A General Characterization of the Early Exercise Premium

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Abstract

Under the (weak) assumption of a Markovian underlying price process, an alternative and intuitive characterization of the early exercise premium is proposed. This new representation involves the first passage time density of the underlying spot price to the exercise boundary and is simply based on the observation that the discounted early exercise premium must be a martingale under the "risk-neutral" measure. The Markov property ensures analytical tractability since it enables the decomposition of the joint density between the first hitting time and the underlying asset price through the convolution of their marginal densities.

The analytical pricing solution proposed for American options is automatically consistent with the "value-matching" condition, is valid for any parameterization of the exercise boundary, and is shown to possess appropriate asymptotic properties. More important, such new valuation framework can be easily transposed from the standard geometric Brownian motion assumption to more general Markovian asset price processes, which can accommodate stochastic volatility and/or stochastic interest rates.

The optimal stopping time density is shown to satisfy a non-linear but one-dimensional integral equation. Using the algorithm suggested by Park and Schuurmann (1976), the first hitting time density of a geometric Brownian motion is obtained for any (time-dependent) specification of the early exercise boundary and tight lower bounds follow for the price of an American option. Several exercise boundary parametric specifications are tested and it is shown that, with only one parameter and at a higher computational speed, it is possible to achieve an accuracy comparable to a 15,000-step binomial tree. The extension to alternative Markovian diffusion processes is left for future research.

Key words: American options, Barrier options, First hitting time, Convolutions.

JEL Classification: G13.

1 Introduction

The inexistence of a closed-form pricing solution for the American put stems from the fact that the option price and the early exercise boundary must be determined simultaneously as the solution of the same free boundary problem set up by McKean (1965). Therefore, the vast literature on this subject, which is reviewed, for instance, in Baroni-Adesi (2005), has only proposed numerical solution methods as well as analytical approximations.

The numerical methods include the finite difference schemes introduced by Brennan and Schwartz (1977) and the binomial model of Cox, Ross and Rubinstein (1979). These methods are both simple and convergent, in the sense that accuracy can be improved by incrementing the number of time or state space steps. However, they are also too time consuming and do not provide the comparative statics attached to an analytical solution. On the other hand, and given the difficulty in finding first passage time densities, the optimal stopping approach initiated with Bensoussan (1984) and Karatzas (1988) has not also produced efficient pricing solutions.

One of the first quasi-analytical approximations is due to Baroni-Adesi and Whaley (1987), who use the quadratic method of MacMillan (1986). Despite its high efficiency and the accuracy improvements brought by subsequent extensions (see, for example, Ju and Zhong (1999)), this method is not convergent. Another non-convergent approach is proposed by Johnson (1983) and Broadie and Detemple (1996). They provide lower and upper bounds for American options, which are based on regression coefficients that are estimated through a time-demanding calibration to a large set of option contracts. Moreover, and as argued by Ju (1998, page 642), this econometric approach can generate less accurate hedging ratios, because the regression coefficients are only optimized for pricing purposes. More recently, Sullivan (2000) approximates the option value function through Chebyshev polynomials and employs a Gaussian quadrature integration scheme at each discrete exercise date. Although the speed and accuracy of the proposed numerical approximation can be enhanced via Richardson extrapolation, its convergence properties are still unknown.

Concerning the convergent pricing methodologies, Geske and Johnson (1984) approximates the American option price through an infinite series of multivariate normal distribution functions. Although the pricing accuracy can be increased as more terms are added, only the first few terms are considered and a Richardson extrapolation scheme is employed in order to reduce the computational burden. Another convergent method, which is also fast and accurate, is the randomization approach of Carr (1998), who also uses Richardson extrapolation. However, one of the main disadvantages of extrapolation schemes is the indetermination of the sign for the approximation error.

Kim (1990), Jacka (1991), Carr, Jarrow and Myneni (1992) and Jamshidian (1992) are in the genesis of the so-called "integral representation method", which provided an analytical representation of the early exercise premium through an integral equation. This approach was also used by Ju (1998) to derive fast and accurate approximate solutions that are based on a multipiece exponential representation of the early exercise boundary. Based on simpler parameterizations of the exercise boundary (which is assumed to be constant or of exponential type), Ingersoll (1998) and Sbuelz (2004) are able to decompose the American put price into a down-and-out European put and a non-deferrable rebate. Hence, they provide closed-form approximations that are fast to implement but not very accurate.

As argued by Carr (1998, page 616) and shown by the numerical experiments run by Broadie and Detemple (1996) and Ju (1998), the most efficient and accurate analytical pricing methods correspond to the econometric approach of Broadie and Detemple (1996), the randomization method of Carr (1998), and the multipiece exponential boundary approximation of Ju (1998). But, given the lower accuracy of the Broadie and Detemple (1996) method with respect to the computation of hedging ratios, the last two approaches seem to be the more promising ones until now. Notice, however, that all the studies already mentioned are based on the Black and Scholes (1973) geometric Brownian motion assumption, and most of them only differ in the specification adopted for the exercise boundary.

Based on the optimal stopping approach, the main purpose of this paper is to derive an alternative characterization of the early exercise premium that is valid for any continuous representation of the exercise boundary and for any Markovian stochastic process describing the dynamics of the underlying asset price. Using the Park and Schuurmann (1976) methodology, it is shown that the first passage time density can be easily recovered under the geometric Brownian motion assumption. Therefore, several parameterizations of the early exercise boundary are tested and new accurate approximations of the American put price are proposed.

Next sections are organized as follows. Based on the optimal stopping formulation of Jacka (1991), section 2 separates the American put into a non-deferrable rebate and an European down-and-out put. In section 3, such "barrier option approach" is shown to be equivalent to the usual decomposition between an European put and an early exercise premium. Moreover, an alternative quasi-analytical and more general characterization is offered for the early exercise premium, and its asymptotic properties are tested. Section 4 provides an efficient algorithm for the computation of a geometric Brownian motion first hitting density, which allows the comparison, in section 6, of the different specifications of the early exercise boundary discussed in section 5. Section 7 concludes.

2 The Barrier Option Approach

The valuation of American options will be pursued in the context of a stochastic intertemporal economy with continuous trading on the time-interval $[t_0, T]$, for some fixed time $T > t_0$, and where uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, \mathcal{Q})$. Throughout the paper, $\mathcal{Q}$ will denote the martingale probability measure obtained when the numeraire of the economy under analysis is taken to be a "money market account" B_t, whose dynamics are governed by the following ordinary differential equation:

$$dB_t = rB_t dt, \tag{1}$$

where r denotes the riskless interest rate, which is assumed to be constant.

Although the alternative representation of the early exercise premium that will be proposed in theorem 1 only requires the underlying asset price process S_t to be Markovian, the forthcoming empirical analysis will be based on the usual geometric Brownian motion assumption, i.e.

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma dW_t^{\mathcal{Q}}, \tag{2}$$

where q represents the dividend yield for the asset price, σ corresponds to the instantaneous volatility (per unit of time) of the asset returns and $W_t^{\mathcal{Q}} \in \Re$ is a standard Brownian motion, initialized at zero and generating the augmented, right continuous, and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$. The underlying asset can be thought as a stock, a stock index, an exchange rate or a financial future, as long as the parameter q is understood as, respectively, a dividend yield, an average dividend yield, the foreign default-free interest rate or the domestic risk-free interest rate.

Hereafter, the analysis will be focused on the valuation of an American put on the asset price S, with strike price K, and with maturity date T, whose time-t ($\leq T$) value will be denoted by $P_t(S, K, T)$.[1] Since the American put can be exercised at any time during its life, it is well known

[1] The American call option can be valued in a similar fashion or, under the geometric Brownian motion assumption, using the parity result derived by McDonald and Schroder (1998, equation 1).

-see, for example, Karatzas (1988, theorem 5.4) or Shreve (2004, equation 8.4.1)- that its price is the solution to an optimal stopping problem, i.e.:

$$P_{t_0}(S,K,T) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathcal{Q}} \left\{ e^{-r[(T\wedge\tau)-t_0]} (K - S_{T\wedge\tau})^+ \middle| \mathcal{F}_{t_0} \right\}, \tag{3}$$

where $\mathcal{T}$ is the set of all stopping times for the filtration $\mathbb{F}$ generated by the underlying price process and taking values in $[t_0, \infty]$, while $\mathbb{E}_{\mathcal{Q}}(X|\mathcal{F}_t)$ denotes the expected value of the random variable X, conditional on $\mathcal{F}_t$, and computed under the equivalent martingale measure $\mathcal{Q}$.[2]

Following, for instance, Carr et al. (1992, equations 1.2 and 1.3), for each time $t \in [t_0, T]$ there exists a *critical asset price* E_t bellow which the American put price equals its intrinsic value and, therefore, early exercise should occur. That is

$$P_t(S,K,T) = (K - S_t)^+ \text{ if } S_t \leq E_t, \tag{4}$$

and

$$P_t(S,K,T) > (K - S_t)^+ \text{ if } S_t > E_t. \tag{5}$$

Consequently, Jacka (1991) argues that the optimal exercise policy should be to exercise the American put option the first time the underlying asset price falls to its critical level. Representing by

$$\tau_e := \inf\{u \geq t_0 : S_u = E_u\} \tag{6}$$

the first passage time of the underlying asset price to its moving boundary and considering that the American option is still alive at the valuation date (i.e. $S_{t_0} > E_{t_0}$), equation (3) can then be restated as:[3]

$$\begin{aligned} P_{t_0}(S,K,T) &= \mathbb{E}_{\mathcal{Q}} \left\{ e^{-r[(T\wedge\tau_e)-t_0]} (K - S_{T\wedge\tau_e})^+ \middle| \mathcal{F}_{t_0} \right\} \\ &= \mathbb{E}_{\mathcal{Q}} \left[e^{-r(\tau_e - t_0)} (K - E_{\tau_e}) \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad + e^{-r(T-t_0)} \mathbb{E}_{\mathcal{Q}} \left[(K - S_T)^+ \mathbf{1}_{\{\tau_e \geq T\}} \middle| \mathcal{F}_{t_0} \right], \end{aligned} \tag{7}$$

where the first line of equation (7) follows from identity (6). Notice that $K \geq E_{\tau_e}$ since Van Moerbeke (1976) has shown that the exercise boundary is bounded from above by $\min\left(K, \frac{r}{q}K\right)$.

Equation (7) decomposes the American put into two components. The first one corresponds to the present value of a non-deferrable (and, in general, also non-constant) rebate $(K - E_{\tau_e})$, payable on the optimal stopping time τ_e. The second component is simply the time-t_0 price of an European down-and-out put on the asset S, with strike price K, maturity date at time T and (time-dependent) barrier levels $\{E_t, t_0 \leq t \leq T\}$. Assuming a convenient parametric specification for the barrier function E_t, it is possible to convert equation (7) into a closed-form solution. Such an approach was pursued, for instance, by Ingersoll (1998), using both constant and exponential specifications, and by Sbuelz (2004), also under a constant barrier formulation.

Unfortunately, the time path $\{E_t, t_0 \leq t \leq T\}$ of critical asset prices at which early exercise occurs, which is called the *exercise boundary,* is not known ex ante and, therefore, the assumption of

[2]Similarly, $\mathcal{Q}(\omega|\mathcal{F}_t)$ will represent the probability of event ω, conditional on $\mathcal{F}_t$, and computed under the probability measure $\mathcal{Q}$.

[3]Next formulae make use of the indicator function, which is defined as:

$$\mathbf{1}_{\{\omega\in\Omega\}} = \begin{cases} 1 \Longleftarrow \omega \in \Omega \\ 0 \Longleftarrow \omega \notin \Omega \end{cases}.$$

a specific parametric form for the barrier function simply transforms equation (7) into a lower bound for the true American put option value. Instead, this paper proposes an alternative characterization of the American put price, which is valid for any specification of the exercise boundary. Moreover, by using a more general class of functional forms for the barrier level E_t, it will be possible to obtain tighter lower bounds for the American option price.

3 The Early Exercise Premium

Similarly to Kim (1990), Jacka (1991), and Carr et al. (1992), the American put price can be divided into two components: the corresponding European put price and an *early exercise premium.* For this purpose, and because $\mathbf{1}_{\{\tau_e \geq T\}} = 1 - \mathbf{1}_{\{\tau_e < T\}}$, equation (7) can be rewritten as:

$$\begin{aligned} P_{t_0}(S,K,T) &= \mathbb{E}_{\mathcal{Q}}\left[e^{-r(\tau_e - t_0)} (K - E_{\tau_e}) \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad + e^{-r(T-t_0)} \mathbb{E}_{\mathcal{Q}}\left[(K - S_T)^+ \middle| \mathcal{F}_{t_0} \right] \\ &\quad - e^{-r(T-t_0)} \mathbb{E}_{\mathcal{Q}}\left[(K - S_T)^+ \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right]. \end{aligned}$$

And, since

$$e^{-r(T-t_0)} \mathbb{E}_{\mathcal{Q}}\left[(K - S_T)^+ \middle| \mathcal{F}_{t_0} \right] := p_{t_0}(S,K,T) \tag{8}$$

can be understood (under a deterministic interest rate setting) as the time-t_0 price of the corresponding European put (with technical features identical to the ones of the American option under analysis), then

$$\begin{aligned} P_{t_0}(S,K,T) &= p_{t_0}(S,K,T) \\ &\quad + \mathbb{E}_{\mathcal{Q}}\left[e^{-r(\tau_e - t_0)} (K - E_{\tau_e}) \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right] \\ &\quad - e^{-r(T-t_0)} \mathbb{E}_{\mathcal{Q}}\left[(K - S_T)^+ \mathbf{1}_{\{\tau_e < T\}} \middle| \mathcal{F}_{t_0} \right]. \end{aligned} \tag{9}$$

The last two terms on the right-hand-side of equation (9) correspond to the early exercise premium, for which a quasi-analytical solution will be proposed in the next lines.

3.1 An Alternative Characterization

The next theorem provides the main result of the paper.

Theorem 1 *Assuming that the underlying asset price process S_t is Markovian and that the interest rate r is constant, the time-t_0 value of an American put $P_{t_0}(S,K,T)$ on the asset price S, with strike price K, and with maturity date T can be decomposed into the corresponding European put price $p_{t_0}(S,K,T)$ and the early exercise premium $eep_{t_0}(S,K,T)$, i.e.*

$$P_{t_0}(S,K,T) = p_{t_0}(S,K,T) + eep_{t_0}(S,K,T), \tag{10}$$

with

$$eep_{t_0}(S,K,T) := \int_{t_0}^{T} e^{-r(u-t_0)} \left[(K - E_u) - p_u(E,K,T) \right] \mathcal{Q}\left(\tau_e \in du \middle| \mathcal{F}_{t_0} \right), \tag{11}$$

and where $\mathcal{Q}(\tau_e \in du | \mathcal{F}_{t_0})$ represents the probability density function of the first passage time τ_e, as defined by equation (6).

Proof. Noticing that the only random variable contained in the second term on the right-hand-side of equation (9) is the first passage time, then

$$\mathbb{E}_{\mathcal{Q}}\left[e^{-r(\tau_e - t_0)}\left(K - E_{\tau_e}\right)\mathbf{1}_{\{\tau_e < T\}}\middle|\,\mathcal{F}_{t_0}\right] = \int_{t_0}^{T} e^{-r(u-t_0)}\left(K - E_u\right)\mathcal{Q}\left(\tau_e \in du \middle|\, \mathcal{F}_{t_0}\right). \tag{12}$$

Concerning the third term on the right-hand-side of equation (9), it is necessary to consider the joint density of the two random variables involved: the first passage time τ_e and the terminal asset price S_T. Hence,

$$\mathbb{E}_{\mathcal{Q}}\left[\left(K - S_T\right)^+ \mathbf{1}_{\{\tau_e < T\}}\middle|\,\mathcal{F}_{t_0}\right] = \int_{\Re}\left(K - S\right)^+ \mathcal{Q}\left(S_T \in dS, \tau_e < T \middle|\, \mathcal{F}_{t_0}\right), \tag{13}$$

where the integration can be restricted to the domain $\Re_+$ if the geometric Brownian motion assumption is imposed. Because the underlying asset price is assumed to be a Markov process, the joint density contained in equation (13) is simply the convolution between the density of the first passage time τ_e and the transition probability density function of the terminal asset price S_T:

$$\mathcal{Q}\left(S_T \in dS, \tau_e < T \middle|\, \mathcal{F}_{t_0}\right) = \int_{t_0}^{T} \mathcal{Q}\left(S_T \in dS \middle|\, S_u = E_u\right)\mathcal{Q}\left(\tau_e \in du \middle|\, \mathcal{F}_{t_0}\right). \tag{14}$$

Therefore, combining equations (13) and (14),

$$\begin{aligned}
& \mathbb{E}_{\mathcal{Q}}\left[\left(K - S_T\right)^+ \mathbf{1}_{\{\tau_e < T\}}\middle|\,\mathcal{F}_{t_0}\right] \\
= & \int_{t_0}^{T}\left[\int_{\Re}\left(K - S\right)^+ \mathcal{Q}\left(S_T \in dS \middle|\, S_u = E_u\right)\right]\mathcal{Q}\left(\tau_e \in du \middle|\, \mathcal{F}_{t_0}\right) \\
= & \int_{t_0}^{T} \mathbb{E}_{\mathcal{Q}}\left[\left(K - S_T\right)^+ \middle|\, S_u = E_u\right]\mathcal{Q}\left(\tau_e \in du \middle|\, \mathcal{F}_{t_0}\right). \qquad (15)
\end{aligned}$$

Moreover, considering equation (8) for $t_0 = u$ and $S_u = E_u$, the expectation contained in the right-hand-side of equation (15) can be expressed in terms of an European put price:

$$\mathbb{E}_{\mathcal{Q}}\left[\left(K - S_T\right)^+ \mathbf{1}_{\{\tau_e < T\}}\middle|\,\mathcal{F}_{t_0}\right] = \int_{t_0}^{T} e^{r(T-u)} p_u\left(E, K, T\right)\mathcal{Q}\left(\tau_e \in du \middle|\, \mathcal{F}_{t_0}\right). \tag{16}$$

Finally, combining equations (9), (12) and (16), the early exercise representation (11) follows. ■

Under the usual geometric Brownian motion assumption, the term $p_u\left(E, K, T\right)$ can be computed using the Merton (1973) formulae and, consequently, equation (11) yields a closed-form solution to the early exercise premium (modulo to the specification of the first passage time density). Notice, however, that the proof of theorem 1 only relies on the much weaker assumption of a Markovian asset price. That is, the early exercise representation (11) is still valid for other asset price processes beyond the standard stochastic differential equation (2).

The representation offered by theorem 1 is also amenable to an intuitive interpretation. Since[4]

$$\lim_{S \to E_u} P_u\left(S, K, T\right) := P_u\left(E, K, T\right) = K - E_u,$$

then equation (11) can be rewritten as

$$eep_{t_0}\left(S, K, T\right) = \int_{t_0}^{T} e^{-r(u-t_0)}\left[P_u\left(E, K, T\right) - p_u\left(E, K, T\right)\right]\mathcal{Q}\left(\tau_e \in du \middle|\, \mathcal{F}_{t_0}\right).$$

[4] See proposition 1.

Using equation (10), today's early exercise premium can now be easily understood as the discounted expectation of the early exercise premium at the first passage time:

$$eep_{t_0}(S,K,T) = \mathbb{E}_{\mathcal{Q}}\left[e^{-r(\tau_e - t_0)} eep_{\tau_e}(E,K,T)\, \mathbf{1}_{\{\tau_e<T\}} \middle| \mathcal{F}_{t_0} \right]. \tag{17}$$

That is, the discounted early exercise premium, being the difference of two option prices, is, as expected, a martingale under measure $\mathcal{Q}$.

Such an interpretation is substantially different from the one implicit in the characterization of the American put already offered by Kim (1990), Jacka (1991), Carr et al. (1992) and Jamshidian (1992). For all these authors, the early exercise premium corresponds to the compensation that the option' holder would require (in the *stopping region*) in order to postpone exercise until the maturity date. Under the geometric Brownian motion assumption and for some early exercise boundary specifications -see, for example, Ju (1998) for a multipiece exponential formulation- it is possible to obtain closed-form solutions for such early exercise representation. Alternatively, the new characterization offered by theorem 1 can be applied for any early exercise boundary specification and under any Markovian process for the underlying asset price.

3.2 Asymptotic properties

Before moving towards an explicit approximation of the American put price and in order to investigate its limits, the asymptotic properties of the early exercise representation (11) are first explored.

Proposition 1 *Under the assumptions of theorem 1, the early exercise premium and the American put value satisfy the following boundary conditions:*

$$\lim_{r\downarrow 0} eep_t(S,K,T) = 0, \tag{18}$$

$$P_T(S,K,T) = (K - S_T)^+, \tag{19}$$

$$\lim_{S\uparrow\infty} P_t(S,K,T) = 0, \tag{20}$$

and

$$\lim_{S\downarrow E_t} P_t(S,K,T) = K - E_t. \tag{21}$$

Proof. See appendix A. ■

In order to facilitate the comparison against the previous literature, the usual results follow once the geometric Brownian motion case is adopted.

Proposition 2 *Under the geometric Brownian motion assumption (2) the American put value function given by theorem 1 converges, in the limit, to the perpetual put formulae given by McKean (1965) or Merton (1973), and satisfies, for $S_t > E_t$ and $t \leq T$, the Black-Scholes partial differential equation*

$$\mathcal{L}P_t(S,K,T) = 0, \tag{22}$$

where $\mathcal{L}$ is the parabolic operator

$$\mathcal{L} := \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + (r-q)\, S\frac{\partial}{\partial S} - r + \frac{\partial}{\partial t}. \tag{23}$$

Proof. See appendix B. ■

The relevance of propositions 1 and 2 emerges from the fact that the American put price is, under the geometric Brownian motion assumption, the unique solution of the initial value problem represented by the partial differential equation (22) and by the boundary conditions (19) to (21) -see, for instance Jacka (1991, proposition 2.3.1). Moreover, according to equation (21) and contrary to the characterization offered by Kim (1990), Jacka (1991), Carr et al. (1992) and Jamshidian (1992), the American put representation contained in theorem 1 is automatically consistent with the so-called *value-matching condition* (no matter the specification adopted for the exercise boundary).

However, it is well known, at least since the analysis of McKean (1965), that in order to uniquely determine both the American put value and the exercise boundary, the initial value problem represented by equations (19) to (22) must be transformed into a larger free boundary problem through the inclusion of an additional *high contact condition*:

$$\lim_{S \downarrow E_t} \frac{\partial P_t(S, K, T)}{\partial S} = -1. \tag{24}$$

As with all previous early exercise representations, the general solution proposed in theorem 1 is not automatically consistent with equation (24) for all exercise boundary specifications. In order to incorporate equation (24) into the valuation problem, Ju (1998) restricted the optimal exercise boundary to a multipiece specification, which was determined by the iterative solution of successive value-match and high contact conditions. However, and as proposition 4 will reveal, it would be too time-consuming to apply the high contact condition to theorem 1, if no restriction is to be imposed to the exercise boundary.

4 The First Passage Time Density

To implement the new American put value representation offered by theorem 1, it is necessary to compute the first passage time density of the underlying asset price to the moving exercise boundary. Except for some crude critical asset price specifications, as for example the constant and exponential functional forms used by Ingersoll (1998), the optimal stopping time density is not known in closed-form. Following Kuan and Webber (2003), this section shows that such first passage time density can be efficiently computed, under the geometric Brownian motion assumption and for any exercise boundary specification, through the numerical method proposed by Park and Schuurmann (1976). For the sake of brevity, the extension to alternative Markovian diffusion processes is left for future research.

4.1 An Integral Equation Representation

Next proposition is based on Park and Schuurmann (1976, theorem 1) and provides a non-linear integral equation for the optimal stopping time density under consideration.

Proposition 3 *Under the assumptions of theorem 1, under the dynamics of equation (2), and considering that the optimal exercise boundary is a continuous function of time, the first passage time density of the underlying asset price to the moving exercise boundary is the implicit solution of*[5]

$$\int_{t_0}^{u} \Phi\left(\frac{E_v^z - E_u^z}{\sqrt{u-v}}\right) \mathcal{Q}\left(\tau_e \in dv \middle| \mathcal{F}_{t_0}\right) = \Phi\left(-\frac{E_u^z}{\sqrt{u-t_0}}\right), \tag{25}$$

[5] Actually, it would suffice to consider a "sectionally continuous" function, meaning that at each point s of discontinuity $E_s = \min\left(E_{s-}, E_{s+}\right)$.

for $u \in [t_0, T]$ *and where*

$$E_v^z := \frac{\ln\left(\frac{S_{t_0}}{E_v}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(v - t_0)}{\sigma}, \tag{26}$$

with $\Phi(\cdot)$ *representing the cumulative density function of the univariate standard normal distribution.*

Proof. Solving the stochastic differential equation (2), then

$$S_v = S_{t_0} \exp\left[\left(r - q - \frac{\sigma^2}{2}\right)(v - t_0) - \sigma Z_v^{\mathcal{Q}}\right],$$

where

$$Z_v^{\mathcal{Q}} := -\int_{t_0}^{u} dW_s^{\mathcal{Q}}$$

is still a canonical Brownian motion (under measure $\mathcal{Q}$). Therefore and using definition (26), the distribution of the first hitting time for the asset price can be written in terms of the previous Wiener process:

$$\mathcal{Q}(\tau_e \leq u | \mathcal{F}_{t_0}) = \mathcal{Q}\left[\sup_{t_0 \leq v < u}\left(Z_v^{\mathcal{Q}} - E_v^z\right) \geq 0 \middle| \mathcal{F}_{t_0}\right], \tag{27}$$

for $u \in [t_0, T]$.

Assuming that the (modified) exercise boundary E_v^z is continuous on $[t_0, u]$, then theorem 1 of Park and Schuurmann (1976) can be applied to equation (27), yielding the following integral equation:

$$\mathcal{Q}(\tau_e \leq u | \mathcal{F}_{t_0}) = \Phi\left(-\frac{E_u^z}{\sqrt{u - t_0}}\right) + \int_{t_0}^{u} \Phi\left(\frac{E_u^z - E_v^z}{\sqrt{u - v}}\right) \mathcal{Q}(\tau_e \in dv | \mathcal{F}_{t_0}). \tag{28}$$

Since $\mathcal{Q}(\tau_e \leq u | \mathcal{F}_{t_0}) = \int_{t_0}^{u} \mathcal{Q}(\tau_e \in dv | \mathcal{F}_{t_0})$ and attending to the symmetry of the normal distribution function, equation (25) follows immediately from equation (28). ■

4.2 The Park and Schuurmann (1976) Algorithm

Based on proposition 3, it is now possible to price the American put option, for any continuous specification of the exercise boundary and through the numerical solution of equations (11) and (25). For this purpose, next proposition summarizes the "standard partition" method proposed by Park and Schuurmann (1976).[6]

Proposition 4 *Under the assumptions of proposition 3 and dividing the time-interval* $[t_0, T]$ *into* N *sub-intervals of (equal) size* $h := \frac{T - t_0}{N}$*, then*

$$\begin{aligned} eep_{t_0}(S, K, T) \quad = \quad & \sum_{j=1}^{N} e^{-r\frac{(2j-1)h}{2}} \left[K - E_{t_0 + \frac{(2j-1)h}{2}} - p_{t_0 + \frac{(2j-1)h}{2}}(E, K, T)\right] \\ & \left[\mathcal{Q}(\tau_e = t_0 + jh) - \mathcal{Q}(\tau_e = t_0 + (j-1)h)\right], \end{aligned} \tag{29}$$

[6] For long option maturities, accuracy can be improved via the more time-consuming "variable step-size" approach suggested by Park and Schuurmann (1980). However, the numerical results presented in section 6 show that such an improvement is negligible even for time-to-maturities of twenty years.

where the probabilities $\mathcal{Q}(\tau_e = t_0 + jh)$ are obtained from the following recurrence relation:

$$\begin{aligned} & \mathcal{Q}(\tau_e = t_0 + kh) \qquad (30) \\ = \quad & \mathcal{Q}(\tau_e = t_0 + (k-1)h) + \left\{ \Phi\left[\frac{E^z_{t_0 + \frac{(2k-1)h}{2}} - E^z_{t_0+kh}}{\sqrt{\frac{h}{2}}} \right] \right\}^{-1} \left\{ \Phi\left(-\frac{E^z_{t_0+kh}}{\sqrt{kh}} \right) \right. \\ & \left. - \sum_{j=1}^{k-1} \Phi\left[\frac{E^z_{t_0 + \frac{(2j-1)h}{2}} - E^z_{t_0+kh}}{\sqrt{\frac{[2(k-j)+1]h}{2}}} \right] \left[\mathcal{Q}(\tau_e = t_0 + jh) - \mathcal{Q}(\tau_e = t_0 + (j-1)h) \right] \right\}, \end{aligned}$$

for $k = 1, \ldots, N$, and with $\mathcal{Q}(\tau_e = t_0) = 0$.

Proof. See appendix C. ■

5 Specification of the Exercise Boundary

The pricing solution offered by theorem 1 depends on the specification adopted for the exercise boundary $\{E_t, t_0 \leq t \leq T\}$. Although such optimal exercise policy is not known ex ante (i.e. before the solution of the pricing problem), its main characteristics have been already established in the literature: *i*) The exercise boundary is a continuous function of time -see, for instance, Jacka (1991, propositions 2.2.4 and 2.2.5); *ii*) E_t is a non-decreasing function of time t -see Jacka (1991, proposition 2.2.2); *iii*) The exercise boundary is bounded from above by $E_T = \min\left(K, \frac{r}{q}K\right)$ -as stated in Van Moerbeke (1976); and *iv*) $\lim_{t \uparrow \infty} E_t = E_\infty$, where E_∞ represents the (constant) critical asset price for the perpetual American put case.

As described by Ingersoll (1998, page 89), in order to price the American put it is necessary to chose a parametric family $\mathcal{E}$ of exercise policies $E_t(\underline{\theta})$, where each policy is characterized by a n-dimensional vector of parameters $\underline{\theta} \in \Re^n$. Then, the early exercise value (as given by equation (11)) is expressed as a function of $\underline{\theta}$, and maximized with respect to the parameters. Since the chosen family $\mathcal{E}$ may not contain the optimal exercise boundary, the resulting American put price constitutes a lower bound for the true option value.

Of course, the more general is the specification adopted for the exercise boundary the smaller should be the approximation error associated to the American put price estimate. Moreover, the chosen parametric family should, at least, satisfy the requirements (i)-(iv) described at the start of this section. However, the parametric families already proposed in the literature have been chosen not for their generality but because they provide fast analytical pricing solutions. In order to measure the accuracy improvement provided by more general families of exercise policies, section 6 will consider the following parametric families:

1. Constant exercise boundary:
$$E_t(\underline{\theta}) = \theta_1, \; \theta_1 > 0. \qquad (31)$$
This is the simplest specification one can adopt and was already used by Ingersoll (1998) and Sbuelz (2004). Although it yields a closed-form solution for equation (11), such exercise boundary can not simultaneously satisfy the previously stated features (iii) and (iv).

2. Exponential family:
$$E_t(\underline{\theta}) = \theta_1 \exp(\theta_2 t), \; \theta_1, \theta_2 > 0. \qquad (32)$$
This specification, already proposed by Ingersoll (1998), also yields an analytical solution for equation (11) but, again, can not simultaneously satisfy requirements (iii) and (iv).

3. Exponential-constant family:

$$E_t(\underline{\theta}) = \theta_1 + \exp(\theta_2 t), \ \theta_2 > 0. \tag{33}$$

This new parameterization corresponds to a simple modification of equation (32) and has never been proposed in the literature. Nevertheless, section 6 will show that it produces smaller pricing errors than equation (32), for the same number of parameters.

4. Polynomial family:

$$E_t(\underline{\theta}) = \sum_{i=1}^{n} \theta_i t^{i-1}. \tag{34}$$

Because the exercise boundary is assumed to be continuous and defined on the closed interval $[t_0, T]$, the Weierstrass approximation theorem implies that E_t can be uniformly approximated, for any desired accuracy level, by the polynomial (34). By increasing the degree of the polynomial (and, therefore, the number of parameters to be estimated), this new class of exercise policies allows the pricing error to be arbitrarily reduced. Section 6 will reveal that with only five parameters (that is, a polynomial of degree 4) it is possible to obtain smaller pricing errors than with the alternative specifications already proposed in the literature.

5. CJM family:

$$E_t(\underline{\theta}) = \min\left(K, \frac{r}{q}K\right)\exp\left(-\theta_1\sqrt{T-t}\right) + E_\infty\left[1 - \exp\left(-\theta_1\sqrt{T-t}\right)\right], \ \theta_1 \geq 0. \tag{35}$$

Equation (35) corresponds to an exponentially weighted average between the upper bound and the perpetual limit of the exercise boundary, and fulfills all the requirements (i)-(iv). Such specification was proposed by Carr et al. (1992, page 93) but has never been tested since it does not yield an analytical solution for the American put price. Next section will show that, with only one parameter, the magnitude of the pricing errors produced by this specification is similar to the one associated to the best parameterizations already available in the literature.

6 Numerical Results

In order to test the influence of the exercise boundary specification on the early exercise value, all the parametric families described in section 5 will be compared for different constellations of the pricing model coefficients contained in equation (2). For this purpose, the maximization of the early exercise value (with respect to the parameters defining the exercise policy) will be implemented through the Powell's method, as described in Press, Flannery, Teukolsky and Vetterling (1994, section 10.5). This method only requires evaluations of the function to be maximized and, therefore, it is faster than a *conjugate gradient* or a *quasi-Newton* algorithm. Nevertheless, it is always possible to use a more robust optimization method that also requires evaluations of the derivatives of the function to be maximized, because the derivatives of the first passage time density can be computed through a recurrence relation similar to equation (30).[7]

Table 1 compares, in terms of both accuracy and efficiency, the valuation of (short maturity) American put options under different specifications of the exercise boundary and using the option' parameters contained in Broadie and Detemple (1996, table 1) and Ju (1998, table 1). Accuracy is

[7] Details available upon request.

measured by the average percentage error (over the twenty contracts considered) of each valuation approach and with respect to the exact American option price. This proxy of the "true" American put value (fourth column) is computed through the binomial tree model with 15,000 time steps, as suggested by Broadie and Detemple (1996, page 1222). Efficiency, that is the computational speed of each valuation method, is evaluated by the total CPU time (expressed in seconds) spent to value the whole set of contracts considered.[8]

To have an idea about the magnitude of the early exercise value associated to each American option contract, the third column of table 1 shows the price of the corresponding European put contracts, which is computed via the Merton (1973) formulae. The American put prices produced by the analytical pricing solutions associated to the constant and exponential boundary specifications (fifth and sixth columns), as given by equations (31) and (32), respectively, are obtained from Ingersoll (1998, sections 4 and 5). For comparison purposes, the last column of table 1 contains the American put prices generated by the three-point multipiece exponential function method proposed by Ju (1998, page 636). As already mentioned in section 1, there are three methods in the literature that seem to dominate the other American pricing approaches in terms of accuracy and efficiency: the regression bounds of Broadie and Detemple (1996), the randomization approach of Carr (1998), and the multipiece exponential boundary approximation of Ju (1998). The choice of the multipiece exponential approximation as a benchmark for the best pricing methods already proposed in the literature follows from Ju (1998, tables 3 and 5): it is faster than the Carr (1998) approach (for the same accuracy level); and much more accurate, for hedging purposes, than the lower and upper bound approximation of Broadie and Detemple (1996).

All the other early exercise boundary approximations (i.e. from the seventh to the tenth column of table 1) are implemented through proposition 4 and with $N = 2^8$. For the exponential-constant (seventh column) and polynomial (of degree 3 and 4, on the eight and ninth columns, respectively) boundary specifications, the parameter corresponding to the constant term in equations (33) and (34) is initialized at the Baroni-Adesi and Whaley (1987) estimate (and at zero, for the other parameters). For the CJM exercise boundary approximation, the initial guess of the single parameter involved in equation (35) is also set at zero.

Tables 2 and 3 present the same information, but for medium and long maturity option contracts, respectively, and yield results similar to the ones contained in table 1 as a consequence of the asymptotic property described in proposition 2. In general, one may conclude that the fastest approximations (in terms of CPU time) are the constant, the exponential, and the three-point multipiece exponential specifications: they all possess computational times bellow 0.1 seconds for all the range of contracts under consideration. However, the pricing errors generated by the constant and the exponential parameterizations can be significative. For instance, in table 2 the constant exercise policy possess a mean percentage pricing error bellow -0.5%, while the average mispricing of the exponential parameterization equals -10 basis points. As expected, the pricing errors produced by the specifications described in section 5 are negative because any approximation of the optimal exercise policy can only yield a lower bound for the true American put price.[9]

With the same number of parameters as the already known exponential approximation, the new exponential-constant parameterization can yield pricing errors about three times smaller, as shown in table 1. More interesting, the CJM approximation suggested by Carr et al. (1992) and now tested, can be about four times faster than the exponential-constant specification (since only

[8] All computations are made by running Pascal programs on an Intel Pentium 4 2.80GHz processor and under a Linux operating system.

[9] The only exception corresponds to the approximation suggested by Ju (1998), for which the pricing errors are consistently positive. This behavior might be explained by the non-uniform convergence of the Richardson extrapolation employed.

one parameter must be estimated), and possesses an accuracy similar to the three-point multipiece exponential approach: the average pricing errors are between one and three basis points. This result is relevant since the CJM approximation satisfies all the requirements described in section 5 for the early exercise boundary specification.

Finally, table 1 shows that the implementation of a polynomial approximation of degree 4 is able to provide smaller pricing errors than the Ju (1998) approach, but at the expense of a prohibitive computational effort. Of course and as shown by table 4, the accuracy of a polynomial specification can be always improved by increasing its degree. Table 4 applies different polynomial parameterizations to a random sample of 1,250 American put options generated as in Ju (1998, table 3). With a five-degree polynomial it is possible to obtain an average absolute percentage error (computed against a binomial tree model with 15,000 time steps) of only one basis point and a maximum absolute percentage error of about 4 basis points.

Overall, taking into consideration both accuracy and efficiency, the best pricing methodology is still the multipiece exponential approach of Ju (1998). Even though such parameterization does not obey to the requirements enunciated in section 5, it seems to be flexible enough to capture the behavior of the critical asset prices. Notice that for the same level of accuracy, the three-point multipiece exponential specification involves six unknown parameters, while the CJM approximation provides only one degree of freedom. Nevertheless, the disparity of pricing errors contained in tables 1, 2 and 3 shows that the early exercise premium depends significantly (if not critically) on the specification adopted for the early exercise boundary.

7 Conclusions

The main theoretical contribution of this paper consisted in deriving an alternative characterization of the early exercise premium, which is valid for any Markovian representation of the underlying asset price and for any parameterization of the exercise boundary. Moreover, the proposed characterization is shown to be automatically consistent with the value-matching condition and to possess appropriate asymptotic properties.

Under the geometric Brownian motion assumption, several parameterizations of the exercise boundary were tested. The disparity of results produced by such different specifications implies that the pricing accuracy depends on the parameterization adopted. Nevertheless, it is shown that the single-parameter specification suggested by Carr et al. (1992, page 93) is as accurate as the six-parameter approximation proposed by Ju (1998), being this latter approach much more efficient.

Concerning further research and since the analytical pricing of American options under the geometric Brownian motion process is already well established through the randomization approach of Carr (1998) or the multipiece exponential boundary approximation of Ju (1998), the characterization proposed in theorem 1 can be more fruitfully applied under alternative (but Markovian) stochastic processes for the underlying asset price. For this purpose to be accomplished in an efficient way, it is only required that the selected price process provides a viable valuation method for European options and for the first passage time density.

A Appendix: Proof of Proposition 1

Concerning the boundary condition (18), since

$$\begin{aligned}\lim_{r\downarrow 0} E_T &= \min\left(K, \lim_{r\downarrow 0}\frac{r}{q}K\right)\\ &= 0\end{aligned}$$

and because the exercise boundary $\{E_t, t_0 \le t \le T\}$ is a non-decreasing function of t, then

$$\lim_{r\downarrow 0} E_u = 0,\ \forall u \in [t_0, T]\,. \tag{36}$$

Combining equations (11) and (36),

$$\lim_{r\downarrow 0} eep_{t_0}(S,K,T) = \int_{t_0}^{T} e^{-0(u-t_0)}\left[K - \lim_{r\downarrow 0} p_u(0,K,T)\right]\lim_{r\downarrow 0}\mathcal{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_{t_0}\right). \tag{37}$$

Finally, since $\left[e^{-r(T-t)}K - S_t\right]^+ \le p_t(S,K,T) \le e^{-r(T-t)}K$ follows from straightforward no-arbitrage arguments, then $\lim_{r\downarrow 0} p_u(0,K,T) = K$ and, therefore, equation (37) can be rewritten as

$$\begin{aligned}\lim_{r\downarrow 0} eep_{t_0}(S,K,T) &= \int_{t_0}^{T}(K-K)\lim_{r\downarrow 0}\mathcal{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_{t_0}\right)\\ &= 0.\end{aligned}$$

The terminal condition (19) follows immediately from equation (10) because $p_T(S,K,T) = (K-S_T)^+$ and $eep_T(S,K,T) = 0$.

Concerning the boundary condition (20) and because $\lim_{S\uparrow\infty} p_t(S,K,T) = 0$, equation (10) yields:

$$\begin{aligned}&\lim_{S\uparrow\infty} P_t(S,K,T) \qquad (38)\\ &= \int_t^T e^{-r(u-t)}\left[(K-E_u) - p_u(E,K,T)\right]\lim_{S\uparrow\infty}\mathcal{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_t\right).\end{aligned}$$

Since $\lim_{S\uparrow\infty} S_u = \infty$, $\forall u \ge t$ and for any reasonable Markov process S_u, then

$$\begin{aligned}\lim_{S\uparrow\infty}\mathcal{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_t\right) &= \lim_{S\uparrow\infty}\mathcal{Q}\left(S_u = E_u \wedge \inf_{t\le v<u}(S_v - E_v) > 0\middle|\,\mathcal{F}_t\right)\\ &= 0,\end{aligned} \tag{39}$$

because the exercise boundary is independent of the current asset price and finite. Combining equations (38) and (39), the boundary condition (20) is obtained.

Finally, the *value-matching condition* (21) is also easily derived from equation (10):

$$\begin{aligned}&\lim_{S\downarrow E_t} P_t(S,K,T) \qquad (40)\\ &= p_t(E,K,T) + \int_t^T e^{-r(u-t)}\left[(K-E_u) - p_u(E,K,T)\right]\lim_{S\downarrow E_t}\mathcal{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_{t_0}\right).\end{aligned}$$

Since

$$\lim_{S\downarrow E_t}\mathcal{Q}\left(\tau_e \in du\middle|\,\mathcal{F}_{t_0}\right) = \delta(u-t)\,,$$

where $\delta(\cdot)$ is the Dirac-delta function, then equation (40) yields

$$\begin{aligned}\lim_{S\downarrow E_t} P_t(S,K,T) &= p_t(E,K,T) + e^{-r(t-t)}\left[(K-E_t) - p_t(E,K,T)\right] \\ &= K - E_t.\end{aligned}$$

■

B Appendix: Proof of Proposition 2

Starting with the perpetual American put option and since, in this case, the critical asset price is a time-invariant constant, that is $E_u = E_\infty, \forall u \in [t_0, T]$, the limit of equation (10), as the option's maturity date tends to infinity, is given by

$$\begin{aligned}&\lim_{T\uparrow\infty} P_t(S,K,T) \\ = &\lim_{T\uparrow\infty} p_t(S,K,T) + \lim_{T\uparrow\infty}\int_{t_0}^{T} e^{-r(u-t)}\left[(K-E_\infty) - p_u(E_\infty,K,T)\right]\mathcal{Q}\left(\tau_e \in du|\,\mathcal{F}_t\right).\end{aligned}$$

Furthermore, Merton (1973, corollary 2) shows that the fair value of a perpetual European put option is equal to zero and, consequently,

$$\begin{aligned}\lim_{T\uparrow\infty} P_t(S,K,T) &= (K-E_\infty)\int_{t_0}^{\infty} e^{-r(u-t_0)}\mathcal{Q}\left(\tau_e \in du|\,\mathcal{F}_{t_0}\right) \\ &= (K-E_\infty)\,\mathbb{E}_{\mathcal{Q}}\left[\left.e^{-r(\tau_e - t)}\right|\mathcal{F}_t\right]. \qquad (41)\end{aligned}$$

Finally, solving the stochastic differential equation (2) and redefining the optimal stopping time τ_e as

$$\begin{aligned}\tau_e &= \inf\left\{u \geq t : S_u = E_\infty\right\} \\ &= \inf\left\{u \geq t : -\frac{1}{\sigma}\left(r - q - \frac{\sigma^2}{2}\right)(u-t) - \int_t^u dW_v^{\mathcal{Q}} = \frac{1}{\sigma}\ln\left(\frac{S_t}{E_\infty}\right)\right\},\end{aligned}$$

the (dividend-adjusted) Merton (1973, page 174) solution follows after applying theorem 8.3.2 of Shreve (2004):

$$\lim_{T\uparrow\infty} P_t(S,K,T) = (K-E_\infty)\left(\frac{E_\infty}{S_t}\right)^{\gamma},\ S_t > E_\infty, \qquad (42)$$

where

$$E_\infty = \frac{\gamma}{1+\gamma}K, \qquad (43)$$

and with

$$\gamma := \frac{r - q - \frac{\sigma^2}{2} + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2}.$$

Applying the parabolic operator $\mathcal{L}$ to equation (10) and using the Leibniz's rule,

$$\begin{aligned} & \mathcal{L}P_t(S,K,T) \\ = & \mathcal{L}p_t(S,K,T) \\ & +\int_t^T re^{-r(u-t)}\left[(K-E_u)-p_u(E,K,T)\right]\mathcal{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right) \\ & +\int_t^T e^{-r(u-t)}\left[(K-E_u)-p_u(E,K,T)\right]\mathcal{L}\mathcal{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right) \\ & -e^{-r(t-t)}\left[(K-E_t)-p_t(E,K,T)\right]\mathcal{Q}\left(\tau_e=t|\,\mathcal{F}_t\right). \end{aligned} \tag{44}$$

Because $\mathcal{L}p_t(S,K,T)=0$, considering that $\mathcal{Q}(\tau_e=t|\,\mathcal{F}_t)=0$ since proposition 2 assumes that $S_t>E_t$, and using definition (23), equation (44) can be simplified into

$$\begin{aligned} & \mathcal{L}P_t(S,K,T) \\ = & \int_t^T e^{-r(u-t)}\left[(K-E_u)-p_u(E,K,T)\right]\left(\frac{\partial}{\partial t}+\mathcal{A}\right)\mathcal{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right), \end{aligned} \tag{45}$$

where

$$\mathcal{A}:=\frac{\sigma^2S^2}{2}\frac{\partial^2}{\partial S^2}+(r-q)\,S\frac{\partial}{\partial S}$$

is the infinitesimal generator of S. Since

$$\left(\frac{\partial}{\partial t}+\mathcal{A}\right)\mathcal{Q}\left(\tau_e\in du|\,\mathcal{F}_t\right)=0$$

can be interpreted as a Kolmogorov backward equation, then the partial differential equation (22) is obtained. ■

C Appendix: Proof of Proposition 4

Equation (29) is simply the discretization of equation (11) for the partition $t_0<t_1<\ldots<t_N=T$, where $h=t_j-t_{j-1}$ $(j=1,\ldots,N)$, $t_j=t_0+jh$, and $u=\frac{t_j+t_{j-1}}{2}$.

Applying the same discretization to equation (25), then

$$\sum_{j=1}^{k}\Phi\left[\frac{E^z_{t_0+\frac{jh+(j-1)h}{2}}-E^z_{t_0+kh}}{\sqrt{kh-\frac{jh+(j-1)h}{2}}}\right]\left[\mathcal{Q}\left(\tau_e=t_0+jh\right)-\mathcal{Q}\left(\tau_e=t_0+(j-1)\,h\right)\right]=\Phi\left(-\frac{E^z_{t_0+kh}}{\sqrt{kh}}\right), \tag{46}$$

for $k=1,\ldots,N$. Finally, solving equation (46) in order to the probability $\mathcal{Q}(\tau_e=t_0+kh)$, equation (30) arises. ■

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Table 1: Comparison of different approximations for American put prices with $S_{t_0} = \$100$ and $T - t_0 = 0.5$ years

			American							
				Early exercise boundary specification						
Option parameters	Strike	European	Exact	Constant	Exp.	ExpConst	3-d Polyn.	4-d Polyn.	CJM	EXP3
	80	0.215	0.219	0.218	0.219	0.219	0.219	0.219	0.219	0.220
$r = 7\%$	90	1.345	1.386	1.376	1.385	1.386	1.386	1.386	1.386	1.387
$q = 3\%$	100	4.578	4.783	4.750	4.778	4.781	4.781	4.782	4.781	4.784
$\sigma = 20\%$	110	10.421	11.098	11.049	11.092	11.097	11.097	11.097	11.095	11.099
	120	18.302	20.000	20.000	20.000	19.999	19.999	19.999	19.999	20.000
	80	2.651	2.689	2.676	2.687	2.688	2.688	2.688	2.688	2.690
$r = 7\%$	90	5.622	5.722	5.694	5.719	5.721	5.721	5.721	5.720	5.724
$q = 3\%$	100	10.021	10.239	10.190	10.233	10.237	10.237	10.237	10.236	10.240
$\sigma = 40\%$	110	15.768	16.181	16.110	16.173	16.180	16.179	16.179	16.177	16.183
	120	22.650	23.360	23.271	23.350	23.358	23.358	23.358	23.355	23.362
	80	1.006	1.037	1.029	1.036	1.037	1.037	1.037	1.037	1.038
$r = 7\%$	90	3.004	3.123	3.098	3.120	3.122	3.122	3.122	3.122	3.125
$q = 0\%$	100	6.694	7.035	6.985	7.029	7.034	7.034	7.034	7.032	7.037
$\sigma = 30\%$	110	12.166	12.955	12.882	12.946	12.953	12.953	12.954	12.951	12.957
	120	19.155	20.717	20.650	20.710	20.716	20.716	20.717	20.713	20.719
	80	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664	1.664
$r = 3\%$	90	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495	4.495
$q = 7\%$	100	9.251	9.250	9.251	9.251	9.251	9.251	9.251	9.251	9.251
$\sigma = 30\%$	110	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798	15.798
	120	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706	23.706
Mean Percentage Error				-0.41%	-0.05%	-0.02%	-0.02%	-0.01%	-0.02%	0.02%
CPU (seconds)			451.32	0.01	0.03	130.07	209.01	262.18	30.76	0.08

Table 1 values American put options under different specifications of the exercise boundary. The third column contains European put prices, while the exact American put values (fourth column) are based on the binomial tree model with 15,000 time steps. The fifth and sixth columns report the American put prices associated to the constant and exponential boundary specifications, as given by equations (31) and (32), respectively. The seventh column presents American put prices computed through proposition 4 and based on the exponential-constant parameterization provided by equation (33). The eight and ninth columns are both based on the polynomial boundary specification of equation (34) with three and four degrees of freedom, respectively. The American put prices contained in the tenth column are obtained from the exercise boundary specification of equation (35). The last column presents the American put prices generated by the three-point multipiece exponential function method proposed by Ju (1998).

Table 2: Comparison of different approximations for American put prices with $S_{t_0} = \$100$ and $T - t_0 = 3$ years

			American							
				Early exercise boundary specification						
Option parameters	Strike	European	Exact	Constant	Exp.	ExpConst	3-d Polyn.	4-d Polyn.	CJM	EXP3
	80	2.241	2.580	2.553	2.575	2.578	2.578	2.578	2.579	2.582
$r = 7\%$	90	4.355	5.167	5.121	5.158	5.164	5.164	5.164	5.165	5.169
$q = 3\%$	100	7.386	9.066	9.002	9.054	9.063	9.063	9.064	9.063	9.069
$\sigma = 20\%$	110	11.331	14.443	14.371	14.430	14.441	14.440	14.441	14.440	14.447
	120	16.117	21.414	21.354	21.403	21.412	21.411	21.412	21.411	21.417
	80	10.309	11.326	11.238	11.310	11.321	11.320	11.322	11.320	11.330
$r = 7\%$	90	14.162	15.722	15.609	15.702	15.717	15.715	15.718	15.715	15.727
$q = 3\%$	100	18.532	20.793	20.656	20.770	20.788	20.786	20.789	20.785	20.800
$\sigma = 40\%$	110	23.363	26.495	26.337	26.468	26.489	26.486	26.490	26.485	26.502
	120	28.598	32.781	32.607	32.752	32.776	32.773	32.776	32.771	32.790
	80	4.644	5.518	5.463	5.507	5.514	5.514	5.515	5.514	5.521
$r = 7\%$	90	7.269	8.842	8.766	8.827	8.837	8.837	8.839	8.837	8.845
$q = 0\%$	100	10.542	13.142	13.048	13.124	13.138	13.137	13.139	13.137	13.147
$\sigma = 30\%$	110	14.430	18.453	18.347	18.433	18.449	18.448	18.450	18.447	18.459
	120	18.882	24.791	24.685	24.771	24.787	24.786	24.788	24.785	24.796
	80	12.133	12.145	12.145	12.145	12.145	12.145	12.145	12.145	12.145
$r = 3\%$	90	17.343	17.369	17.367	17.368	17.368	17.368	17.368	17.368	17.368
$q = 7\%$	100	23.301	23.348	23.347	23.348	23.348	23.348	23.348	23.348	23.348
$\sigma = 30\%$	110	29.882	29.964	29.961	29.963	29.963	29.963	29.963	29.963	29.963
	120	36.972	37.104	37.099	37.103	37.103	37.103	37.103	37.103	37.103
Mean Percentage Error				-0.52%	-0.10%	-0.03%	-0.03%	-0.02%	-0.03%	0.02%
CPU (seconds)			448.99	0.01	0.04	107.71	195.85	242.44	39.17	0.08

Table 2 values American put options under different specifications of the exercise boundary. The third column contains European put prices, while the exact American put values (fourth column) are based on the binomial tree model with 15,000 time steps. The fifth and sixth columns report the American put prices associated to the constant and exponential boundary specifications, as given by equations (31) and (32), respectively. The seventh column presents American put prices computed through proposition 4 and based on the exponential-constant parameterization provided by equation (33). The eight and ninth columns are both based on the polynomial boundary specification of equation (34) with three and four degrees of freedom, respectively. The American put prices contained in the tenth column are obtained from the exercise boundary specification of equation (35). The last column presents the American put prices generated by the three-point multipiece exponential function method proposed by Ju (1998).

Table 3: Comparison of different approximations for American put prices with $S_{t_0} = \$100$ and $T - t_0 = 20$ years

			American							
				Early exercise boundary specification						
Option parameters	Strike	European	Exact	Constant	Exp.	ExpConst	3-d Polyn.	4-d Polyn.	CJM	EXP3
	80	1.732	5.584	5.574	5.579	5.581	5.582	5.582	5.583	5.585
$r = 7\%$	90	2.384	8.503	8.493	8.498	8.500	8.501	8.501	8.502	8.504
$q = 3\%$	100	3.141	12.346	12.336	12.341	12.343	12.344	12.344	12.346	12.347
$\sigma = 20\%$	110	3.997	17.261	17.252	17.256	17.258	17.259	17.259	17.260	17.262
	120	4.948	23.400	23.394	23.397	23.399	23.399	23.400	23.401	23.401
	80	8.447	20.378	20.346	20.364	20.370	20.372	20.372	20.375	20.382
$r = 7\%$	90	10.023	25.135	25.101	25.119	25.126	25.128	25.128	25.132	25.138
$q = 3\%$	100	11.656	30.298	30.264	30.282	30.290	30.292	30.292	30.295	30.302
$\sigma = 40\%$	110	13.338	35.857	35.822	35.840	35.848	35.850	35.850	35.853	35.860
	120	15.065	41.797	41.763	41.781	41.789	41.790	41.791	41.794	41.800
	80	2.818	9.864	9.849	9.856	9.858	9.860	9.860	9.862	9.864
$r = 7\%$	90	3.584	13.453	13.438	13.445	13.448	13.449	13.450	13.451	13.454
$q = 0\%$	100	4.423	17.734	17.718	17.725	17.728	17.730	17.730	17.731	17.734
$\sigma = 30\%$	110	5.331	22.745	22.730	22.736	22.740	22.741	22.741	22.743	22.745
	120	6.303	28.526	28.512	28.517	28.520	28.522	28.522	28.523	28.525
	80	27.973	32.959	32.924	32.954	32.958	32.959	32.959	32.957	32.962
$r = 3\%$	90	32.769	39.123	39.084	39.117	39.121	39.123	39.123	39.121	39.127
$q = 7\%$	100	37.655	45.523	45.480	45.516	45.521	45.521	45.523	45.521	45.528
$\sigma = 30\%$	110	42.615	52.137	52.091	52.129	52.135	52.136	52.137	52.135	52.143
	120	47.635	58.948	58.899	58.939	58.946	58.947	58.947	58.945	58.954
Mean Percentage Error				-0.10%	-0.04%	-0.02%	-0.02%	-0.02%	-0.01%	0.01%
CPU (seconds)			445.21	0.01	0.02	78.38	276.31	370.89	20.77	0.08

Table 3 values American put options under different specifications of the exercise boundary. The third column contains European put prices, while the exact American put values (fourth column) are based on the binomial tree model with 15,000 time steps. The fifth and sixth columns report the American put prices associated to the constant and exponential boundary specifications, as given by equations (31) and (32), respectively. The seventh column presents American put prices computed through proposition 4 and based on the exponential-constant parameterization provided by equation (33). The eight and ninth columns are both based on the polynomial boundary specification of equation (34) with three and four degrees of freedom, respectively. The American put prices contained in the tenth column are obtained from the exercise boundary specification of equation (35). The last column presents the American put prices generated by the three-point multipiece exponential function method proposed by Ju (1998).

Table 4: Accuracy of the polynomial specification for a large and random sample of American puts

	Polynomial specifications			
	Second degree	Third degree	Fourth degree	Fifth degree
Percentage Errors				
mean	-0.0197%	-0.0154%	-0.0127%	-0.0108%
maximum	0.0014%	0.0014%	0.0014%	0.0014%
minimum	-0.0725%	-0.0585%	-0.0497%	-0.0432%
99th percentile	0.0008%	0.0009%	0.0009%	0.0009%
1st percentile	-0.0656%	-0.0526%	-0.0447%	-0.0390%
Absolute Percentage Errors				
mean	0.0198%	0.0155%	0.0128%	0.0109%
maximum	0.0725%	0.0585%	0.0497%	0.0432%
minimum	0.0000%	0.0000%	0.0000%	0.0000%
99th percentile	0.0656%	0.0526%	0.0447%	0.0390%

Table 4 reports the pricing errors associated to the valuation of 1,250 randomly generated American put options through different polynomial parameterizations of the exercise boundary (34). The strike price is always set at $100 while the other option features were generated from uniform distributions and within the following intervals: volatility between 10% and 60%; interest rate and dividend yield between 0% and 10%; underlying spot price between $70 an $130; and, time-to-maturity ranging from 0.0 to 3.0 years. The pricing errors produced by the polynomial specifications were computed against the binomial tree model with 15,000 time steps.