditional loss rate is approximated as,

$$\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M) = 1_{D_i}(\tilde{V}_i | e_M) \left(\left(Dr_{i0} + (1 - Dr_{i0}) \left(\frac{1}{n} \sum_{k=1}^n 1_{A_{ik}}(\tilde{Z}_i | e_M) \right) \right) \left(\frac{1}{n} \sum_{j=1}^n 1_{B_{ij}}(\tilde{Y}_i | e_M) \right) \right). \quad (15)$$

Because the conditional distributions of the latent factors are independent, the law of iterated expectations implies,

$$\begin{aligned} & E\left(1_{D_i}(\tilde{V}_i | e_M) \cdot 1_{A_{ik}}(\tilde{Z}_i | e_M) \cdot 1_{B_{ij}}(\tilde{Y}_i | e_M)\right) \\ &= E\left(1_{D_i}(\tilde{V}_i | e_M)\right) \cdot E\left(1_{A_{ik}}(\tilde{Z}_i | e_M)\right) \cdot E\left(1_{B_{ij}}(\tilde{Y}_i | e_M)\right) \quad \forall i, k, j. \end{aligned} \quad (16)$$

The Loss Rate on an Asymptotic Portfolio of Revolving Credits

Consider a portfolio composed of N accounts that have identical credit limits, $M_i = M$, identical initial drawn balances, $DR_{i0} M_i = DR_0 M$, identical latent factor correlations, $\{\rho_V, \rho_X, \rho_Y\}$, and identical default thresholds, $D_i = D$. Assume that all credits' end-of-period draw rates, $\tilde{\delta}_i$, and loss rates given default, $\tilde{\lambda}_i$, are taken from respective unconditional distributions that are identical across credits (the distributions for $\tilde{\delta}_i$ and $\tilde{\lambda}_i$ may differ). Under these assumptions, the $2n+1$ threshold values in expression (15) are identical across individual credits, and indicator function subscript i no longer is necessary, $1_{D_i}(\tilde{V}_i) = 1_D(\tilde{V}_i)$, $1_{A_{ij}}(\tilde{Z}_i) = 1_{A_j}(\tilde{Z}_i)$, and $1_{B_{ij}}(\tilde{Y}_i) = 1_{B_j}(\tilde{Y}_i)$ for $j = 1, 2, 3, \dots, n$. The loss rate for an individual credit will depend on the identity of the credit, but the dependence arises only through the idiosyncratic risk factors in the latent variables \tilde{V}_i, \tilde{Z}_i , and \tilde{Y}_i , and so the subscript can be eliminated, $\Lambda_i^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n) = \Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n)$.

Let $\tilde{\tilde{V}}$ represent the vector $(\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_N)$ and define $\tilde{\tilde{Y}}$ and $\tilde{\tilde{Z}}$ analogously. Let $\Lambda_p^A(\tilde{\tilde{V}}, \tilde{\tilde{Z}}, \tilde{\tilde{Y}}, n | e_M)$ represent the approximate loss rate on the portfolio of N accounts conditional on a realization of e_M , and n increments in the step-function approximation,

$$\Lambda_p^A(\tilde{V}, \tilde{Z}, \tilde{Y}, n | e_M) = \left(\frac{\sum_{i=1}^N (M \cdot \Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M))}{N \cdot M} \right) = \left(\frac{\sum_{i=1}^N (\Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M))}{N} \right). \quad (17)$$

Recall that $\Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M)$ is independent of $\Lambda^A(\tilde{V}_j, \tilde{Z}_j, \tilde{Y}_j, n | e_M)$ for all $i \neq j$ and the conditional loss rates for individual credits are identically distributed. Thus, the Strong Law of Large Numbers requires, for any admissible value of e_M ,

$$\lim_{N \rightarrow \infty} \Lambda_p^A(\tilde{V}, \tilde{Z}, \tilde{Y}, n | e_M) = \lim_{N \rightarrow \infty} \left(\frac{\sum_{i=1}^N \Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M)}{N} \right) \xrightarrow{a.s.} E(\Lambda^A(\tilde{V}_i, \tilde{Z}_i, \tilde{Y}_i, n | e_M)). \quad (18)$$

Independence among the conditional indicator functions implies,

$$\lim_{N \rightarrow \infty} \Lambda_p^A(\tilde{V}, \tilde{Z}, \tilde{Y}, n | e_M) \xrightarrow{a.s.} E(1_D(\tilde{V}_i | e_M)) \left(Dr_0 + (1 - Dr_0) \left(\frac{1}{n} \right) \sum_{j=1}^n E(1_{A_j}(\tilde{Z}_j | e_M)) \right) \left(\left(\frac{1}{n} \right) \sum_{j=1}^n E(1_{B_j}(\tilde{Y}_j | e_M)) \right). \quad (19)$$

Substitution of the expressions for the conditional expectations yields,

$$\lim_{N \rightarrow \infty} \left(\Lambda_p^A(\tilde{V}, \tilde{Z}, \tilde{Y}, n | e_M) \right) \xrightarrow{a.s.} \Phi \left(\frac{D - \sqrt{\rho_V} e_M}{\sqrt{1 - \rho_V}} \right) \cdot \left(Dr_0 + (1 - Dr_0) \left(\frac{1}{n} \right) \sum_{j=1}^n \Phi \left(\frac{A_j - \sqrt{\rho_Z} e_M}{\sqrt{1 - \rho_Z}} \right) \cdot \left(\left(\frac{1}{n} \right) \sum_{j=1}^n \Phi \left(\frac{B_j - \sqrt{\rho_Y} e_M}{\sqrt{1 - \rho_Y}} \right) \right) \right). \quad (20)$$

Expression (20) is an approximation for the inverse of the conditional distribution function for an asymptotic portfolio's loss rate evaluated at $e_M \in (-\infty, +\infty)$. Propositions 2 and 3 in the Appendix can be applied to show that, in the limit, as $n \rightarrow \infty$, the approximation converges to the true underlying asymptotic portfolio conditional loss rate that is consistent with the model assumptions.

The only random factor driving the unconditional portfolio loss rate distribution is common latent factor, \tilde{e}_M . As a consequence, an approximation for an asymptotic portfolio's loss rate density function is defined by the implicit function,

$$\tilde{\Lambda}_p^A \sim \left\{ \Phi \left(\frac{D - \sqrt{\rho_V} e_M}{\sqrt{1 - \rho_V}} \right) \cdot \left(Dr_0 + (1 - Dr_0) \left(\frac{1}{n} \sum_{j=1}^n \Phi \left(\frac{A_j - \sqrt{\rho_Z} e_M}{\sqrt{1 - \rho_Z}} \right) \cdot \left(\frac{1}{n} \sum_{j=1}^n \Phi \left(\frac{B_j - \sqrt{\rho_Y} e_M}{\sqrt{1 - \rho_Y}} \right) \right) \right) \right) \cdot \phi(e_M) \right\},$$

for $e_M \in (-\infty, \infty)$. (21)

Many risk management applications make direct use of the inverse conditional loss distribution. For example, expression (20) can be used to estimate capital allocations or estimate the probability that cumulative portfolio credit losses will breach tranche loss rate thresholds in basket credit risk transfer products and securitizations.

Calculation of the Critical Values of a Portfolio's Loss Rate Distribution

Many risk management functions require estimates for portfolio loss rates that are consistent with a particular cumulative probability threshold. Consider for example the portfolio loss rate that exceeds a proportion, α , of all potential portfolio credit losses (or alternatively, a loss rate exceeded by at most $1 - \alpha$ of all potential portfolio losses). Because the portfolio loss rate function is decreasing in e_M , expression (20) evaluated at $e_M = \Phi^{-1}(1 - \alpha)$ is the loss rate consistent with a cumulative probability of α . Using the identity $\Phi^{-1}(1 - \alpha) = -\Phi^{-1}(\alpha)$, and the definitions of the latent variable thresholds in Table 1 and 2, it follows that an approximation for the portfolio loss rate consistent with a cumulative probability of α is,

$$\Lambda_p^A(\tilde{V}, \tilde{Z}, \tilde{Y}, n | e_M = \Phi^{-1}(1-\alpha)) \xrightarrow{a.s.} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V} \Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}}\right) \left(Dr_0 + (1-Dr_0) \left(\frac{1}{n}\right) A(\alpha) \right) \left(\left(\frac{1}{n}\right) B(\alpha) \right) \quad (22)$$

for $\alpha \in [0,1]$, where,

$$A(\alpha) = \sum_{i=1}^n \Phi\left(\frac{\Phi^{-1}\left(1-\Omega\left(\frac{i-1}{n}\right)\right) + \sqrt{\rho_Z} \Phi^{-1}(\alpha)}{\sqrt{1-\rho_Z}}\right) \quad (23)$$

$$B(\alpha) = \sum_{j=1}^n \Phi\left(\frac{\Phi^{-1}\left(1-\Theta\left(\frac{j-1}{n}\right)\right) + \sqrt{\rho_Y} \Phi^{-1}(\alpha)}{\sqrt{1-\rho_Y}}\right) \quad (24)$$

Interpretation

The first term in expression (22), $\Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V} \Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}}\right)$, is the inverse of asymptotic portfolio's cumulative default rate distribution³ evaluated at a probability of α .

When *EAD* and *LGD* are both constant as they are in the Vasicek ASFM framework,

$\Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V} \Phi^{-1}(\alpha)}{\sqrt{1-\rho_V}}\right) \cdot LGD \cdot EAD$, is the formula used to estimate a capital allocation

with a coverage rate of α . For example, this loss rate formula (with $\alpha = .999$) is used to calculate minimum regulatory capital requirements in the Basel II AIRB approach. The interpretation is that when capital is set at this level, 99.9 percent of all potential portfolio credit losses will be less than the capital allocation.

³ The default rate distribution is the probably distribution of the random proportion of credits in an asymptotic portfolio that default each period.

The remaining terms in expression (22) modify the Vasicek portfolio loss rate formula to account for stochastic credit exposures, $\left(Dr_0 + (1 - Dr_0)\left(\frac{1}{n}\right)A(\alpha)\right)$, and stochastic losses given default $\left(\left(\frac{1}{n}\right)B(\alpha)\right)$. Additional interpretation for these expressions follows.

Consider the case of a fixed-term loan, $Dr_0 = 1$, so there is no revolving account balance. Expression (22) then becomes,

$$\Lambda_p^A\left(\tilde{V}, \tilde{Z}, \tilde{Y}, n \mid e_M = -\Phi^{-1}(\alpha)\right) \xrightarrow{a.s.} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V} \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_V}}\right) \left(\left(\frac{1}{n}\right)B(\alpha)\right). \quad (25)$$

Proposition 2 in the appendix shows that $\left(\frac{1}{n}\right)B(\alpha)$ is an approximation for the portfolio's

LGD rate conditional on a realized value of the common factor,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)B(\alpha) = E(\tilde{\lambda} \mid e_M = \Phi^{-1}(1 - \alpha)). \quad (26)$$

Proposition 3 in the appendix shows as $\rho_Y \rightarrow 0$, and $n \rightarrow \infty$, the portfolio's conditional *LGD* rate converges in probability to the expected value of an individual credit's unconditional *LGD* distribution,

$$\lim_{n \rightarrow \infty} \left(\lim_{\rho_Y \rightarrow 0} \left(\frac{1}{n}\right)B(\alpha)\right) = E(\tilde{\lambda}). \quad (27)$$

Thus, when individual losses in default are uncorrelated, expression (25) is an approximation for the expression,

$$\lim_{n \rightarrow \infty} \Lambda_p^A\left(\tilde{V}, \tilde{Z}, \tilde{Y}, n \mid e_M = -\Phi^{-1}(\alpha)\right) \xrightarrow{a.s.} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V} \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_V}}\right) E(\tilde{\lambda}). \quad (28)$$

Proposition 4 in the appendix shows, as $\rho_Y \rightarrow 1$ and $n \rightarrow \infty$, the asymptotic portfolio's

conditional *LGD* rate, $\left(\frac{1}{n}\right)B(\alpha)$, converges in probability to $\Theta^{-1}(\alpha)$.

$$\lim_{n \rightarrow \infty} \left(\lim_{\rho_Y \rightarrow 1} \left(\frac{1}{n} \right) B(\alpha) \right) = \Theta^{-1}(\alpha) \quad (29)$$

If $\lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) B(\alpha) = E(\tilde{\lambda} | e_M = \Phi^{-1}(1-\alpha))$ represents an asymptotic portfolio's *LGD*

rate conditional on a common factor realization $e_M = \Phi^{-1}(1-\alpha)$, it follows that the probability density of the asymptotic portfolio's *LGD* rate is defined by the implicit function,

$$\left\{ \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) B(\alpha), \phi(\Phi^{-1}(\alpha)) \right\}, \text{ for all } \alpha \text{ over the range } [0,1].$$

The remaining term in expression (22), $\left(DR_0 + (1 - DR_0) \left(\frac{1}{n} \right) A(\alpha) \right)$, is an approximation for the conditional utilization rate relative to the asymptotic portfolio's total committed line of credit. Assume each account's *LGD* is constant equal to LGD_0 , that each account has an initial drawn exposure, $DR_0 M$, and each account has the potential to take down up to $(1 - DR_0)M$ in additional credit from a remaining open line. Under these assumptions, expression (22) simplifies to,

$$\Lambda_p^A \left(\tilde{V}, \tilde{Z}, \tilde{Y}, n | e_M = -\Phi^{-1}(\alpha) \right) \xrightarrow{a.s.} \Phi \left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V} \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_V}} \right) \left(DR_0 + (1 - DR_0) \left(\frac{1}{n} \right) A(\alpha) \right) LGD_0. \quad (30)$$

Applying Proposition 2 to evaluate $\left(\frac{1}{n}\right)A(\alpha)$ as $n \rightarrow \infty$, it can be shown,

$\lim_{n \rightarrow \infty} \frac{1}{n} A(\alpha) \rightarrow E(\tilde{\delta} | e_M = -\Phi^{-1}(\alpha))$, and thus the portfolio's conditional *EAD* is given by,

$\lim_{n \rightarrow \infty} \left(DR_0 + (1 - DR_0) \left(\frac{1}{n} \right) A(\alpha) \right) \xrightarrow{a.s.} \left(DR_0 + (1 - DR_0) E(\tilde{\delta} | e_M = -\Phi^{-1}(\alpha)) \right)$. Clearly $\left(\frac{1}{n}\right)A(\alpha)$ is

an approximation for the portfolio's conditional draw rate.

When draw rate realizations are uncorrelated, Proposition 3 can be applied to show,

$\left(DR_0 + (1 - DR_0) \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) A(\alpha) \right) \xrightarrow{a.s.} DR_0 + (1 - DR_0) \cdot E(\tilde{\delta}_i)$, and so the portfolio's *EAD* is the

expected value of an individual credit's overall utilization rate. Application of Proposition 4

will show, as $\rho_Y \rightarrow 1$ and $n \rightarrow \infty$, the asymptotic portfolio's conditional draw rate

converges in probability to $\Omega^{-1}(\alpha)$. Finally, the probability density of the portfolio's draw

rate is defined by the implicit function, $\left\{ \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) A(\alpha), \phi(\Phi^{-1}(\alpha)) \right\}$, for α over the range

$[0, 1]$.

To summarize the interpretation of expression (22), an asymptotic portfolio's loss rate

consistent with a cumulative probability of α is the product of: (i) the asymptotic portfolio's

conditional default rate distribution evaluated at $e_M = \Phi^{-1}(1 - \alpha)$; (ii) the asymptotic

portfolio's conditional total credit facility utilization rate distribution evaluated at

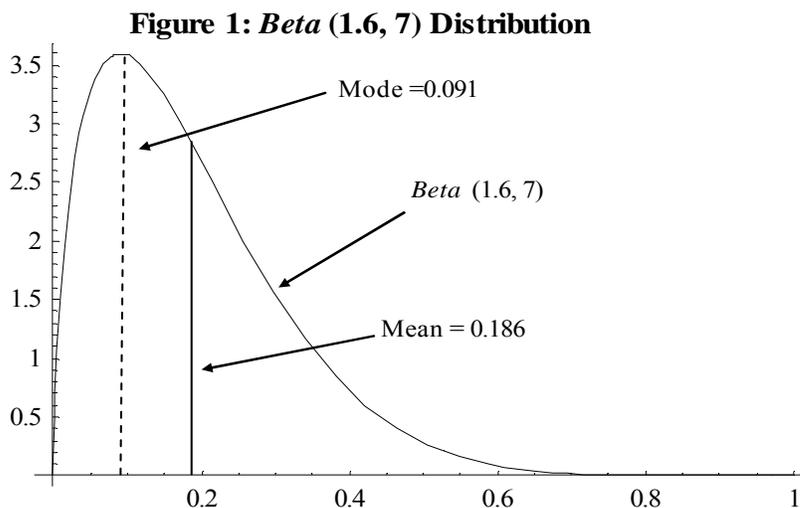
$e_M = \Phi^{-1}(1 - \alpha)$; and, (iii) the asymptotic portfolio's conditional loss rate given default rate

distribution evaluated at $e_M = \Phi^{-1}(1 - \alpha)$. Expressions (23) and (24) are, respectively, step

function approximations for an asymptotic portfolio's conditional draw rate and conditional loss rate given default.

5. EXAMPLES OF UNCONDITIONAL DRAW RATE AND LOSS GIVEN DEFAULT DISTRIBUTIONS

In this section, the step function algorithm is applied to approximate three alternative unconditional distributions that could be used to represent distributions for either individual draw rates or *LGD* rates depending on the specific application. The three distributions are all members of the beta distribution family, but the beta parameters are selected so that one unconditional distribution is skewed right, one is symmetric, and one is skewed left. The analysis demonstrates that both the skewness and the correlation of individual *LGD* and *EAD* distributions are important determinants of the shape of the asymptotic portfolio loss rate distribution. The description of the distributions is independent of what quantities they may represent, so to simplify the discussion we describe them as unconditional *LGD* distributions.



Example 1: Positively Skewed Distribution

The first distribution is the Beta distribution with the first parameter (alpha) equal to 1.6 and the second parameter (beta) equal to 7. The density function is,

$$\begin{aligned} \tilde{\lambda} &\sim \text{Beta}(1.6, 7, \tilde{\lambda}) \\ \text{Beta}(1.6, 7, \lambda) &= \frac{\Gamma(8.6)}{\Gamma(1.6)\Gamma(7)} \lambda^{0.6} (1-\lambda)^6, \quad \text{for } 0 < \lambda < 1 \end{aligned} \quad (31)$$

$\Gamma(b) = \int_0^{\infty} y^{b-1} e^{-y} dy$, $b > 0$, is the mathematical gamma function. This unconditional

distribution is skewed right and might represent the random draw rates on revolving corporate credits or the loss given default rates on wholesale bank loans or alt-A mortgages.

The $\text{Beta}(1.6, 7, \tilde{\lambda})$ probability density is plotted in Figure 1.

Assume individual *LGD* rates are distributed $\text{Beta}(1.6, 7, \tilde{\lambda})$ and loss rates are driven by the single common factor specification described earlier. Figure 2 plots an asymptotic portfolio's *LGD* distribution for alternative correlation assumptions assuming individual

LGDs are distributed $\text{Beta}(1.6, 7, \tilde{\lambda})$. The asymptotic portfolio's *LGD* distribution is

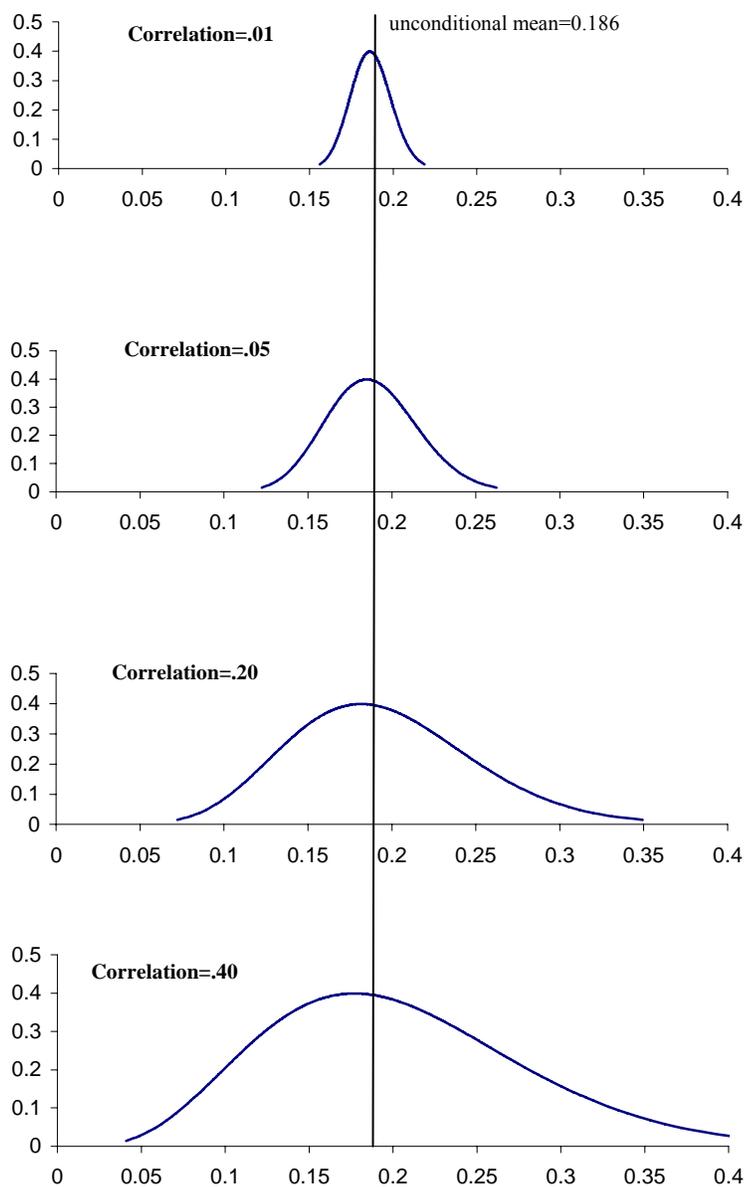
approximated using the step function approach, $\left\{ \left(\frac{1}{n} \right) B(\alpha), \phi(\Phi^{-1}(\alpha)) \right\}, \forall \alpha \in [0, 1]$, using

$n = 2500$.

When individual credit loss rates are uncorrelated, the portfolio's unconditional *LGD* distribution converges to $E(\text{Beta}(1.6, 7, \tilde{\lambda})) = 0.1862$. As correlation among individual *LGD* realizations increases, the range of the portfolio *LGD* distribution increases and the distribution becomes increasingly positively skewed. When the correlation among individual

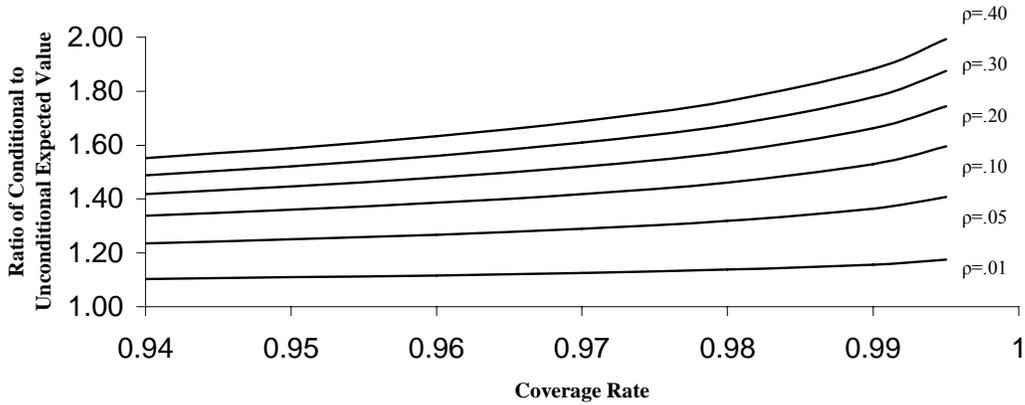
credit's *LGDs* is 1, there is no longer any ability to diversify *LGD* risk within the portfolio and, the asymptotic portfolio *LGD* distribution converges to the $Beta(1.6, 7, \tilde{\lambda})$ distribution.

Figure 2: Asymptotic Portfolio Unconditional *LGD* or Draw Rate Distribution for Alternative Correlations when Individual Credits are Distributed *Beta* (1.6, 7)



Consider the upper tail values of an asymptotic portfolio's unconditional *LGD* distribution. Let $\tilde{\lambda}_p$ represent the asymptotic portfolio's random *LGD* rate and $\Theta_p(\tilde{\lambda}_p, \rho_Y)$ represent its associated unconditional cumulative distribution function. The notation indicates that the asymptotic portfolio's *LGD* distribution is determined in part by the correlation between realizations in individual credits' loss rates. The loss rate consistent a cumulative probability of α is, $\Theta_p^{-1}(\alpha, \rho_Y) = \lim_{n \rightarrow \infty} \frac{1}{n} B(\alpha) = E(\tilde{\lambda} | e_M = -\Phi^{-1}(\alpha))$.

Figure 3: Effect of Correlation on Large Loss Realizations of an Asymptotic Portfolio *LGD* Distribution when Individual Credits are Distributed *Beta* (1.6,7)



For a given cumulative probability α , as the correlation among individual credit's *LGD* realizations increases, the magnitude of the asymptotic portfolio's conditional *LGD* increases, or alternatively, $\frac{\partial \Theta_p^{-1}(\alpha, \rho_Y)}{\partial \rho_Y} > 0$. Figure 3 plots, for alternative coverage rates and correlation assumptions, the ratio, $\frac{\Theta_p^{-1}(\alpha, \rho_Y)}{E(\tilde{\lambda}_p)}$. This ratio can be applied as a multiplier to correct the simple Vasicek model unexpected loss measure (calculated using the expected

value of the *LGD* distribution) for the systematic risk that arises from correlation in individual *LGD* distributions.

Example 2: Negatively Skewed Distribution

The second unconditional distribution considered is the *Beta* distribution with parameters $\alpha=4$ and $\beta=1.1$. This probability density is negatively skewed,

$$\tilde{\lambda} \sim \text{Beta}(4, 1.1, \tilde{\lambda})$$

$$\text{Beta}(1.6, 7, \pi) = \frac{\Gamma(5.1)}{\Gamma(4)\Gamma(1.1)} \lambda^3 (1-\lambda)^{0.1}, \text{ for } 0 < \lambda < 1. \quad (32)$$

This distribution might be representative of individual draw rates or *LGD* rates on sub-prime credit card accounts or other revolving retail credits. The $\text{Beta}(4, 1.1, \tilde{\lambda})$ density function is plotted in Figure 4.

Figure 4: Beta (4, 1.1) Distribution

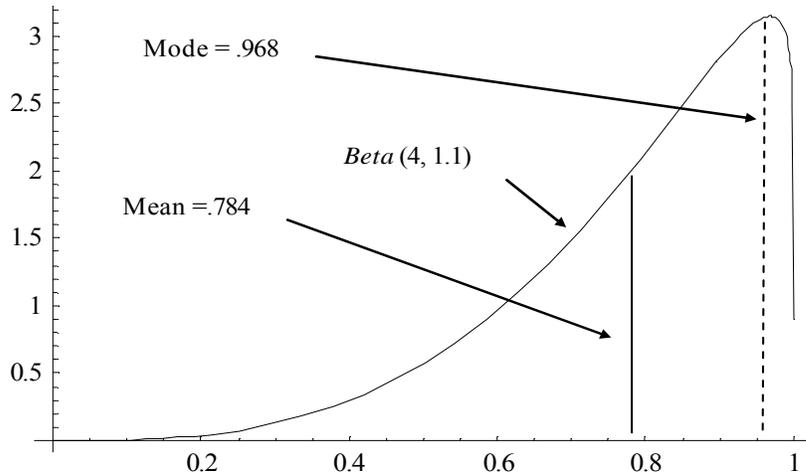


Figure 5 plots the asymptotic portfolio *LGD* distribution that is generated under different correlations assumptions when individual *LGDs* are distributed, $\tilde{\lambda} \sim \text{Beta}(4, 1.1, \tilde{\lambda})$. The unconditional portfolio *LGD* distribution is approximated using the step function approach with $n = 2500$. When individual *LGD* realizations are uncorrelated, *LGD* risk is

completely diversified and the asymptotic portfolio's *LGD* distribution converges to $E(\text{Beta}(4, 1.1, \tilde{\lambda})) = 0.7845$. As the correlation among obligors' *LGD* rates increases, the asymptotic portfolio's *LGD* distribution becomes increasingly negatively skewed, converging in the extreme to the $\text{Beta}(4, 1.1, \tilde{\lambda})$ distribution for $\rho_Y = 1$ (not shown).

For a given unexpected loss coverage rate α , $\Theta_p^{-1}(\alpha, \rho_Y)$ increases as the correlation increases, but not as dramatically as it does in the case of the $\text{Beta}(1.6, 7, \tilde{\lambda})$ distribution. As the correlation increases, $\Theta_p(\tilde{\lambda}_p, \rho_Y)$ becomes more negatively skewed, and the negative skewness dampens the effect of an increase in correlation. This relationship is illustrated in Figure 6 where the ratio, $\frac{\Theta_p^{-1}(\alpha, \rho_Y)}{E(\tilde{\lambda}_p)}$, is plotted for alternative correlations. A comparison of Figures 3 and 6 will show that correlation has a much larger impact on the upper tail values of $\Theta_p^{-1}(\alpha, \rho_Y)$ when individual *LGD* distributions are positively skewed.

Figure 5: Unconditional Asymptotic Portfolio LGD or Draw Rate Distribution for Alternative Correlations when Individual Credits are Distributed *Beta* (4, 1.1)

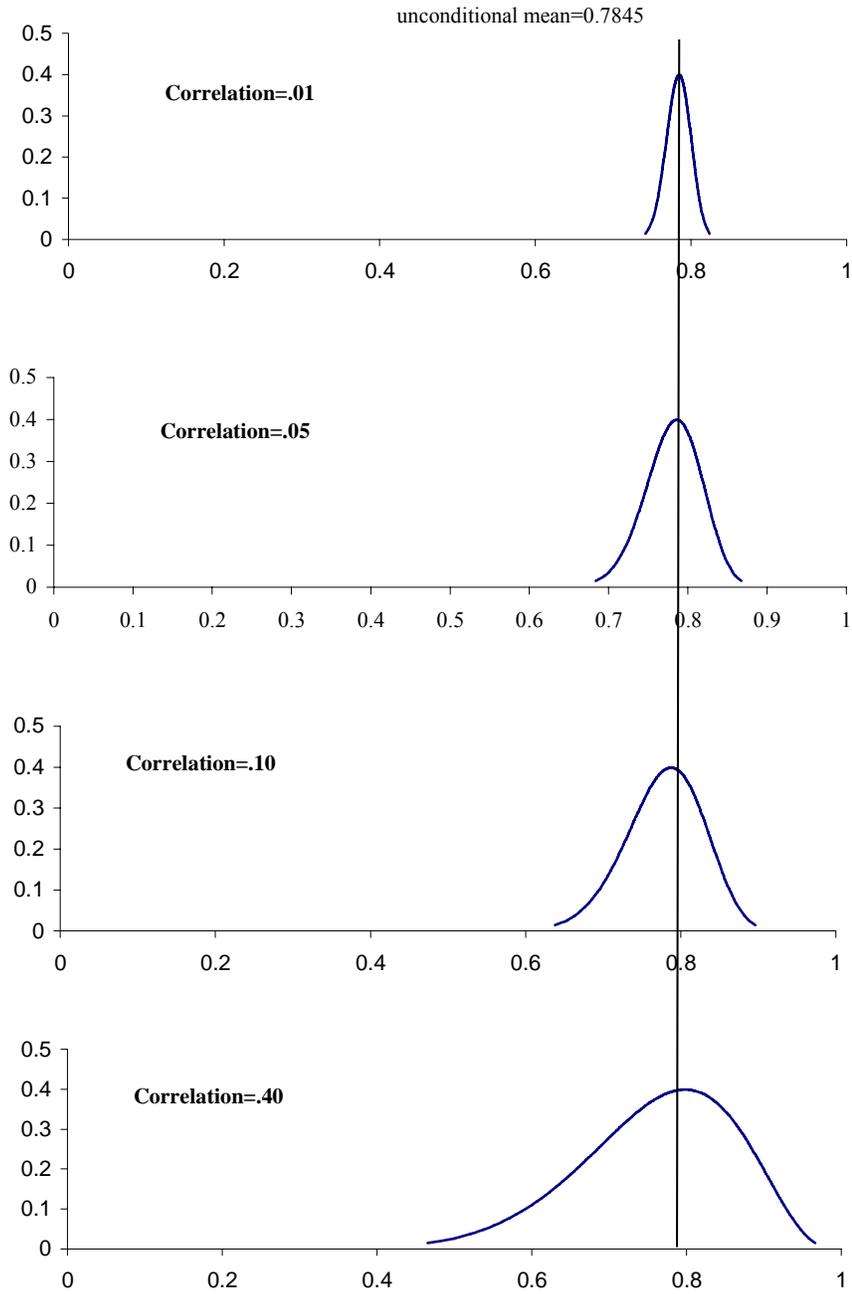


Figure 6: Effect of Correlation on Large Loss Realizations of an Asymptotic Portfolio LGD Distribution when Individual Credits are Distributed *Beta* (4, 1.1)

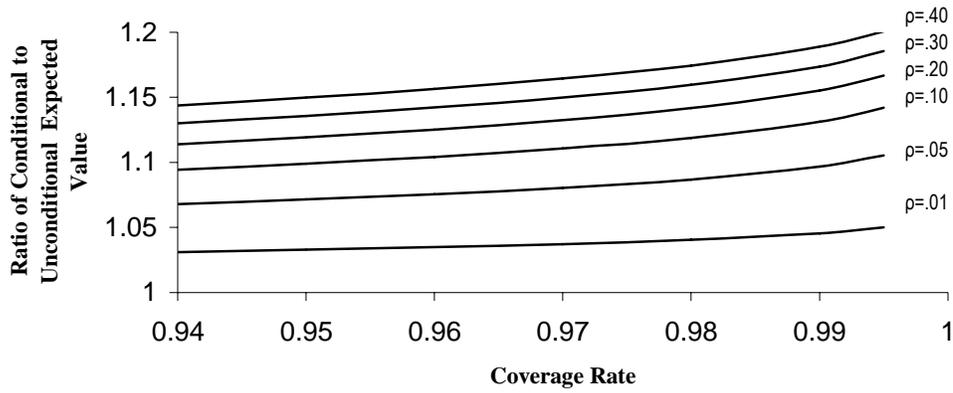
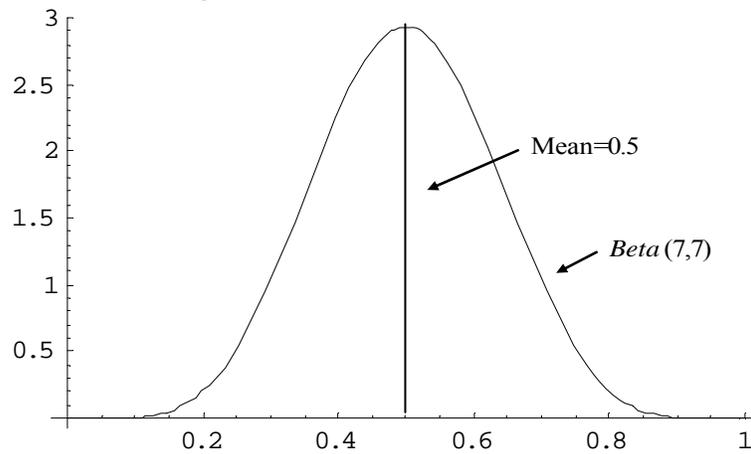


Figure 7: *Beta* (7, 7) Distribution



Example 3: Symmetric Distribution

Example 3 is the Beta distribution with parameters alpha=7 and beta=7,

$$\tilde{\lambda} \sim \text{Beta}(7,7, \tilde{\lambda})$$

$$\text{Beta}(7,7, \lambda) = \frac{\Gamma(14)}{\Gamma(7)\Gamma(7)} \lambda^6 (1-\lambda)^6, \text{ for } 0 < \lambda < 1 \quad (33)$$

This distribution is symmetric and might be representative of the distribution of *LGD* rates on investment grade corporate debt. The $Beta(7,7,\tilde{\lambda})$ density function is plotted in Figure 7.

The relationship between correlation and upper tail values of the asymptotic portfolio's *LGD* distribution, $\Theta_p^{-1}(\alpha, \rho_Y)$, is illustrated in Figure 8. Figure 8 plots, for alternative cumulative probabilities and correlation assumptions, $\frac{\Theta_p^{-1}(\alpha, \rho_Y)}{E(\tilde{\lambda}_p)}$. These implied multipliers are larger than those pictured in Figure 6, but smaller than those pictured in Figure 3 illustrating again the importance of interaction between the unconditional distribution's skewness and the correlation among the Gaussian latent factors.

Figure 8: Correlation Effects on the Conditional Expected Value of the Beta(7,7) Distribution

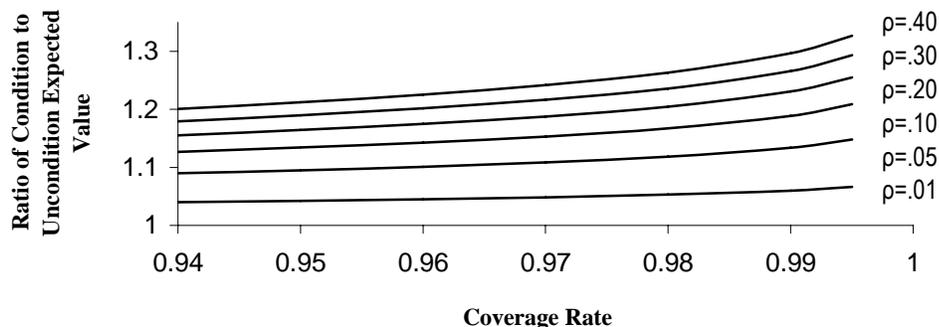
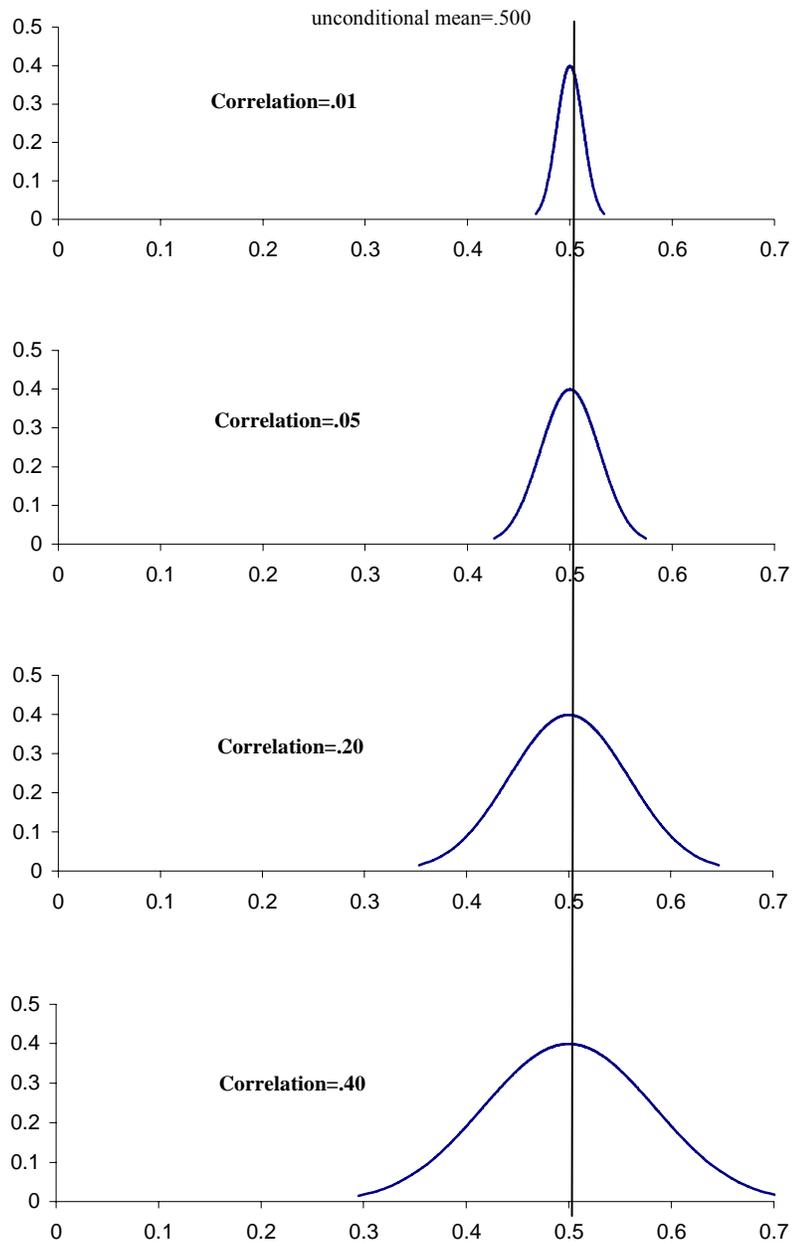


Figure 9 plots, for different correlation assumptions, the unconditional asymptotic portfolio *LGD* distribution approximation ($n = 2500$) that is generated when *LGD*s are distributed $\tilde{\lambda} \sim Beta(7,7,\tilde{\lambda})$. The distribution converges to $E(Beta(7,7,\tilde{\lambda})) = 0.5$ when individual *LGD* realizations are uncorrelated. As the correlation increases, the range of the

asymptotic portfolio *LGD* distribution increases, and the portfolio *LGD* distribution converges to $Beta(7,7,\tilde{\lambda})$ as $\rho_Y \rightarrow 1$.

Figure 9: Unconditional Asymptotic Portfolio LGD or Draw Rate Distribution for Alternative Correlations when Individual Credits are Distributed *Beta* (7, 7)



6. EXAMPLES OF ASYMPTOTIC PORTFOLIO UNCONDITIONAL LOSS RATE DISTRIBUTIONS

In this section, we apply the step function algorithm to approximate the unconditional loss rate distributions for alternative asymptotic credit portfolios. The approximations are based on $n = 2500$. The examples are intended to resemble portfolios that include both fixed-term loans and revolving credit facilities for both wholesale and retail credits.

Published evidence on the shape and correlations of individual credits' *LGD* and *EAD* distributions is limited. In the case *EAD*, few studies characterize the shape of exposure distributions and no study has attempted to estimate the strength of *EAD* correlation in a structural model.⁴ A larger number of studies focus on the distribution of *LGD* rates, but the evidence is still sparse and much of it is specialized to default rates for agency-rated credits.

Most studies investigating *LGD* correlation behavior investigate linear times series correlation estimates between observed default frequencies and default recovery rates. Only one study estimates a structural model *LGD* correlation parameter. Frye (2000b) estimates ρ_Y to be about 20 percent for agency-rated bonds, but his estimate is based on a structural model that assumes that *LGD* distributions are symmetric. It seems likely that alternative specifications for *LGD* that include significant skew in the unconditional *LGD* distribution would produce more modest estimates of correlation, but such issues have yet to be studied. Also, as noted by Carey and Gordy (2004), most *LGD* correlation estimates have been

⁴ Araten and Jacobs (2001) provide simple descriptive statistics on a sample of Chase revolving facilities. Jiménez, Lopez and Saurina (2006) provide an aggregated histogram of all corporate *EADs* derived from the Spanish credit registry for all credit institution loans in excess of 6000 Euros over a period spanning 1984-2005.

derived from rating-agency bond data, and the correlations for different liability classes are likely to differ according to firm capital structure characteristics and the identity of important stakeholders, including the presence (or absence) of significant banking interests.

A review of the publicly available literature suggests that the shape of individual unconditional *LGD* and *EAD* distributions as well as the magnitudes of their correlations is an open issue. This study will not contribute to the calibration debate but instead will consider asymptotic portfolio loss rate distributions for a number of alternative parameterizations.

Portfolio Loss Distribution Example 1: Portfolio of Term Loans

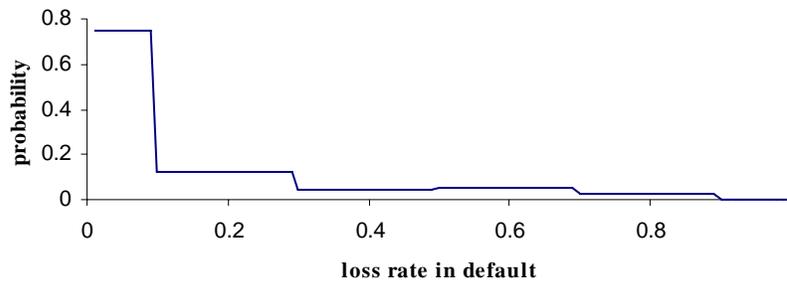
The first example is chosen to represent the portfolio loss rate distribution that may arise on a portfolio of term loans comprised of non investment-grade senior secured credits. Figure 10 plots the distribution of projected *LGD* rates on loans that receive a recovery rating by *FitchRatings*.⁵ A large share of the *FitchRatings* sample of credits are secured first-lien loans which in part explains the favorable recovery rate distribution. This forward-looking *LGD* rate distribution is not conditioned on any realized state of the economy and so it proxies for an unconditional *LGD* distribution.

The distribution in Figure 10, while not very granular, is broadly similar to the $Beta(1.6, 7, \lambda)$ distribution we will use to represent the *LGD* distribution for individual secured first-lien loans. To construct the asymptotic portfolio loss rate distribution for this class of exposures, we assume that all loans are fully drawn ($EAD=1$) and individual credits have an unconditional probability of default of 0.5 percent. The default correlation parameter

⁵ Figure 10 is constructed from information provided in *FitchRatings* (2006).

is set at 20 percent ($\rho_V = 0.20$) to reflect the wholesale nature of these credits and the calibration used in the Basel AIRB capital framework.

Figure 10: Projected LGD Distribution Loans Rated by *FitchRatings*



The asymptotic portfolio loss distribution is plotted for different *LGD* correlation assumptions in Figure 11. The alternative panels in Figure 11 clearly highlight the importance of systematic risk in recovery rates. As the correlation between individual credit *LGD* rates increases, the skewness of the asymptotic portfolio's loss rate distribution increases markedly. As correlation increases from 0 to 10 percent, the 99.5 percent critical value of the portfolio loss rate distribution increases by almost 60 percent. When individual *LGD* correlations are 20 percent, the portfolio 99.5 percent loss-coverage rate is about 87.5 percent larger than the estimate produced by the simple Vasicek ASRF formula (from the top panel of Figure 11) that assumes uncorrelated *LGDs*.

Portfolio Loss Distribution Example 2: Portfolio of Revolving Senior Unsecured Credits

The second example is chosen to represent the loss rate distribution of an asymptotic portfolio of revolving senior unsecured bank loans made to investment-grade obligors. The example assumes that portfolio obligors begin the period with a 30 percent facility utilization rate and draw on their remaining credit line over the subsequent period. Because these are

wholesale credits, we use the Basel II default correlation assumption, $\rho_V = 0.20$. We examine the shape of an asymptotic portfolio loss rate distribution under alternative correlations assumptions for *LGD* and *EAD*.

Figure 11: Asymptotic Portfolio Loss Rate Distributions for Fixed *EAD U* nder Alternative Correlation Assumptions. Individual Credits have *PD* =0.5% and Unconditional *LGD* s~Beta(1.6,7).

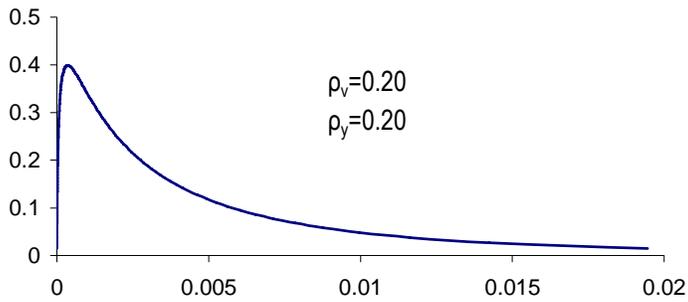
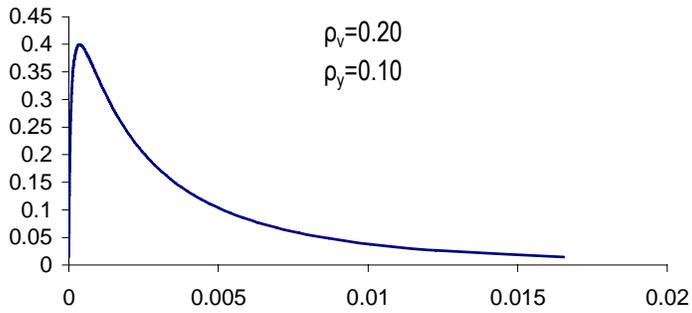
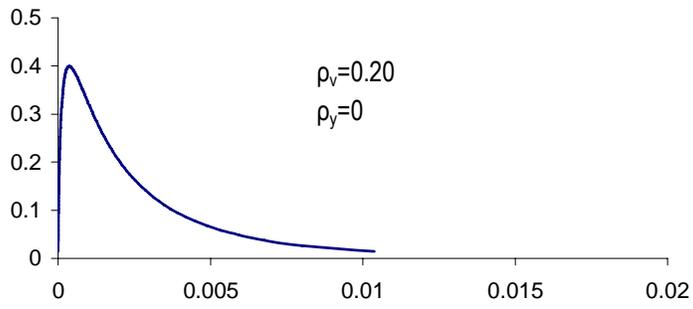
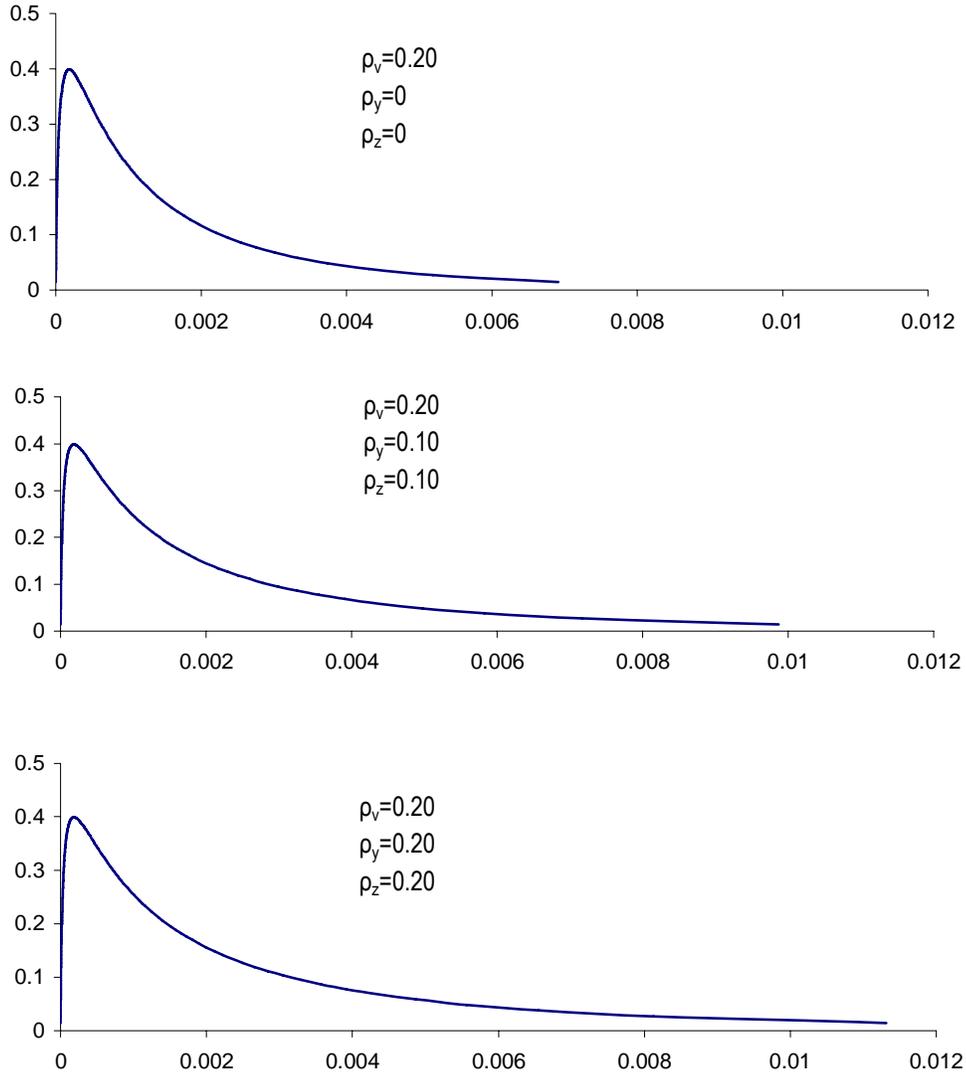


Figure 12: Asymptotic Portfolio Loss Rate Distributions for Alternative Correlations. Individual Credits have PD=0.25%, 30% Initial Utilization, and 70% Revolving Balances, with Unconditional Draw Rates~Beta (1.6,7) and Unconditional LGDs~Beta (7,7).



Altman (2006, Table 2) reports data that suggests that the historical loss rate distribution on senior unsecured bank loans is very close to symmetric, with an average loss rate of about 50 percent and a standard deviation of about 25 percent. The $Beta(7,7,\lambda)$ distribution provides a close approximation to this *LGD* distribution. Araten and Jacobs

(2001, Table1) estimate for the Chase data they examine, a BBB+/BBB- rated credit has, on average, about a 55 percent loan equivalent value 1-year prior to default. We are not aware of any study that further characterizes the exposure distribution on these types of facilities, but the assumption of an initial utilization rate of 30 percent and a $Beta(1.6,7,\lambda)$ draw rate distribution matches both the mean of the Araten and Jacobs *EAD* data and conventional wisdom that suggests that bankers are at least partially successful at limiting takedowns by distressed obligors. We assume an unconditional default rate of 0.25 percent.

Figure 12 plots estimates of the asymptotic portfolio loss rate distribution under alternative assumptions for *LGD* and *EAD* correlations. The panels in Figure 12 show that correlation in individual credit *LGD* and *EAD* distributions has a large effect of on the tails of the portfolio's credit loss distribution. As correlation in *EADs* and *LGDs* increases from 0 to 10 percent (0 to 20 percent), the loss value associated with a 99.5 percent cumulative probability increases by 43 percent (64 percent).

Portfolio Loss Distribution Example 3: Sub-Prime Customer Credit Card Portfolio

This example is intended to mimic a sub-prime credit card portfolio. Unlike the earlier examples, we are unable to reference a published study to anchor our choice of distributional assumptions. Individual accounts are assumed to have an unconditional probability of default of 4 percent, and default correlations are assumed to be 4 percent, consistent with the Basel AIRB treatment of qualified retail exposures. Customers are assumed to begin the period with a 20 percent credit limit utilization. They are assumed to draw on the remaining 80 percent of their credit limit randomly, with their draw rate modeled

using the $Beta(4,1.1, \lambda)$ distribution. Because these are unsecured credits, recovery rates are low. We model account $LGDs$ using the $Beta(4,1.1, \lambda)$ distribution.

Figure 13: Asymptotic Portfolio Loss Rate Distributions for Alternative Correlations.
Individual Credits have PD=4%, 20% Initial Utilization, and 80% Revolving
Balance, with Unconditional Draw Rates~Beta (4,1.1,1) and Unconditional
LGDs~Beta (4,1.1,1).

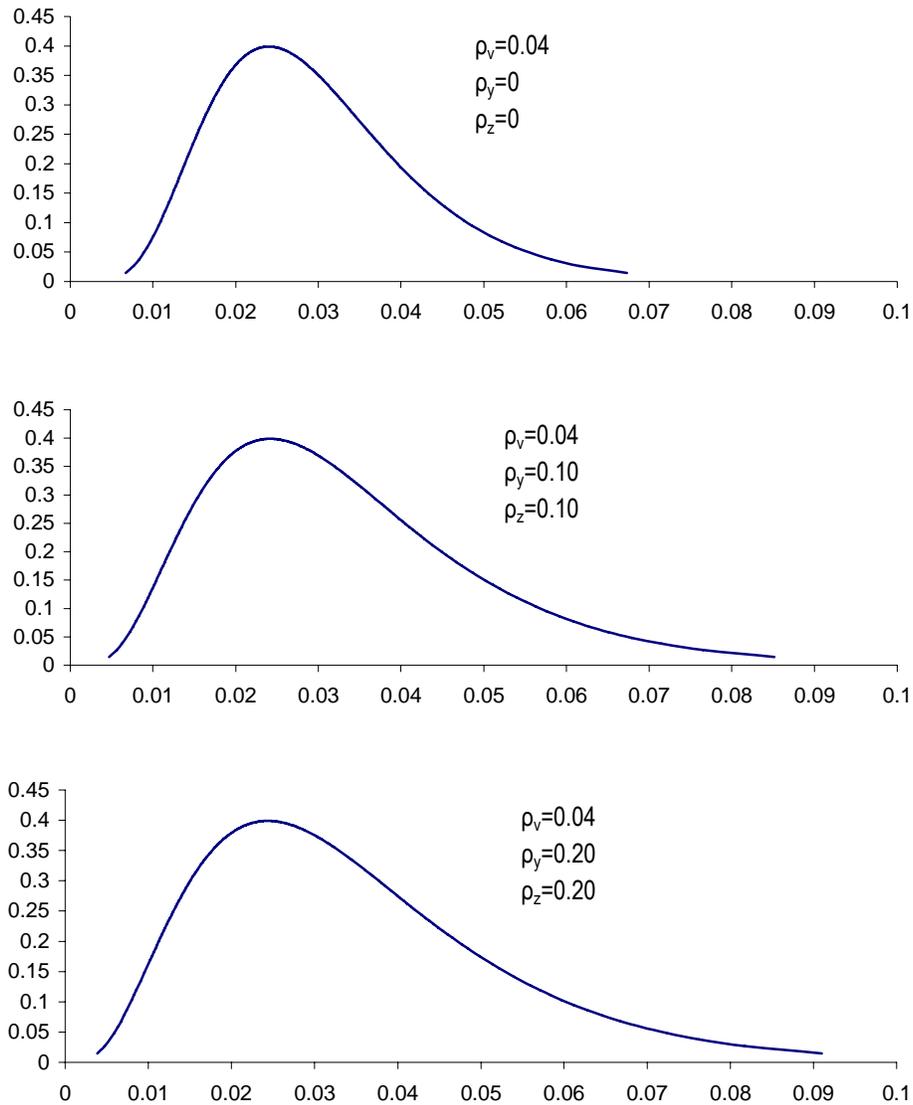


Figure 13 plots estimates of the asymptotic portfolio's loss rate distribution under alternative assumptions for *LGD* and *EAD* correlations. Unlike the earlier two examples, the panels in Figure 13 show that correlation in individual credit *LGD* and *EAD* distributions has a relatively minor effect on the tails of the portfolio's credit loss distribution. As correlation in *EADs* and *LGDs* increases from 0 to 10 percent (0 to 20 percent), the loss value associated with a 99.5 percent cumulative probability increases by only 26 percent (35 percent).

7. CONCLUSIONS

This paper has developed a tractable generalization of the single common factor portfolio credit loss model that includes correlated stochastic exposures and loss rates. The model uses a step function to approximate *LGD* and *EAD* distributions and a generalization of the latent factor framework of Vasicek to model correlations. The new model does not restrict *EAD* or *LGD* distributions or their correlations. The model produces a closed-form approximation for a portfolio's inverse cumulative conditional loss rate that is used to calculate unconditional portfolio loss rate distributions and expressions that can be used to calculate economic capital allocations. Portfolio loss rate distributions are estimated for stylized representations of alternative wholesale and retail portfolios. The results show that the additional systematic risk created by positive correlation in *EAD* and *LGD* distributions increases the skewness of an asymptotic portfolio's loss rate distribution increasing the measured risk of loss in lower tranches of CDOs and securitizations and mandating the need for larger economic capital allocations relative to those calculated using the Vasicek model.

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APPENDIX

Proposition 1: $E(\tilde{\alpha}) = l = 1 - \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) \sum_{j=0}^{n-1} E(1_{E(\tilde{\alpha}, j, n)})$

Using , $\lim_{n \rightarrow \infty} E(1_{E(\tilde{\alpha}, j, n)}) - E(1_{E(\tilde{\alpha}, j-1, n)}) = \Xi \left(\frac{j}{n} \right)$, the mathematical expectation of

$\tilde{\alpha}$ can be written,

$$E(\tilde{\alpha}) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \left(\frac{j}{n} \right) \left(E(1_{E(\tilde{\alpha}, j, n)}) - E(1_{E(\tilde{\alpha}, j-1, n)}) \right) = 1 - \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) \sum_{j=0}^{n-1} E(1_{E(\tilde{\alpha}, j, n)}).$$

Proposition 2: $\lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) B(\alpha) = E(\tilde{\lambda} | e_M = \Phi^{-1}(1 - \alpha))$.

Expressions (4), (10) and the definitions in Table 2 imply, in the limit as $n \rightarrow \infty$,

$$E(\tilde{\lambda} | e_M) = \lim_{n \rightarrow \infty} \left[1 - \left(\frac{1}{n} \right) \sum_{j=1}^n E(1_{A_j}(\tilde{Z}_i | e_M)) \right] = \lim_{n \rightarrow \infty} \left[1 - \left(\frac{1}{n} \right) \sum_{j=1}^n \Phi \left(\frac{A_j - \sqrt{\rho_Y} e_M}{\sqrt{1 - \rho_Y}} \right) \right].$$

Substitution of the definitions of the A_j values from Table 2 and $e_M = \Phi^{-1}(1 - \alpha) = -\Phi^{-1}(\alpha)$

establishes the result.

Proposition 3: $\lim_{n \rightarrow \infty, \rho_Y \rightarrow 0} \left(\lim_{\rho_Y \rightarrow 0} \left(\frac{1}{n} \right) B(\alpha) \right) = E(\tilde{\lambda})$.

Substitution for expression (24) will show,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[\lim_{\rho_Y \rightarrow 0} \left(\frac{1}{n} \right) \sum_{j=1}^n \Phi \left(\frac{\Phi^{-1} \left(1 - \Theta \left(\frac{j-1}{n} \right) \right) + \sqrt{\rho_Y} \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_Y}} \right) \right] \\ &= 1 - \left(\frac{1}{n} \right) \sum_{j=0}^{n-1} \Theta \left(\frac{j}{n} \right) = 1 - \left(\frac{1}{n} \right) \sum_{j=0}^{n-1} E(1_{E(\tilde{\lambda}, j, n)}) \xrightarrow{a.s.} E(\tilde{\lambda}) \end{aligned}$$

Proposition 4: $\lim_{n \rightarrow \infty} \left(\lim_{\rho_Y \rightarrow 1} \left(\frac{1}{n} \right) B(\alpha) \right) = \Theta^{-1}(\alpha)$.

To see this, recall: $\left(\frac{1}{n} \right) B(\alpha) = \frac{1}{n} \sum_{j=1}^n 1_{B_j}(\tilde{Y}_i | e_M = \Phi^{-1}(1-\alpha))$.

When $\rho_Y = 1$, $\tilde{Y}_i = \tilde{e}_M$, and consequently $(\tilde{Y}_i | e_M = \Phi^{-1}(1-\alpha)) = \Phi^{-1}(1-\alpha)$. From Table 2, the indicator function thresholds are defined by, $B_j = \Phi^{-1}\left(1 - \Theta\left(\frac{j}{n}\right)\right)$. The largest increment J for which its indicator function equals one conditional on $\tilde{Y}_i = \Phi^{-1}(1-\alpha)$ is the largest J for which $\Phi^{-1}\left(1 - \Theta\left(\frac{J}{n}\right)\right) \leq \Phi^{-1}(1-\alpha)$. For n sufficiently large, the step function mesh will

become sufficiently fine so that the equality will determine the value of J . From

$\Phi^{-1}(1-\alpha) = \Phi^{-1}\left(1 - \Theta\left(\frac{J}{n}\right)\right)$, it is apparent that, $J = n \cdot \Theta^{-1}(\alpha)$. As a consequence,

$$\lim_{n \rightarrow \infty} \left(\lim_{\rho_Y \rightarrow 1} \left(\frac{1}{n} \right) B(\alpha) \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{j=1}^n 1_{B_j}(\tilde{Y}_i | e_M = \Phi^{-1}(1-\alpha)) \right) = \frac{1}{n} J = \Theta^{-1}(\alpha).$$