Economic Capital for Pooled Market, Credit and Instrument-Specific Risks

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April 2006

Overview

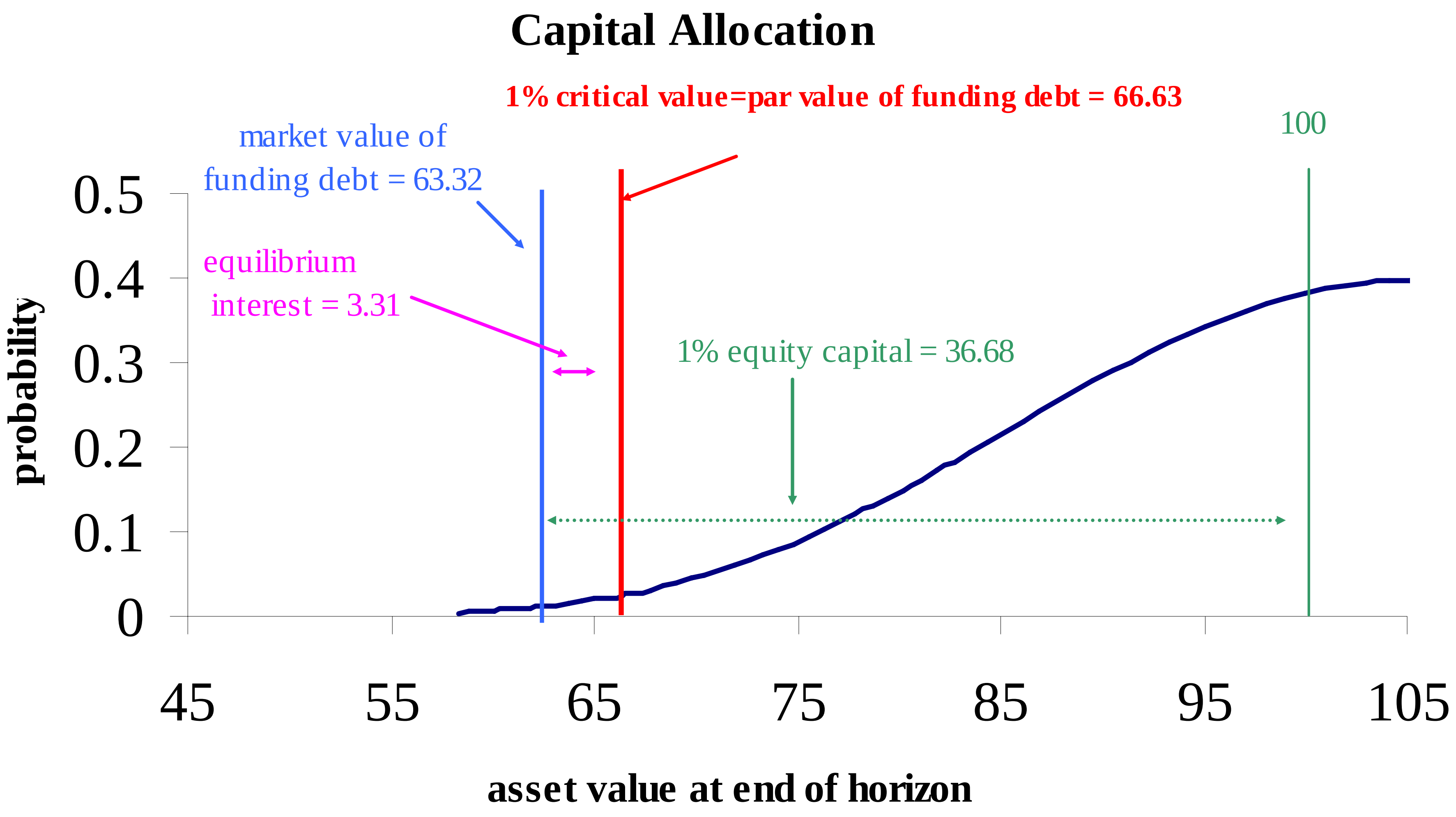
- Foundations of Capital Allocation
- Single Asset Capital Allocation
- Asymptotic Portfolio Models
  - Credit
  - Market
  - Credit and Market
- Concentration and Specific Risks

Capital Allocation Problem

- Given an investment and a time horizon, choose a capital structure that maximizes the use of debt finance, but ensures a survival probability of at least X% (e.g., 99, 99.9)
- *Alternatively:* Choose a capital structure so that the probability of default on the institution's funding debt is at most 1-X%
- **ECONOMIC CAPITAL = EQUITY**
- Capital allocation is a specific formulation of an optimal capital structure problem

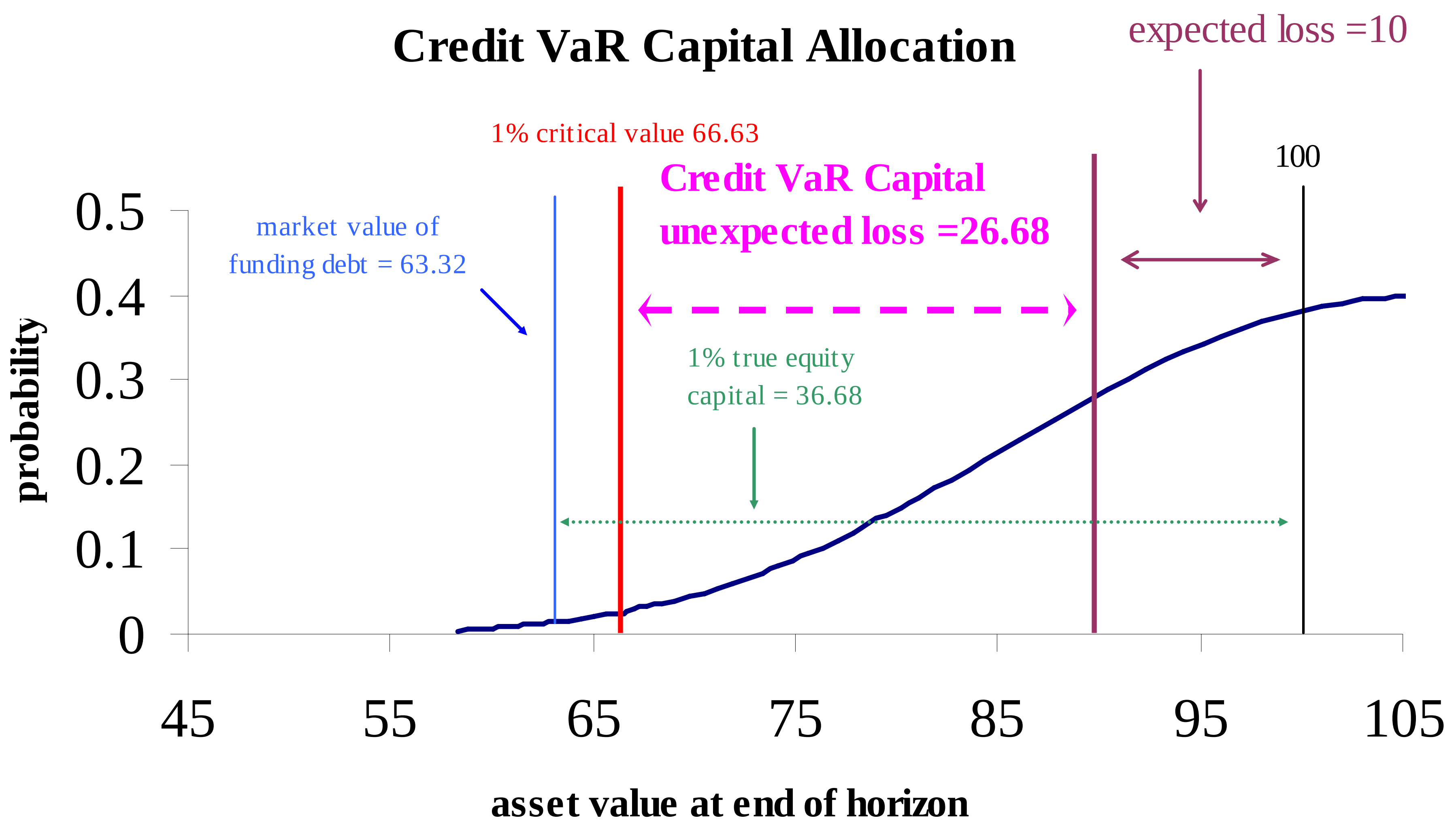
Solution

- Consider the investment portfolio’s distribution of possible values over the future horizon of interest
- Assume the institution funds itself with a discount debt issue


Capital Allocation
1% critical value=par value of funding debt = 66.63
100
market value of
funding debt = 63.32
equilibrium
interest = 3.31
1% equity capital = 36.68
probability
0.5
0.4
0.3
0.2
0.1
0
45
55
65
75
85
95
105
asset value at end of horizon

Standard Credit VaR Capital Allocation

- **Economic capital = Unexpected Loss**
- Credit VaR understates capital needs
  - Need capital for expected loss and unexpected loss
- Credit VaR forgets to pay interest on the funding debt



Credit VaR Capital Allocation
expected loss =10
1% critical value 66.63
100
Credit VaR Capital
unexpected loss =26.68
market value of
funding debt = 63.32
1% true equity
capital = 36.68
0.5
0.4
0.3
0.2
0.1
0
probability
45
55
65
75
85
95
105
asset value at end of horizon

Economic Capital Allocation Steps

- Estimate critical value on the loss tail of the future payoff or return distribution
  - Cumulative tail loss probability = 1-target survival rate
- Critical value = the maximum maturity value of the funding debt that can be issued
- Estimate the initial value of the funding debt

Example BSM Model

- Black-Scholes-Merton model
- Full equilibrium pricing model
  - Equity is a call option on the underlying asset
  - Debt is a risk free claim less a default option
- Underlying asset value follows geometric Brownian Motion

$$\tilde{A}_T = A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \tilde{z}\sigma\sqrt{T}}, \qquad \tilde{z} \sim \phi$$

Distribution of Equity & Debt Returns

- Firm finances itself with a T period discount bond with maturity value =*par*
- Equity value at time T

$$Min[0, \tilde{A}_T - par]$$

- Debt value at time t

$$Min[\tilde{A}_T, par]$$

Credit Risk Capital

- Let
  - A_0 be the initial underlying asset value
  - *Par* be the maturity value of debt
  - r_f the risk free rate
  - ♦ the asset volatility, ❍ the instantaneous drift rate
  - ♋ the target solvency rate
  - *T* the time horizon
  - Funding debt par value for a maturity-T bond

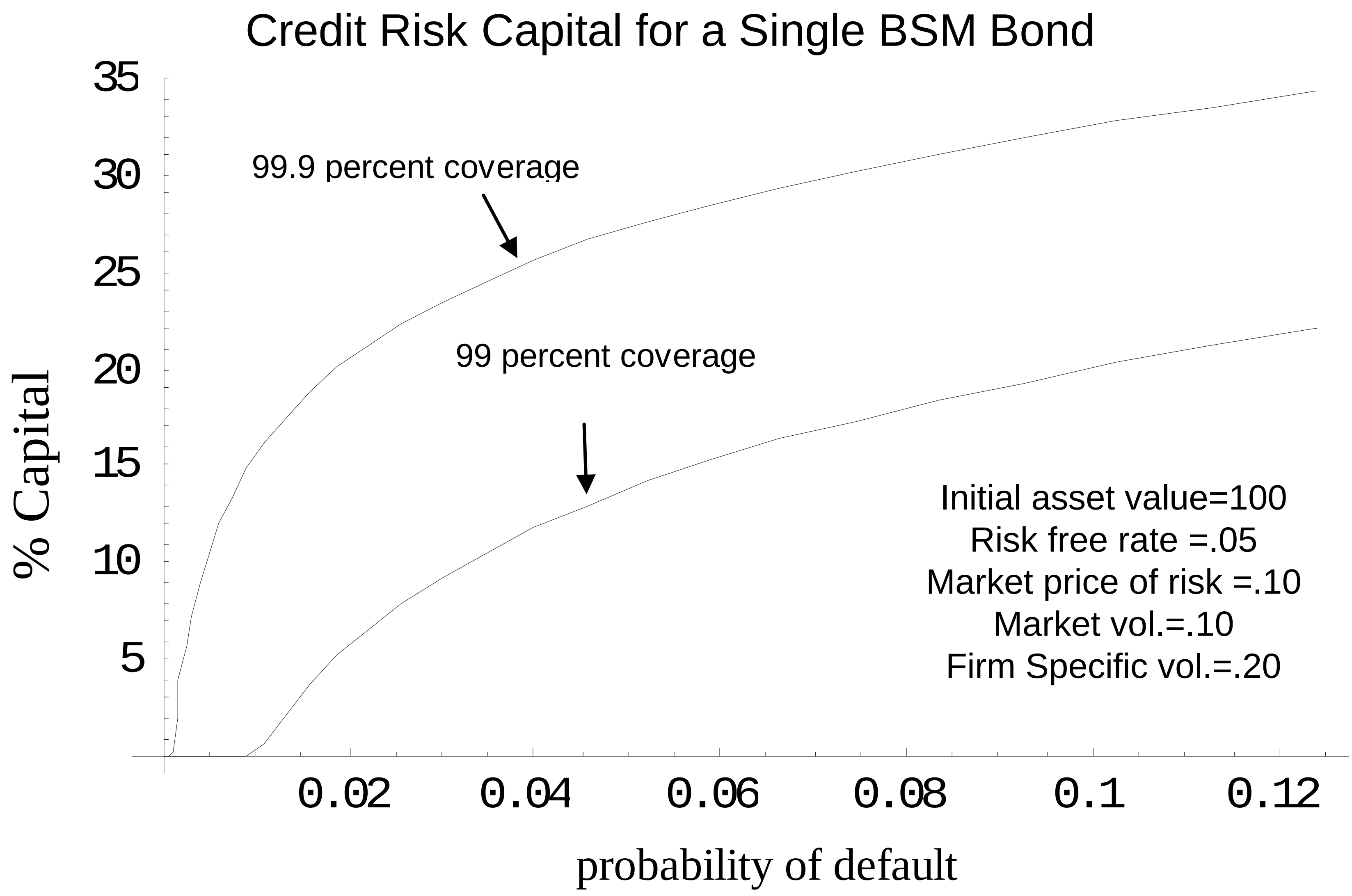
$$Par_{FD} = A_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(1-\alpha)}$$

Closed Form Credit Risk Capital

$$d_1 = \frac{Ln\left(\frac{A_0}{Par_{FD}}\right) + \left(r_f + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},\ d_2 = d_1 - \sigma\sqrt{T}$$

$$d_3 = \frac{Ln\left(\frac{A_0}{Par}\right) + \left(r_f + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},\ d_4 = d_3 - \sigma\sqrt{T}$$

$$Capital = 1 - \left(\frac{Par_{FD}\, e^{-r_f T}\left(1 - \Phi(-d_2)\right) - A_0\Phi(-d_1)}{e^{-r_f T}\, Par\, \Phi(d_4) - A_0\Phi(-d_3)}\right)$$


Credit Risk Capital for a Single BSM Bond
35
30
25
20
15
10
5
% Capital
99.9 percent coverage
99 percent coverage
Initial asset value=100
Risk free rate =.05
Market price of risk =.10
Market vol.=.10
Firm Specific vol.=.20
0.02
0.04
0.06
0.08
0.1
0.12
probability of default

Closed From Equity Capital

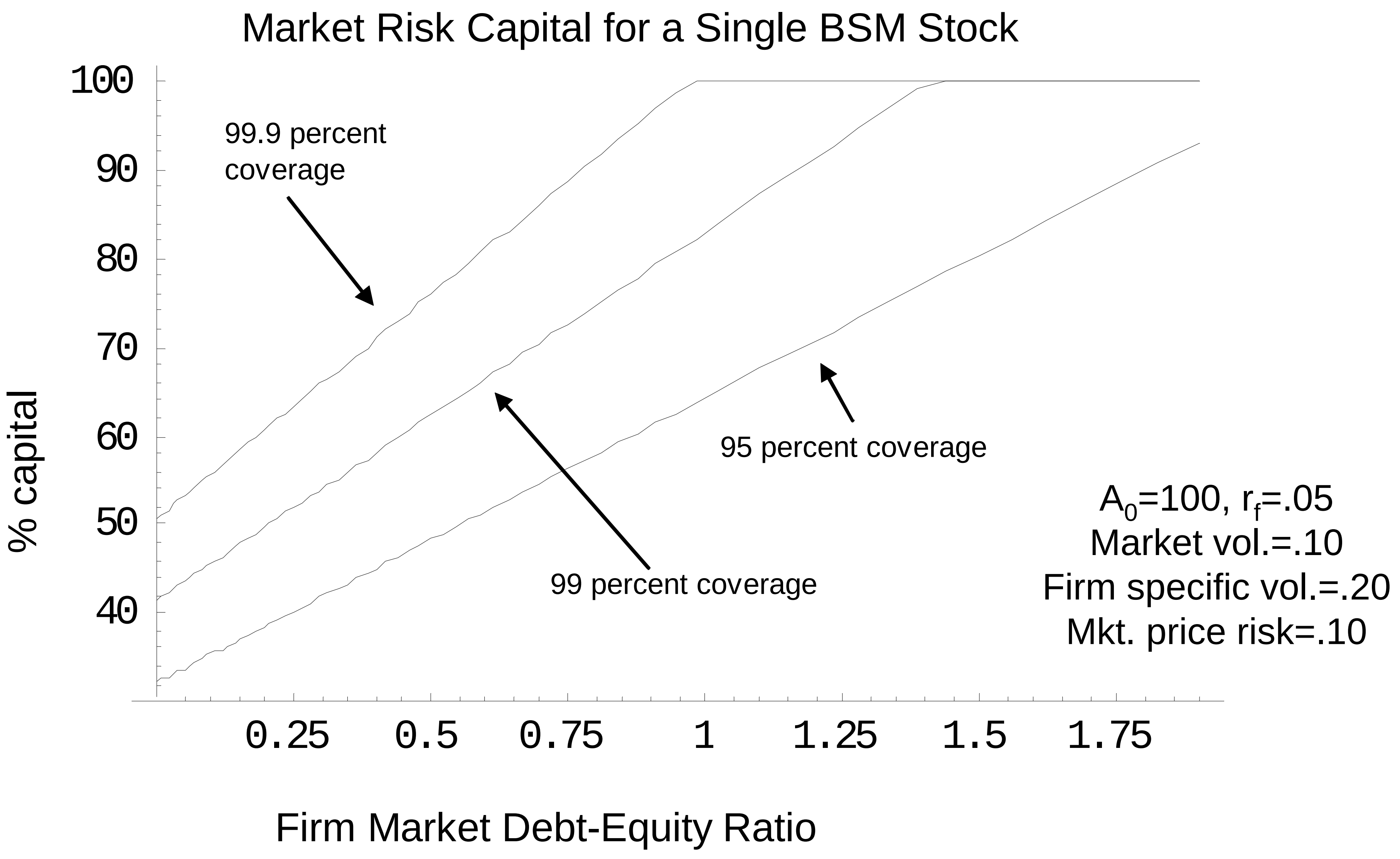
$$c_1 = \frac{Ln\left(\frac{A_0}{Par}\right) + \left(r_f + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \; c_2 = c_1 - \sigma\sqrt{T}$$

$$c_3 = \Phi^{-1}(1-\alpha) + \left(\frac{(\mu - r_f)\sqrt{T}}{\sigma}\right),$$

$$Par_{FD}^{e} = \left[A_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(1-\alpha)} - Par \right]$$

$$Capital =$$

$$1 - \frac{A_0\left(\Phi(c_1) - \Phi\left(-c_3 + \sigma\sqrt{T}\right)\right) - e^{-r_f T} Par\left(\Phi(c_3) - \Phi(-c_2)\right) + e^{-r_f T} Par_{FD}^{e}\,\Phi(-c_3)}{A_0\,\Phi(c_1) - e^{-r_f T} Par\;\Phi(c_2)}$$


Market Risk Capital for a Single BSM Stock
% capital
100
90
80
70
60
50
40
99.9 percent
coverage
99 percent coverage
95 percent coverage
A0=100, rf=.05
Market vol.=.10
Firm specific vol.=.20
Mkt. price risk=.10
0.25
0.5
0.75
1
1.25
1.5
1.75
Firm Market Debt-Equity Ratio

Portfolio Capital

- In most cases, capital reductions arise from diversification effects achieved in a portfolio of investments
- Risk from common factors determine risk and capital
- Idiosyncratic instrument-specific risks can be diversified

Single Common Factor Model

- Use two factor model of underlying assets' future value generating function
    - Common market risk Brownian
    - Firm specific Brownian
- Common market risk factor is priced
    - Equilibrium market price of risk affects drift rate

-Firm-specific factors can be diversified and do

not have a market price of risk associated with them

$$\tilde{A}_T = A_0\, e^{\left(r_f + \lambda\sigma_m - \frac{\left(\sigma_m^2 + \sigma_i^2\right)}{2}\right)T + \sqrt{T}\left(\sigma_m \tilde{z}_m + \sigma_i \tilde{z}_i\right)}$$

Correlations

- Correlation between stock and bond returns are determined by market volatility and idiosyncratic volatility

$$Corr\left[\frac{1}{T}\ln\left(\frac{\tilde{A}_{it}}{A_{i0}}\right),\frac{1}{T}\ln\left(\frac{\tilde{A}_{jt}}{A_{j0}}\right)\right]=\frac{\sigma_M^2}{\left(\sigma_M^2+\sigma_i^{\,2}\right)^{\frac{1}{2}}\left(\sigma_M^2+\sigma_j^{\,2}\right)^{\frac{1}{2}}},\ \forall i,j.$$

Asymptotic Portfolio construction assumes risk characteristics are identical

$\sigma_i = \sigma_j = \bar{\sigma}, \forall i, j,$ the correlations become,

$$Corr\left[\frac{1}{T}\ln\left(\frac{\tilde{A}_{it}}{A_{i0}}\right),\frac{1}{T}\ln\left(\frac{\tilde{A}_{jt}}{A_{j0}}\right)\right]=\frac{\sigma_M^2}{\sigma_M^2+\bar{\sigma}^2}\quad \forall i,j.$$

Asymptotic Portfolio

- Consider a portfolio composed on an infinite number of positions in instruments, debt or equity, with identical risk characteristics
- The only thing that differs among positions is the realizations of their individual idiosyncratic risk factors
- In a very large portfolio, idiosyncratic risk realizations will diversify, and the portfolio returns will converge to a distribution that only depends on the risk of the shared common factor
- It is possible to derive the closed form of this asymptotic return distribution and use this to derive closed form expressions for asymptotic portfolio capital requirements

Asymptotic Portfolio Market Risk Distribution

$$\theta(z_m) = e^{\left(r_f + \lambda\sigma_m - \frac{\sigma^2}{2}\right)T + \sigma_m z_m \sqrt{T}}, \sigma^2 = \sigma_m^2 + \sigma_i^2$$

$$\omega(z_m, A_0, par) = \frac{Ln\left(\frac{par}{A_0}\right) - Ln(\theta(z_m))}{\sigma_i \sqrt{T}},$$

$$c_1 = \frac{Ln\left(\frac{A_0}{par}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, c_2 = c_1 - \sigma\sqrt{T}$$

$$R_T(\tilde{z}_m) = \frac{A_0\theta(\tilde{z}_m)e^{\frac{T\sigma^2}{2}}\Phi\left(-\omega(\tilde{z}_m, A_0, par) + \sigma_i\sqrt{T}\right) - par\ \Phi(-\omega(\tilde{z}_m, A_0, par))}{A_0\Phi(c_1) - e^{-r_f T}par\ \Phi(c_2)} - 1,$$

$$\tilde{z}_m \sim \phi$$

Asymptotic Portfolio Credit Risk Distribution

$$\theta(z_m) = e^{\left(r_f + \lambda\sigma_m - \frac{\sigma^2}{2}\right)T + \sigma_m z_m \sqrt{T}}, \sigma^2 = \sigma_m^2 + \sigma_i^2$$

$$\omega(z_m, A_0, par) = \frac{Ln\left(\frac{par}{A_0}\right) - Ln(\theta(z_m))}{\sigma_i \sqrt{T}},$$

$$c_1 = \frac{Ln\left(\frac{A_0}{par}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, c_2 = c_1 - \sigma\sqrt{T}$$

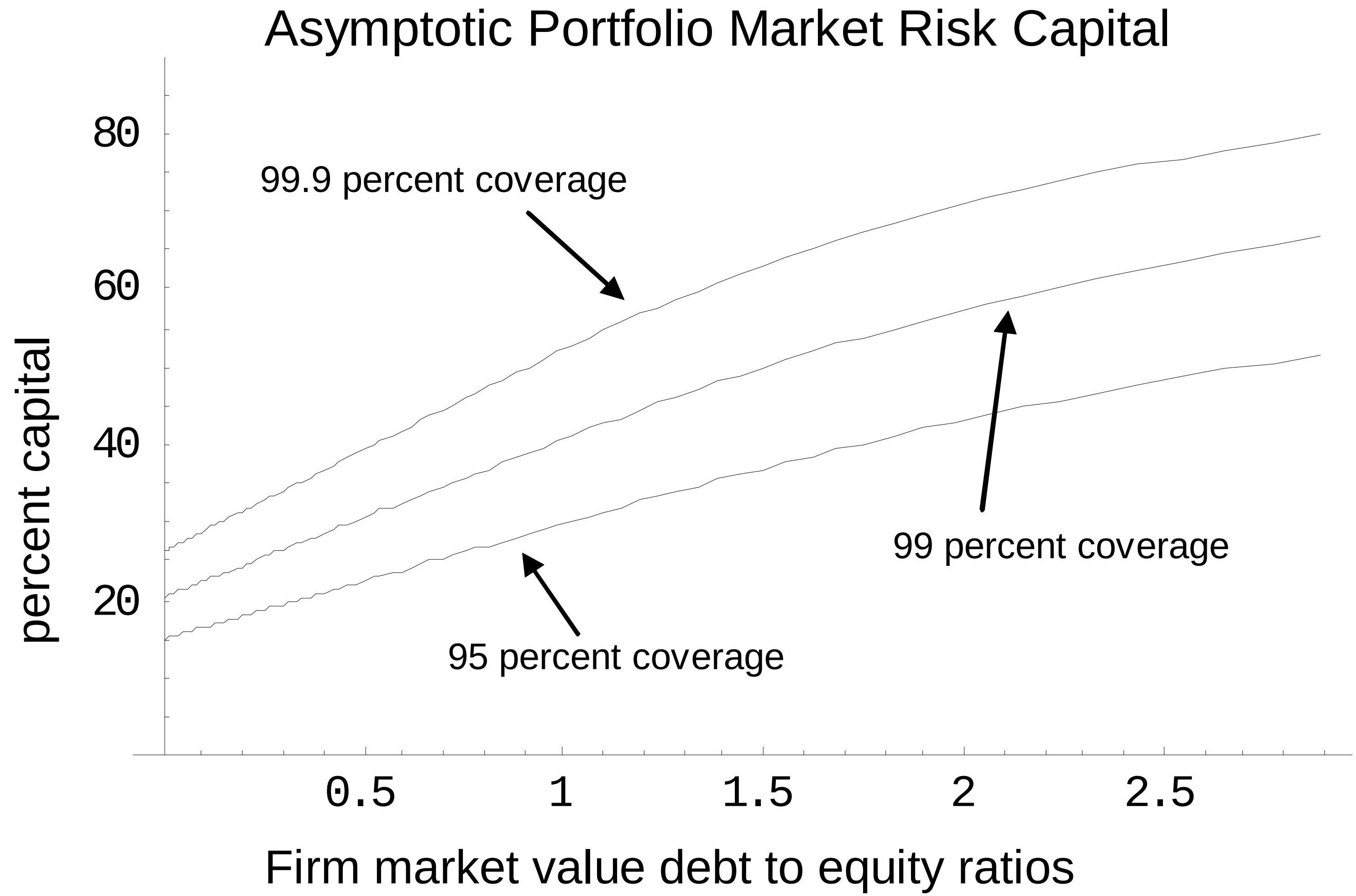
$$R_T(\tilde{z}_m) = \frac{A_0\theta(\tilde{z}_m)e^{\frac{T\sigma^2}{2}}\Phi\left(\omega(\tilde{z}_m, A_0, par) - \sigma_i\sqrt{T}\right) + par\,\Phi(\omega(\tilde{z}_m, A_0, par))}{-e^{-r_f T} par\,\Phi(c_2) - A_0\Phi(-c_1)} - 1,$$

$$\tilde{z}_m \sim \phi$$

Economic Capital

- Use Asymptotic Return Distribution to Identify the par value of funding debt that is consistent with the target solvency rate
- Use risk neutralized asymptotic return distribution to price the funding debt
- Closed-form solution (with an integral) exists for both capital allocation solutions

A_0=100, r_f=.05, mkt vol.=.10, specific vol.=20, corr=.20, mkt price risk=.10

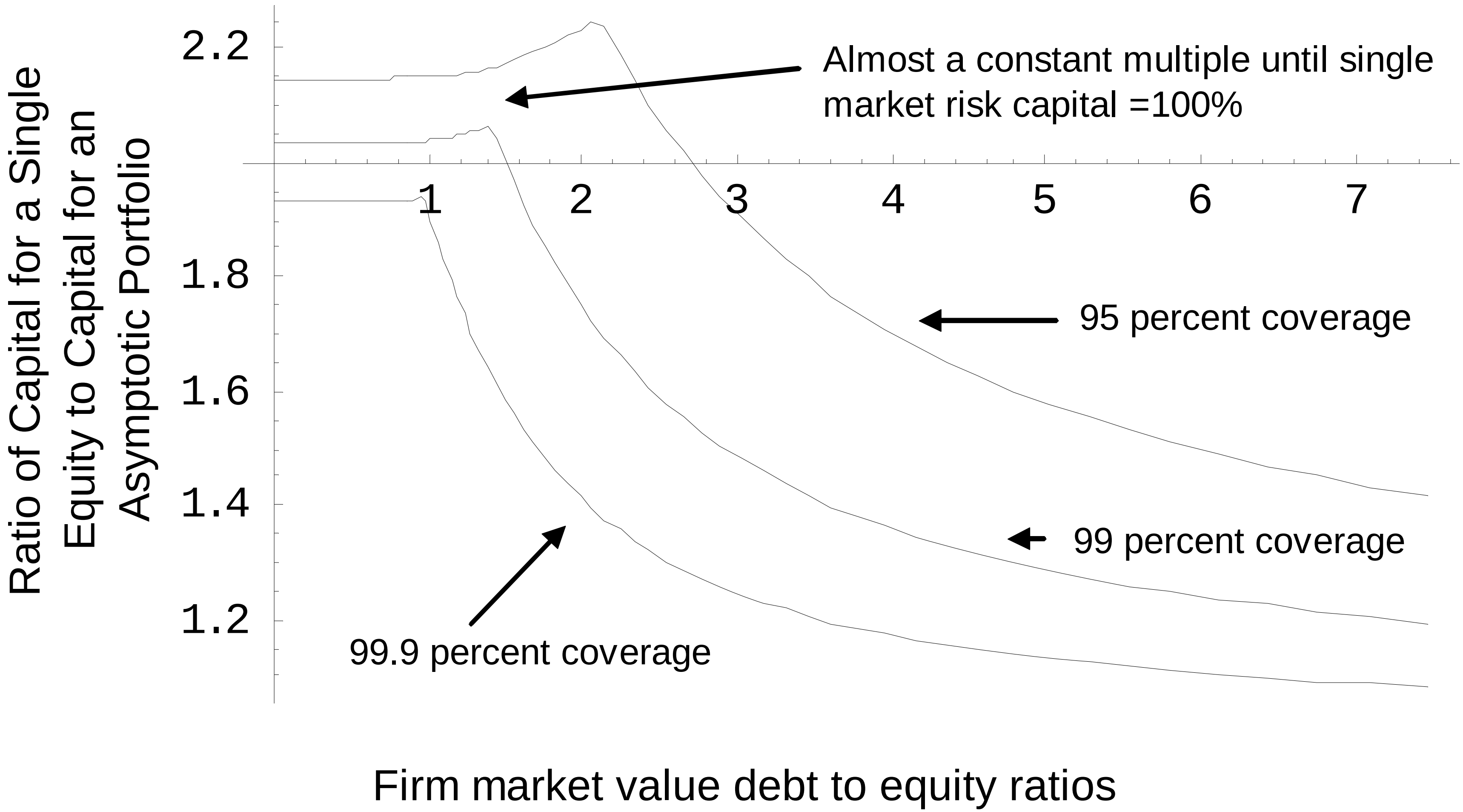


Importance of Specific Risks

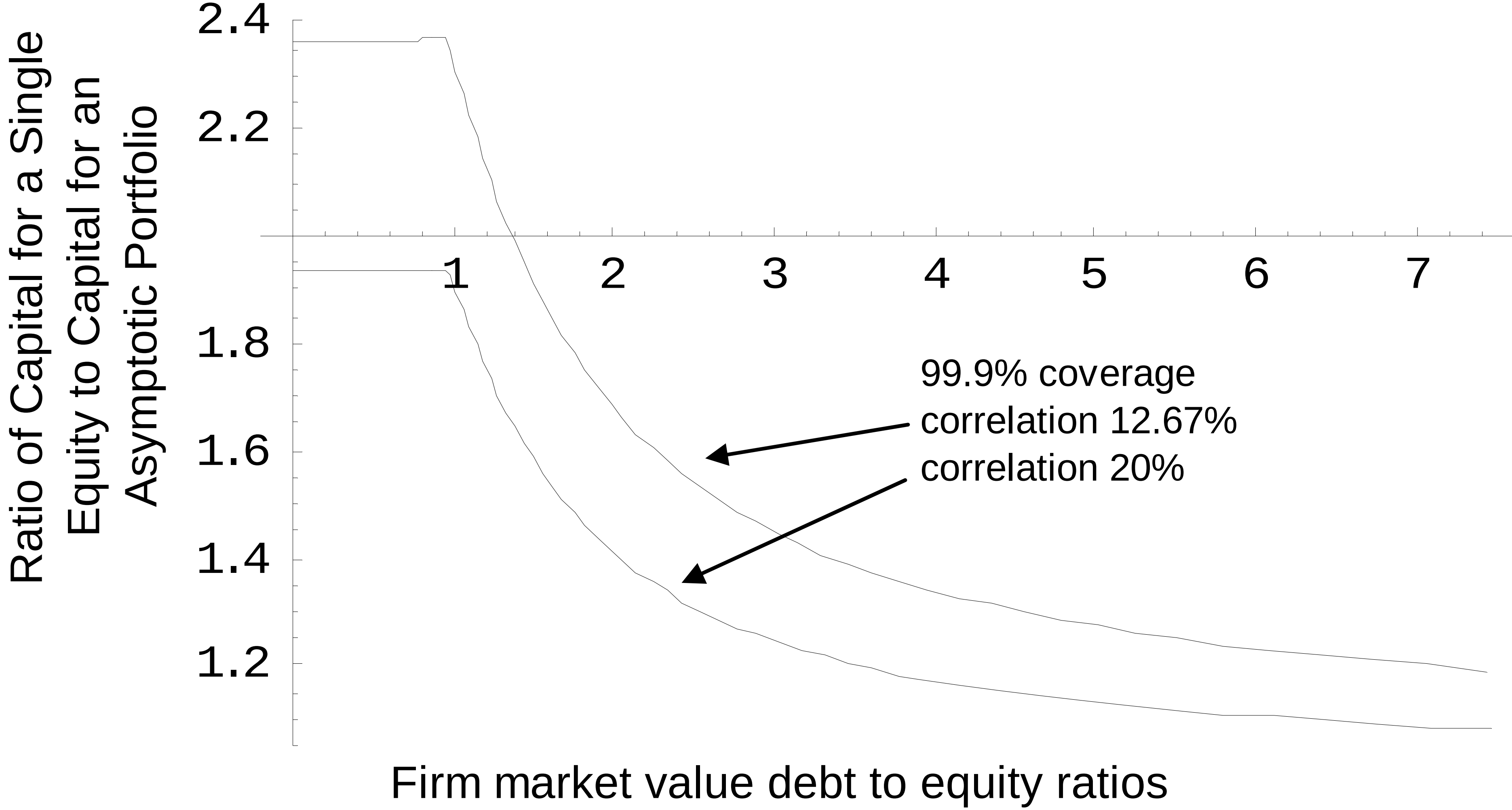
- The relationship between the capital needed for a single position versus the capital needed for a well-diversified portfolio tells us how important specific risk might be for less-diversified portfolios

A_0=100, r_f=.05, mkt vol.=.10, specific vol.=20, corr=.20, mkt price risk=.10

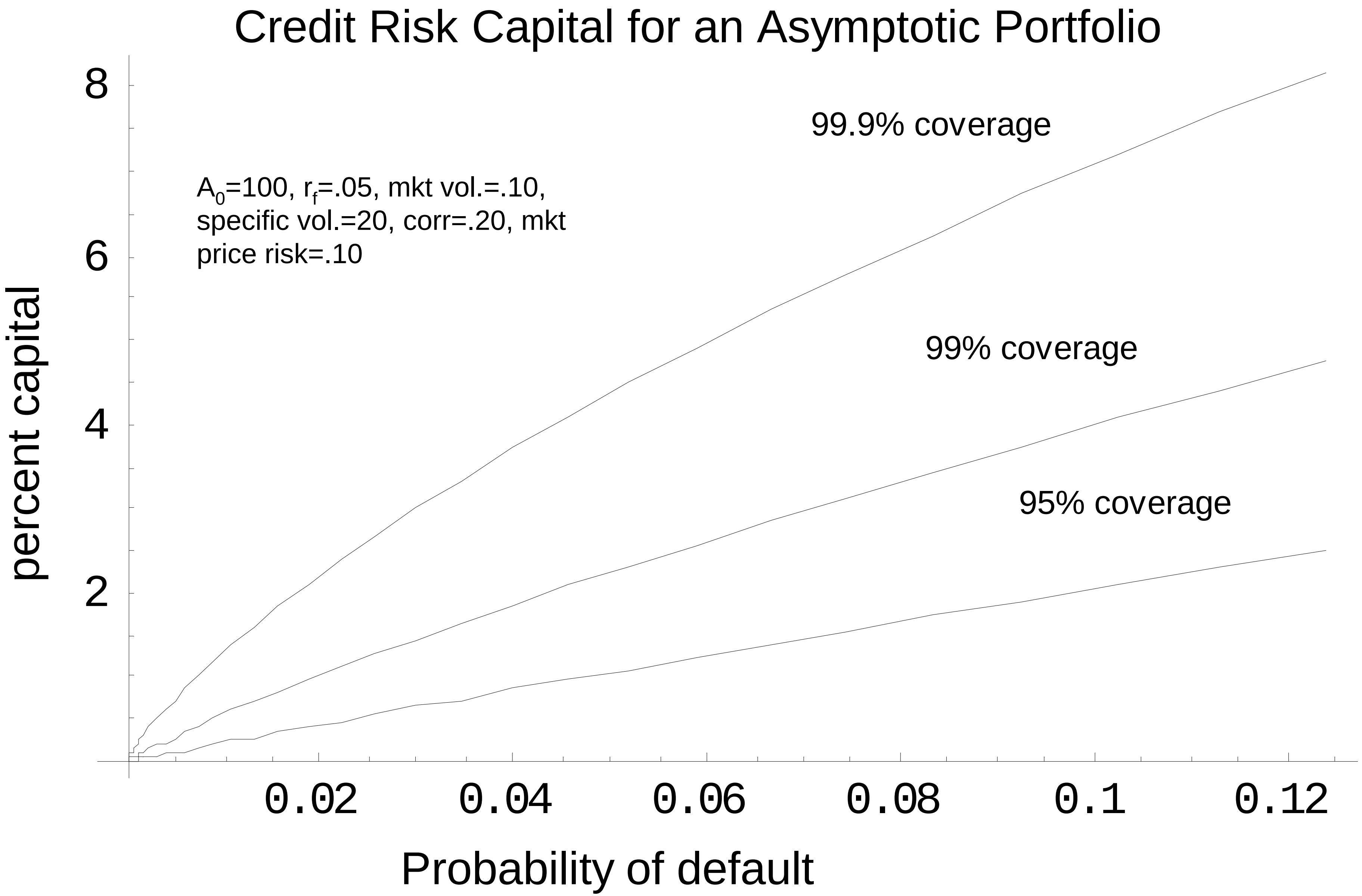
The Potential Importance of Specific Risk for Market Risk Capital

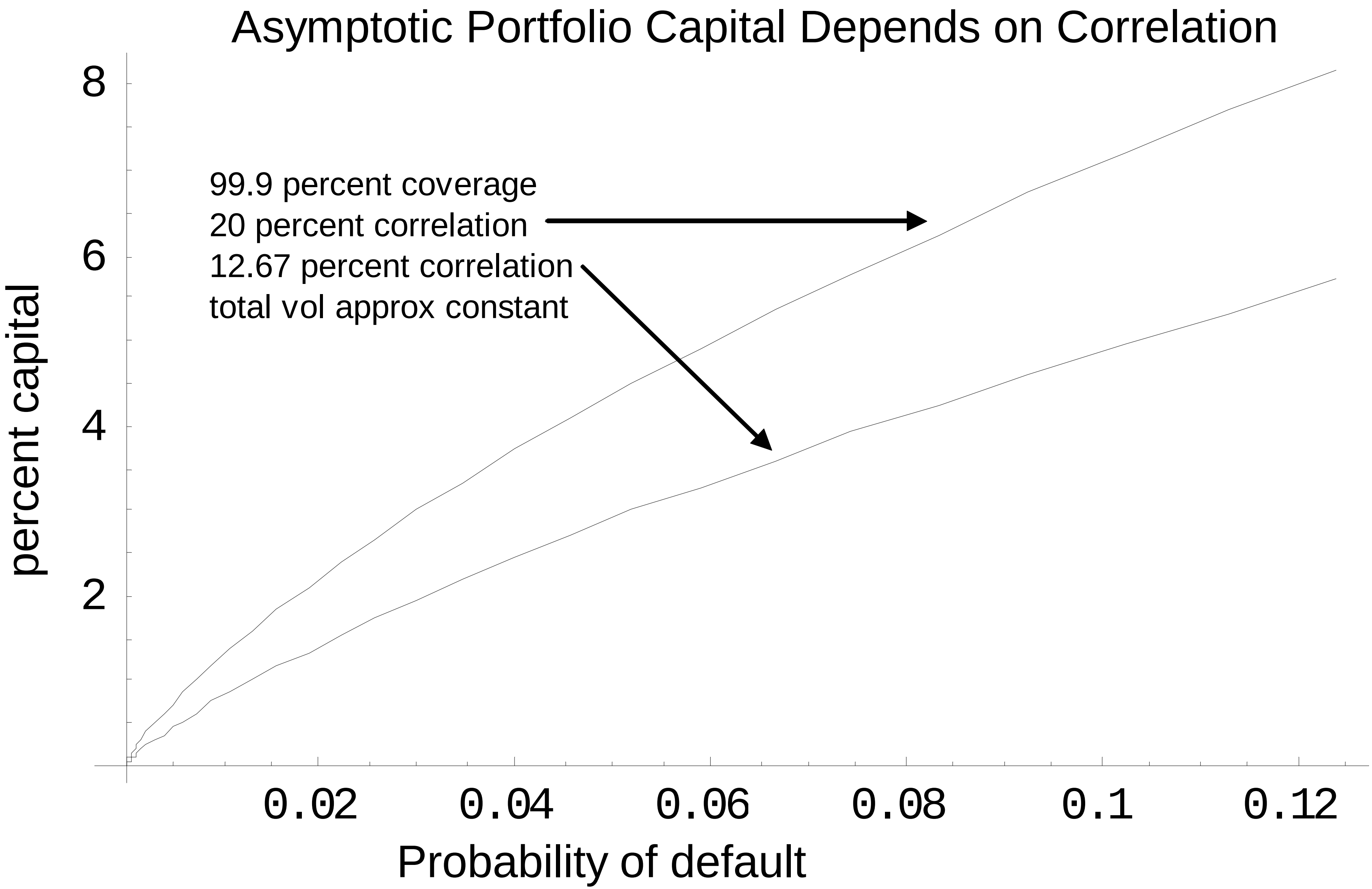


The Potential Importance of Specific Risk Depends on Market Risk Correlations

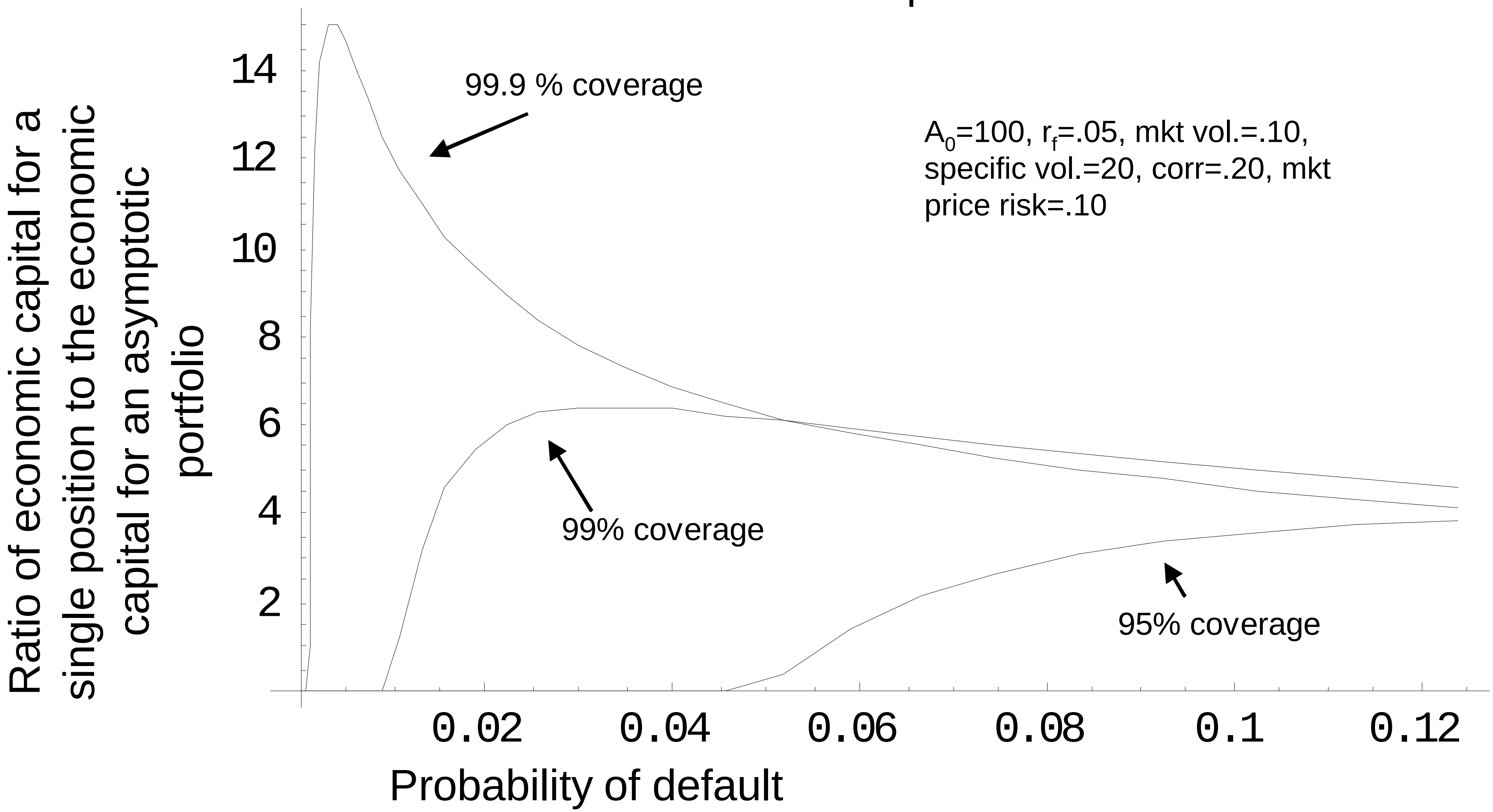


Credit Risk Capital for an Asymptotic Portfolio


Credit Risk Capital for an Asymptotic Portfolio
8
6
4
2
percent capital
99.9% coverage
99% coverage
95% coverage
A0=100, rf=.05, mkt vol.=.10,
specific vol.=20, corr=.20, mkt
price risk=.10
0.02
0.04
0.06
0.08
0.1
0.12
Probability of default


Asymptotic Portfolio Capital Depends on Correlation
99.9 percent coverage
20 percent correlation
12.67 percent correlation
total vol approx constant
percent capital
8
6
4
2
0.02
0.04
0.06
0.08
0.1
0.12
Probability of default

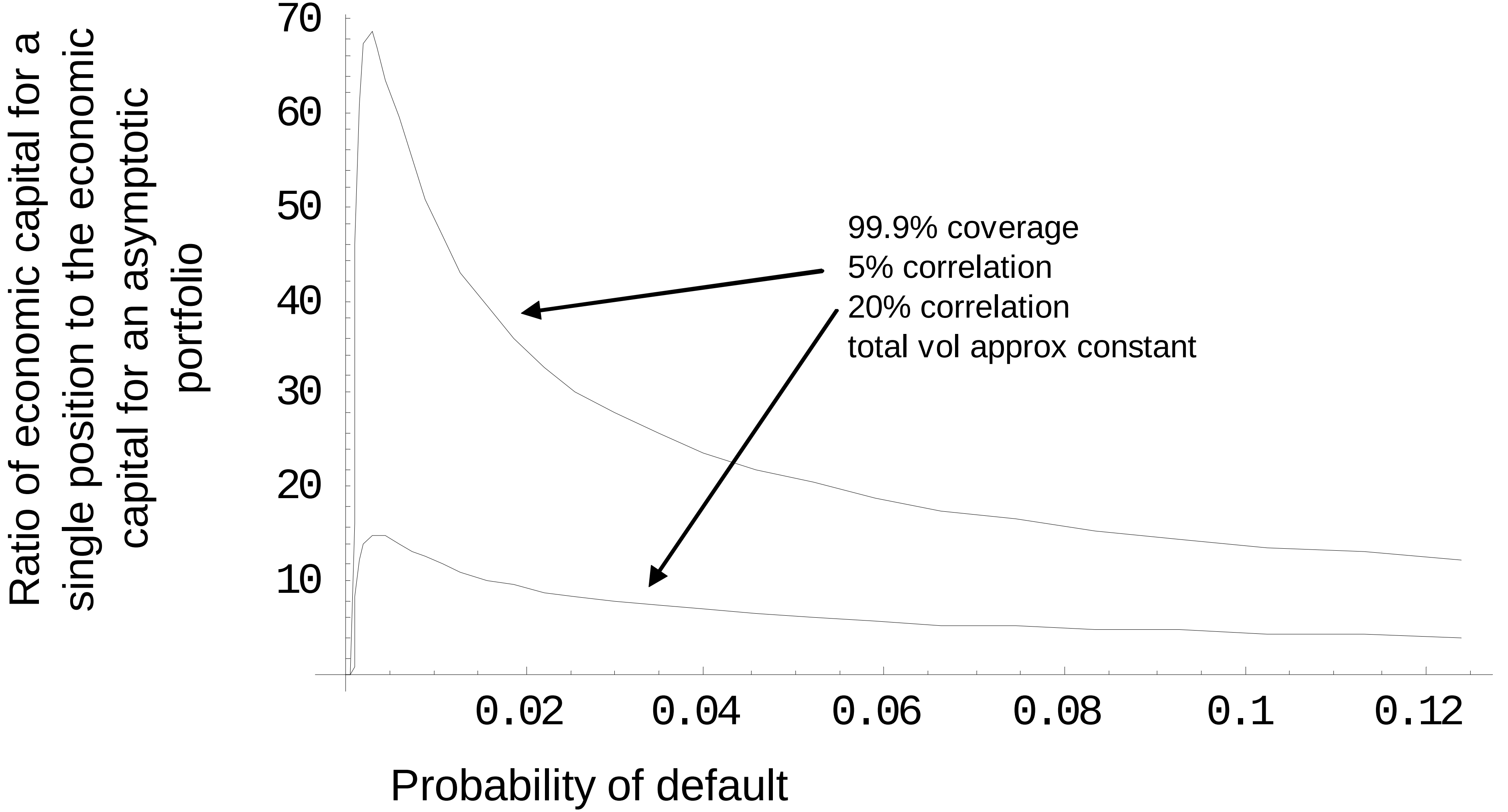
The Potential Importance of Specific Risk for Credit Portfolio Capital Alloctions



Specific Risk (Concentration Risk) is Really Important for Credit Risk Capital Allocation

Especially when targeting 99.9% solvency rate as in Basel II

Portfolio Correlation Characteristics Determine the Potential Importance of Specific Risk

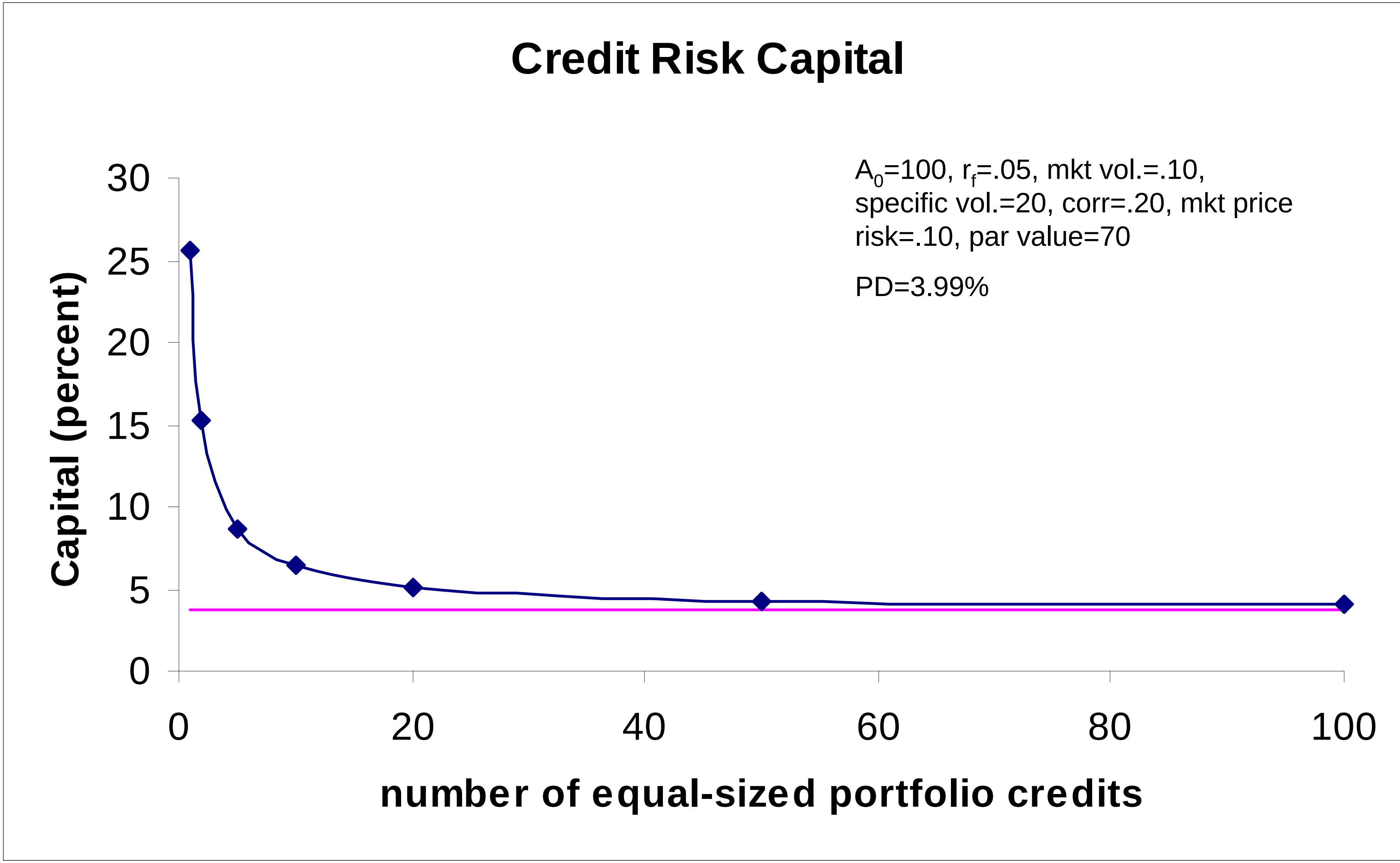


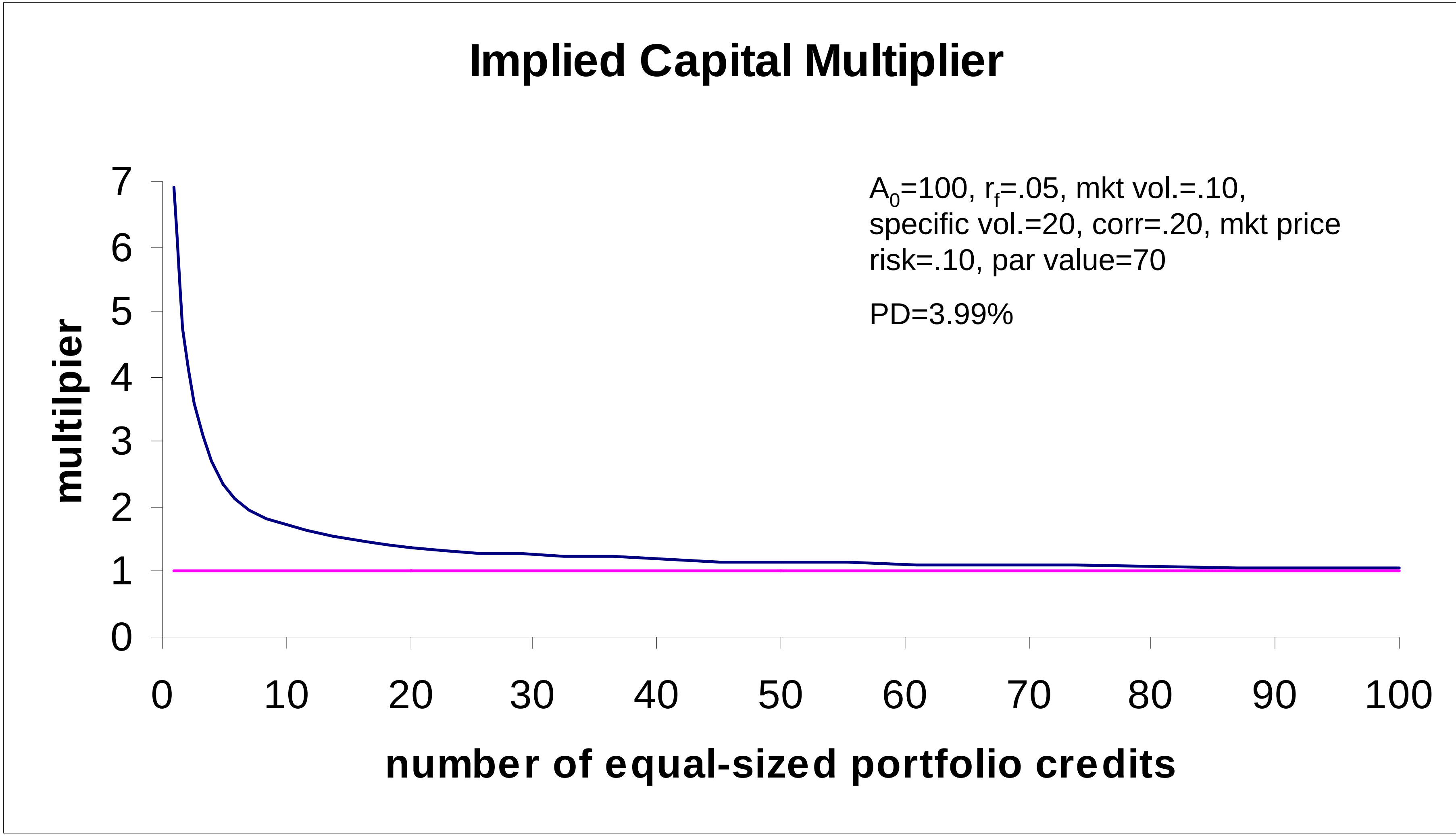
Concentration Risk and Basel II

- Concentration Risk Can be Very Important
  - Potentially most important for retail & mortgages where correlation assumption is very low
  - Important for less well populated bank PD-LGD buckets
- Specific (Concentration Risk) theoretically is much more important for credit than market risk
- **In practice, VaR models may not measure market risk very well, so specific risk is still very important in practice**

Credit Risk Capital Rate of Convergence

- Single credit ❾closed form capital
- Asymptotic portfolio❾ closed form capital
- How about portfolios w/ uniform credit size and number of credits in between 0 and Infinity?
  - No closed form for distribution
  - Monte Carlo techniques
  - Save discussion of technical Monte Carlo issues for another time……


Credit Risk Capital
A0=100, rf=.05, mkt vol.=.10,
specific vol.=20, corr=.20, mkt price
risk=.10, par value=70
PD=3.99%
Capital (percent)
30
25
20
15
10
5
0
0
20
40
60
80
100
number of equal-sized portfolio credits



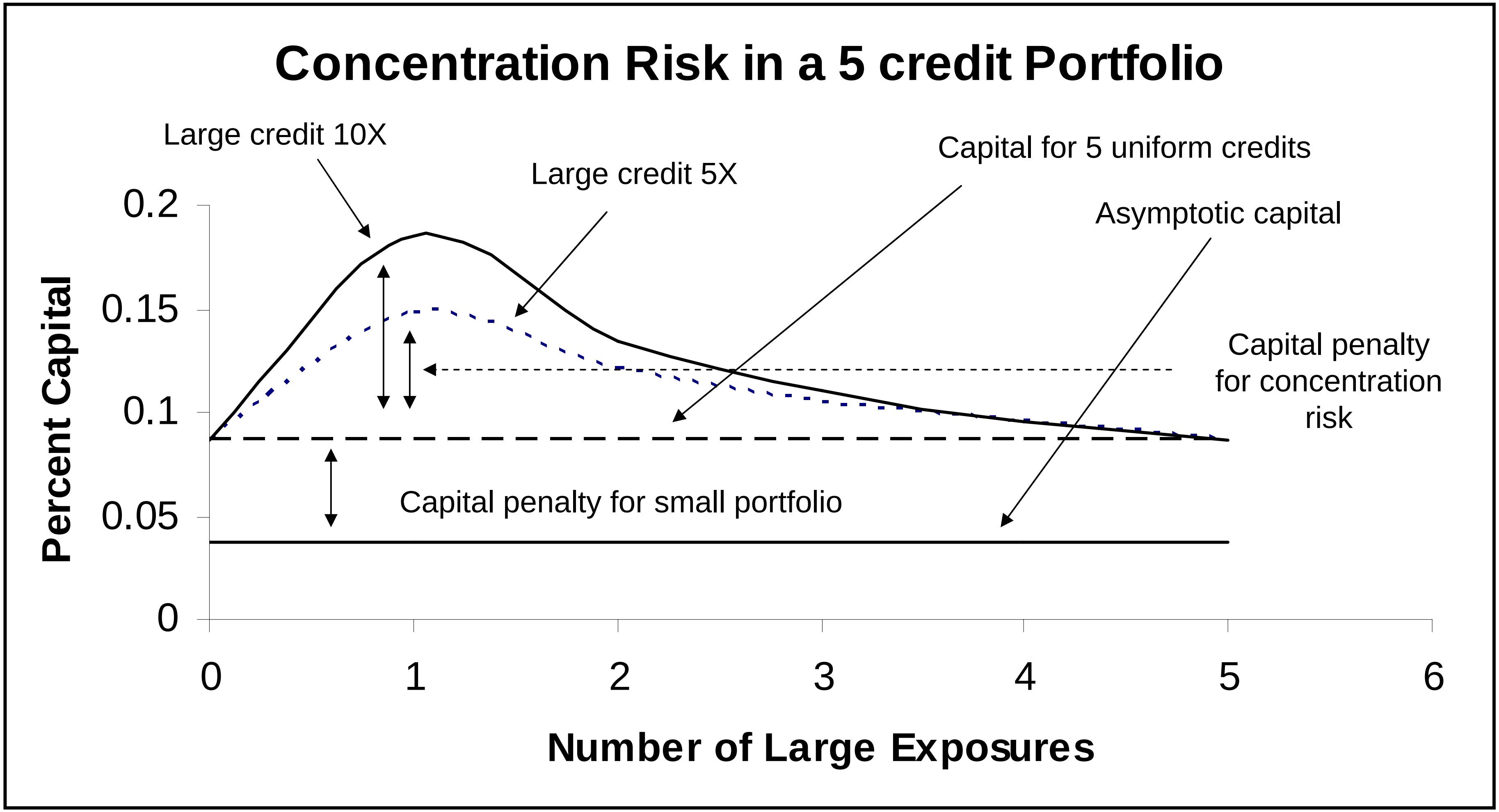
Implied Capital Multiplier
A_0=100, r_f=.05, mkt vol.=.10, specific vol.=20, corr=.20, mkt price risk=.10, par value=70
PD=3.99%
multilpier
0
1
2
3
4
5
6
7
0
10
20
30
40
50
60
70
80
90
100
number of equal-sized portfolio credits

Credit Concentration Risk Capital

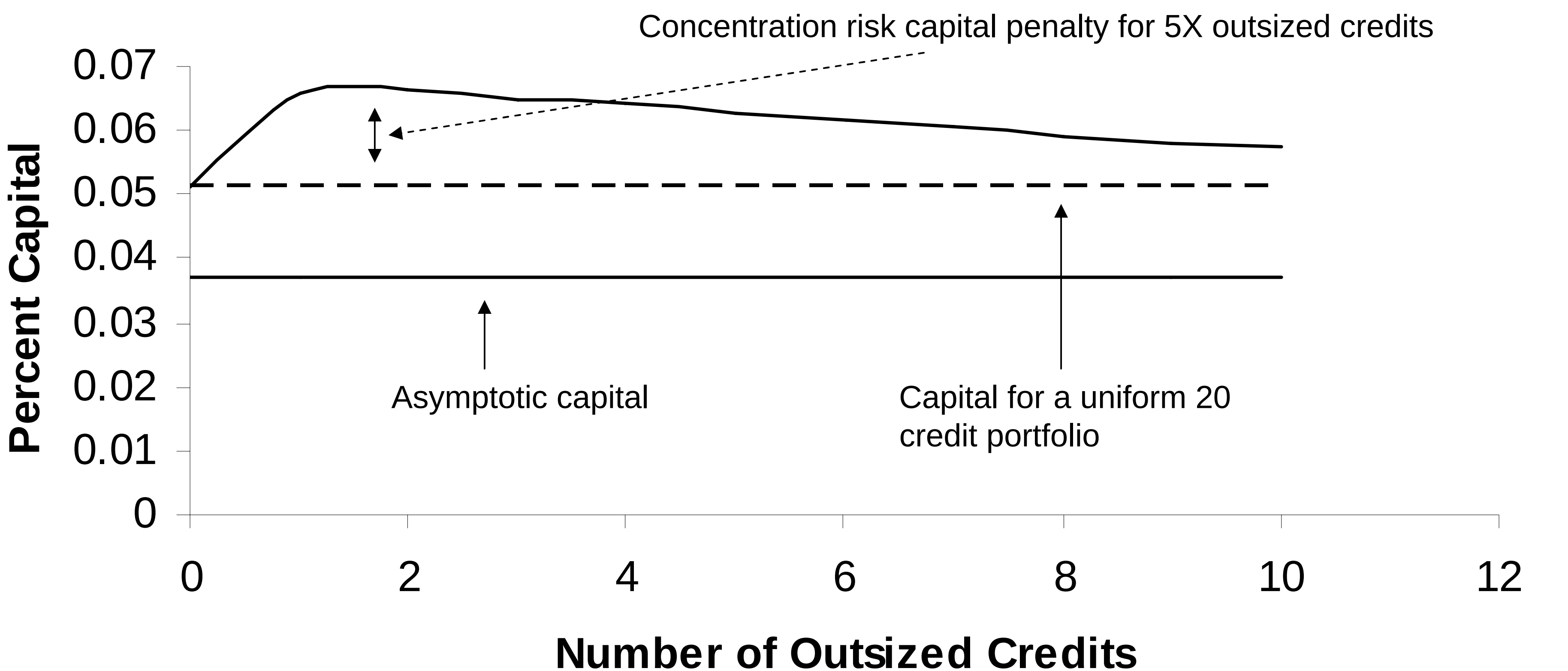
- Prior charts measure capital penalty for less than perfectly diversified portfolios
- Credits are still assumed uniform regarding size and credit risk characteristics
- What if some credits are outsized relative to the average credit?
  - Concentration risk capital

A_0=100, r_f=.05, mkt vol.=.10, specific vol.=20, corr=.20, mkt price risk=.10, par value=70

PD=3.99%



Concentration Risk Capital for 20 Credit Portfolio

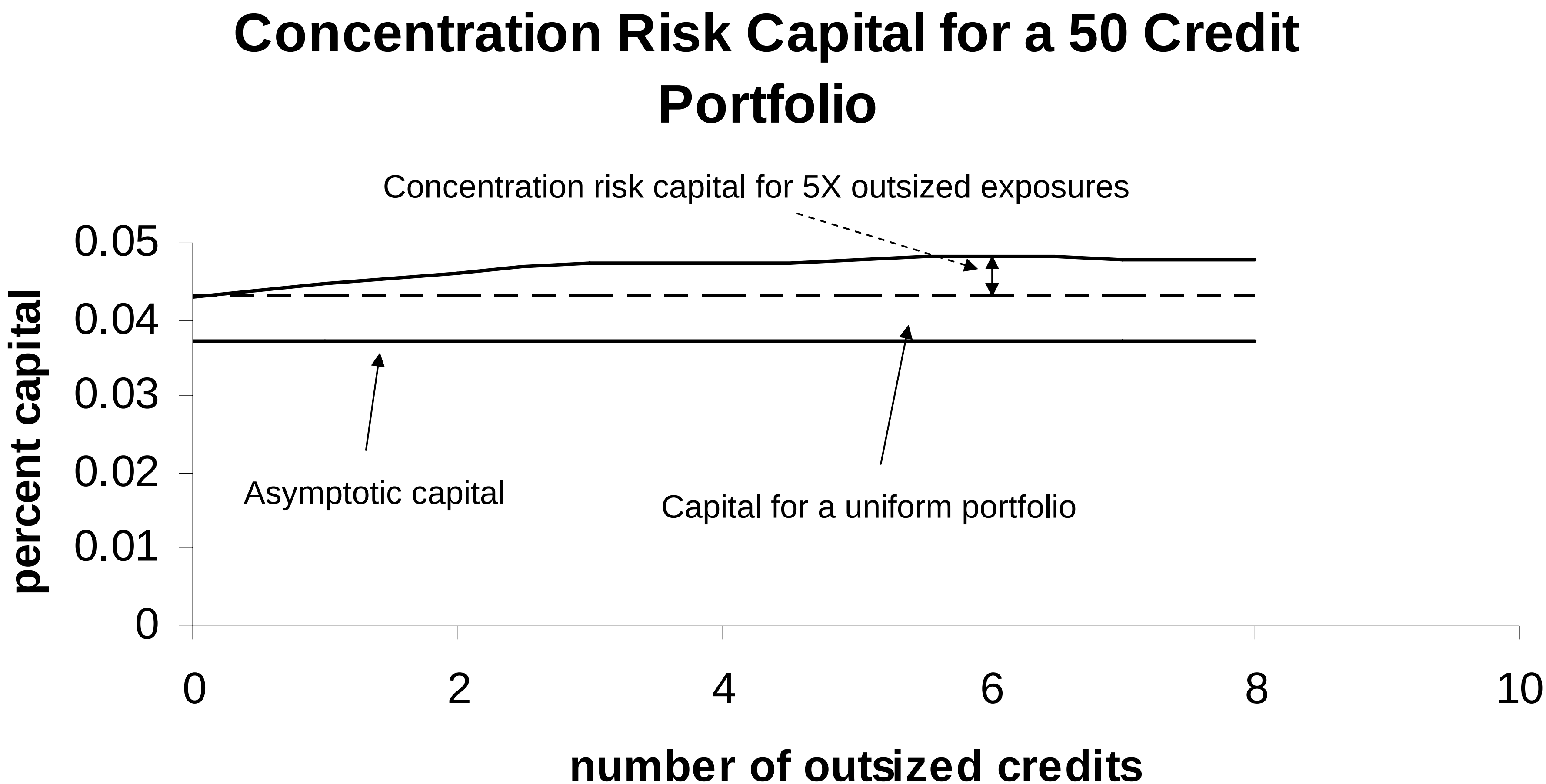


A_0=100, r_f=.05, mkt vol.=.10, specific vol.=20, corr=.20, mkt price risk=.10, par value=70

PD=3.99%

Concentration Risk Capital

- Depends on size of outsized credits
- Depends on Characteristics of remaining portfolio pool of credits
  - Number of credits
  - correlation


Concentration Risk Capital for a 50 Credit Portfolio
Concentration risk capital for 5X outsized exposures
0.05
0.04
0.03
0.02
0.01
0
percent capital
Asymptotic capital
Capital for a uniform portfolio
0
2
4
6
8
10
number of outsized credits

Integrated Market and Credit Risk

- Must use same future value measure
  - Returns
  - Payoffs
- Cannot mix payoffs and Losses
- Must use same time horizon
  - Basel mixes 10-day returns and 1-year losses
  - Nonsense approach

Asymptotic Market & Credit Economic Capital

  - If the market and credit portfolios are each fully diversified relative to idiosyncratic risk

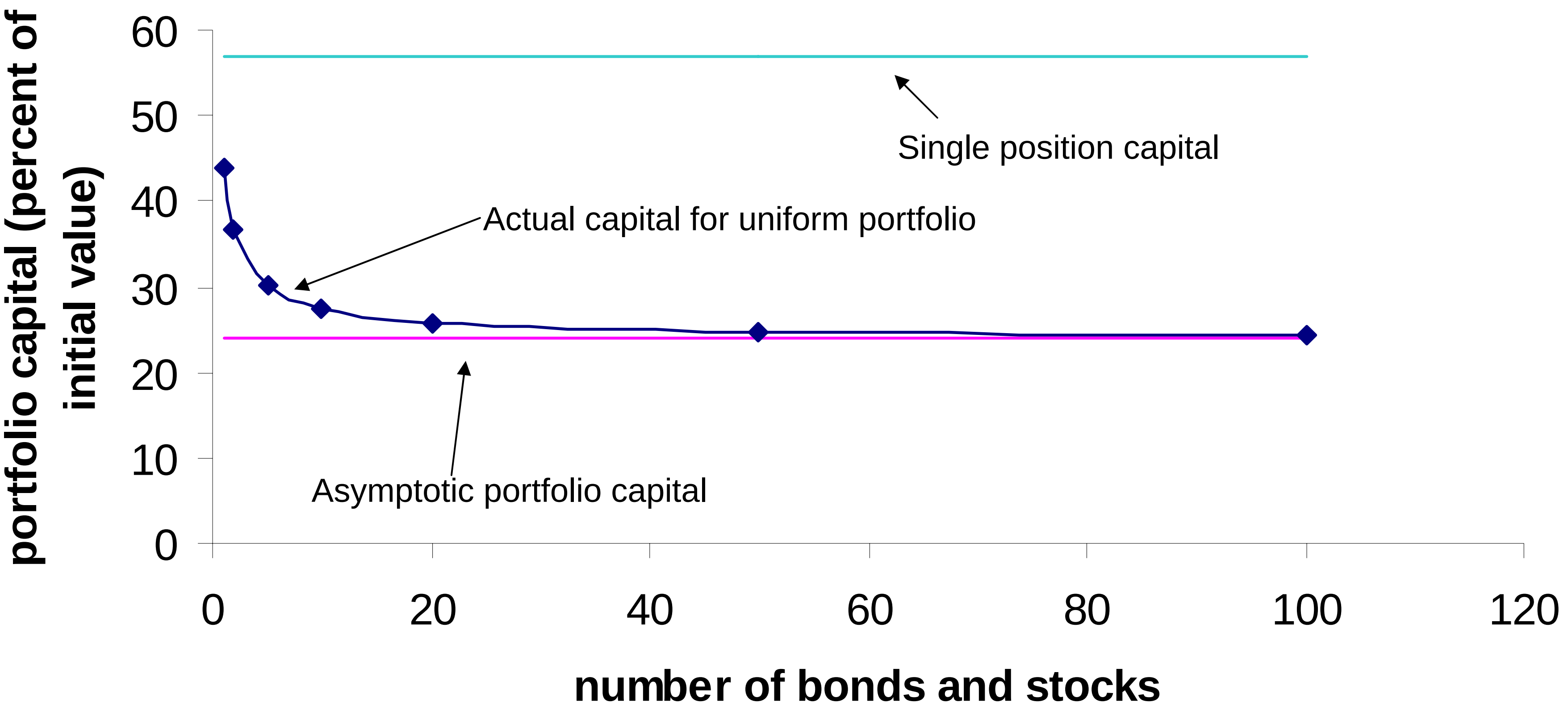
- Market and Credit Risk Asymptotic Capital Allocations are Additive
    - No diversification benefit available
    - Strict Proof in single common factor case
  - Closed Form Capital Allocation Formula
  - Intuition generalizes to multiple common factors

Less Than Fully Diversified Idiosyncratic Risk

- Diversification benefits available between market and credit risk
- No closed form distribution
- Must use Monte Carlo methods
- Lots of precision required to estimate 99.9% coverage

A_0=100, r_f=.05, mkt vol.=.10,
specific vol.=20, corr=.20,
mkt price risk=.10,
bond par value=70,
PD=3.99%,Stock par=50

Integrated Economic Capital



Convergence

- Close to asymptotic capital by about 50 bonds and 50 stocks of uniform size and respective equivalent risk characteristics
- Concentration risks may have a big capital cost

Basel II Issues

- Basel II assumes credit is already full diversified
- Built in bias toward additive nature of market and credit risk capital unless one has already applied a pillar II non-diversification and/or concentration risk capital add-on

Conclusions

- Equilibrium capital allocation model approach
- Expected convergence rates to asymptotic capital properties
- Consistent way to aggregate market and credit risks
- Consistent measure of the importance of concentration risks fro market and credit portfolios