# Affine Structural Models of Corporate Bond Pricing

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**Abstract**

In existing structural models of corporate bond pricing, the underlying asset return volatility is assumed to be constant, the jump component in the return process follows a compound Poisson, and the interest rate is described by a one-factor model. In this paper, we consider an affine class of structural models that can allow for stochastic asset return volatility, a high frequency jump component in the return process, and a multi-factor term structure model. We provide analytical results for both the price of corporate bonds and the real probability of default for this class of models under certain assumptions on the recovery rate and default boundary.

# Affine Structural Models of Corporate Bond Pricing

Assessing and managing credit risk of corporate bonds has been a major area of interest and concern to academics, practitioners, and regulators. One widely used approach to the valuation of corporate bonds is the so-called structural approach based on Black and Scholes (1973) and Merton (1974).[1] Recently there have been a number of empirical studies of structural models using bond data. For instance, Jones and Rosenfeld (1984), Lyden and Saraniti (2000), Delianedis and Geske (2001), Ericsson and Reneby (2001) and Eom, Helwege, and Huang (2004) examine the implications of the models on pricing using individual corporate bond prices; Schaefer and Strebulaev (2004) on hedging; KMV (e.g. Kealhofer and Kurbat (2001) and Leland (2002) on the actual default probability; and Huang and Huang (2002) on both pricing and the actual default probability. Whereas the structural approach has been found quite useful, the empirical evidence has also indicates that standard structural models still have difficulty in accurately predicting spreads or explaining spreads and default rates simultaneously.

The main assumptions made in existing structural models include that the firm's asset return volatility is constant; that the (default-free) interest rate is either constant or follows a one-factor model; and that the jump component in the asset return process is modelled by a compound Poisson process. In this paper, we extend the existing models by relaxing these three assumptions. More specifically, we consider an affine class of structural models of corporate bond pricing, in which the underlying asset volatility can be stochastic, the underlying asset return can include a high-frequency jump component, and the interest rate process can be driven by multi factors. Under certain assumptions on the recovery rate and default boundary, analytical results are available for both corporate bond prices and real default probabilities under this class of models.

The paper is organized as follows. Section 1 considers an affine class of structural models of corporate bond pricing. Section 2 discusses the implementation of models and reports numerical results. Section 2 concludes.

[1] Another popular approach, which is not the focus of this study, is the reduced-form approach of Jarrow and Turnbull (1995) and Duffie and Singleton (1999). See also Das and Tufano (1996), Duffie, Schroder, and Skiadas (1996), Jarrow (2001), Robert Jarrow and Turnbull (1997) and Madan and Unal (1998).

# 1 Models of Corporate Bond Pricing

In this section, we consider an affine class of corporate bond pricing models that can allow for both stochastic volatility and jumps, and for a multi-factor term structure specification as well.

To fix the notation, let $r$ be the interest rate process and $V$ be the firm's asset value process. Denote by $\mathbb{Q}$ the risk-neutral probability measure. The underlying structure of the models considered in our analysis is as follows:

$$\begin{aligned} d\ln V_t &= [r_t - \delta - \sigma_v(t)^2/2 - \mu_k(r_t, \ln V_t)]dt + \sigma_v(t)dW_t^v + dJ_t - \xi dt, & (1) \\ dv(t) &= \kappa(1 - v(t))\,dt + \sigma_v\sqrt{v(t)}dZ_t, & (2) \\ r_t &= y_{1t} + y_{2t} + y_{3t} & (3) \end{aligned}$$

where $v(t) = \sigma_v(t)^2$, the function $\mu_k$—affine in both $r$ and $\ln X$—is non-zero only when $V_t^*$ is stochastic, and $Z_t$ denotes a standard Brownian motion under $\mathbb{Q}$, which can be correlated with the standard Brownian motion $W_t^v$ in the asset return process by $\rho$. Process $J$ is a Leẃy jump process and parameter $\xi$ is such that the compensated $J$ is a Q-Martingale. Note that the long-run mean of the activity rate is normalized to unity in equation (2) for identification purpose. The state variables $y_i, i = 1, \ldots, 3$, that determine the interest rate are assumed to have an affine structure and will be specified later.

## *1.1 Zero-Coupon Bonds*

Consider first the case where default can occur only at maturity $T$. Let $V^*$ be the default boundary.

**Assumption 1** *(i) Default occurs if $V_T < V_T^*$; (ii) In the event of default, the absolute priority rule is followed and there is no bankruptcy cost.*

Under this assumption, we are in the Merton world. The value of the zero-coupon bond can be obtained from the equity value of the firm, which itself is equal to the price of an European call option written on the firm's asset value. Below we consider two special cases.

## 1.2 The Compound Poisson Jump

The jump component $J$ is assumed to follow a compound Poisson process. Both the interest rate and asset return volatility can be stochastic. The value of equity holders can be obtained using results from Duffie, Pan, and Singleton (2000). As shown in DPS, this affine class of jump-diffusion with stochastic volatility models include the models considered in Heston (1993), Bakshi, Cao, and Chen (1997), Bates (2000), and Bakshi and Madan (2000) as special cases.

## 1.3 High-Frequency Jump

The jump component $J$ is assumed to follow a general Levý jump process. The interest rate is assumed to be non-stochastic but asset return volatility can be stochastic. Results can be obtained from Carr and Wu (2002).

In particular, we consider two different jump specifications: the variance gamma and log-stable specifications (c.f. the appendix for more details on these jump models).

The solution to the Fourier transformation of the log asset return is given as follows (c.f. Carr and Wu (2002) or Huang and Wu (2004)). We have

$$\phi_v(u) = E^{\mathbb{Q}}\left[e^{iuv_t}\right] = \exp -B(t)\nu_0 - A(t)), \tag{4}$$

where where

$$B(t) = \frac{2\psi\left(1-e^{-\eta t}\right)}{2\eta - (\eta-\kappa^*)\left(1-e^{-\eta t}\right)}; \tag{5}$$

$$A(t) = \frac{\kappa}{\sigma_v^2}\left[2\ln\left(1-\frac{\eta-\kappa^*}{2\eta}\left(1-e^{-\eta t}\right)\right)+(\eta-\kappa^*)t\right], \tag{6}$$

with

$$\eta = \sqrt{(\kappa^*)^2 + 2\sigma_v^2\psi}, \quad \kappa^* = \kappa - iu\rho\sigma\sigma_v.$$

An inverse Fourier transform can then be used to obtain the equity value. In practice, this inversion can be done more efficiently using the FFT (see Carr and Madan (1999)).

#### *1.3.1 The Longstaff-Schwartzed model with a multi-factor structure*

Assume that under the risk-neutral measure

$$\begin{aligned} d\ln V_t &= (r_t - \delta - \sigma_v^2/2)dt + \sigma_v dW_t && (7) \\ r &= y_1 + y_2 + y_3 && (8) \\ dy_{it} &= (\alpha_i - \beta_i y_{it})\, dt + \sigma_i dZ_{it}, \qquad i = 1, \ldots, 3 && (9) \end{aligned}$$

where $Z_i, i = 1, \ldots, 3$ are independent of each other and $\mathrm{cov}[dZ_{it}, dW_t] = \rho_{vi} dt$. If $y_{2t}$ and $y_{3t}$ zero, we recover the Longstaff-Schwartz model.

The default probability under the forward measure $Q^T$ in this multi-factor model can be calculated similar to the one-factor term structure models. See appendix for more details.

### *1.4 Coupon Bonds*

Consider an $n$-period defaultable bond with unit face value. The bond pays fixed-rate coupons and matures at $T$. Let $c$ be the coupon rate. We consider two assumptions on the recovery rate of the bond in the event of default. (A1) The bond recovery is equal to $w < 1$ (times the face value) and to be received on the first scheduled coupon date after default (discrete-time recovery in the sense of Duffie (1998)). (A2) The bond recovery is equal to a $w$ fraction of an otherwise identical Treasury bond (the Jarrow and Turnbull's (1995) model of recovery of treasury).

Consider the recovery assumption made in (A1) first. Let $P^{\mathrm{RF}}(0,T)$ denote the time-0 value of the bond with discrete-time recovery. We have

$$\begin{aligned} P^{\mathrm{RF}}(0,T) &= c/2\sum_{i=1}^{n} D(0,T_i)(1-Q^i(0,T_i)) + D(0,T_n)(1-Q^n(0,T_n)) \\ &\quad + w\sum_{i=1}^{n} D(0,T_i)\left[Q^i(0,T_i) - Q^{i-1}(0,T_{i-1})\right] \end{aligned} \tag{10}$$

where $Q^i(0,T_i)$ represents the time-0 unconditional default probability by time $T_i$ under the $T_i$-forward measure, $D(0,T_i)$ denotes the time-0 value of a $T_i$-maturity default-free zero-coupon bond, and $T_n = T$. On the RHS of (10), the first two terms represent the payoff conditional on no default, whereas the last term comes from the bond recovery in the event of default.

Consider next the recovery assumption made in (A2). Let $P^{\mathrm{RT}}(0,T)$ denote the time-0 value of the bond with a recovery of treasury. We have

$$P^{\mathrm{RT}}(0,T) = \left(\frac{c}{2}\right)\sum_{i=1}^{n} D(0,T_i)[1-w_\ell Q^i(0,T_i)] + \left(1+\frac{c}{2}\right)D(0,T)[1-w_\ell Q^n(0,T)] \tag{11}$$

where $w_\ell$ is the loss rate. In this approach, each coupon is treated independently and the price of a coupon bond is simply the sum of prices of the independent zeroes. This "portfolio of zeros" approach is used in Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001). The advantage of this approach is that it allows for correlation between the default process and the interest rate.

Pricing formulas given in Eqs. (10) and (11) are fairly general as no assumption about the underlying state processes has yet been made. One can see from the two equations that once probabilities of survival (or default) are known, the price of a defaultable bond is straightforward to calculate. To obtain the default probabilities, however, we need to specify the dynamics of the underlying state variables. Below we consider two specifications.

### 1.5 Asset Return with Constant Volatility

Let $V$ be the firm asset value process and $X$ be a new process $(V_t/V_t^*)_{t\geq 0}$. Assume that under the risk-neutral measure,

$$
\begin{aligned}
d\ln X_t &= [r_t - \delta - \sigma_v^2/2 - \mu_k(r_t, \ln X_t)]dt + \sigma_v dW_t^v + d\left[\sum_{i=1}^{N_t} Z_i\right] - \lambda\xi dt, \qquad (12)\\
dr_t &= \kappa_r(\theta - r_t)\, dt + \sigma_r dW_t^r \qquad (13)
\end{aligned}
$$

where the function $\mu_k$—affine in both $r$ and $\ln X$—is non-zero only when $V_t^*$ is stochastic, $W^v$ and $W^r$ are one-dimensional standard Brownian motions and have a correlation coefficient of $\rho$. Parameters $\kappa_r$, $\theta$, and $\sigma_r$ are the speed of mean-reverting, the long-term mean, and the volatility of the interest rate, respectively.

Let's consider several special cases of the specification given in Eqs. (12) and (13).

#### 1.5.1 The Extended Merton Model

This model is first considered in Eom, Helwege, and Huang (2004). In this model, a coupon bond is treated as if it were a portfolio of zero-coupon bonds, each of which can be priced using the zero-coupon version of the model. The default boundary $V_t^* = K\,\forall t \in \{T_i\}$ and default is triggered if the asset value is below $K$ on coupon dates. However, unlike the Merton model, the interest rate here can be stochastic.

The price of a coupon bond can be written as follows

$$
\begin{aligned}
P^{\mathrm{M}}(0,T) &= \sum_{i=1}^{2T-1} D(0,T_i)E^{\mathbb{Q}}\left[(c/2)I_{\{V_{T_i}\geq K\}} + \min\left(wc/2, V_{T_i}\right) I_{\{V_{T_i}<K\}}\right]\\
&\quad + D(0,T)E^{\mathbb{Q}}\Big[(1+c/2)\,I_{\{V_T\geq K\}} + \min\Big(w(1+c/2), V_T\Big)I_{\{V_T<K\}}\Big] \qquad (14)
\end{aligned}
$$

where $D(0,T_i)$ denotes the time-0 value of a default-free zero-coupon bond maturing at $T_i$, $I_{\{\cdot\}}$ is the indicator function, $E^{\mathbb{Q}}[\cdot]$ is the expectation at time-0 under the $\mathbb{Q}$ measure, and $w$ is the recovery rate.

It is known that

$$E^{\mathbb{Q}}[I_{\{V_t \geq K\}}] = N(d_2(K,t)) \tag{15}$$

$$E^{\mathbb{Q}}\left[I_{\{V_t < K\}} \min(\psi, V_t)\right] = V_0 D(0,t)^{-1} e^{-\delta t} N(-d_1(\psi,t)) + \psi \left[N(d_2(\psi,t)) - N(d_2(K,t))\right] \tag{16}$$

where $\psi \in [0,K]$, $N(\cdot)$ represents the cumulative standard normal function and

$$d_1(x,t) = \frac{\ln\left(\frac{V_0}{xD(0,t)}\right) + (-\delta + \sigma_v^2/2)t}{\sigma_v \sqrt{t}}; \;\; d_2(x,t) = d_1(x,t) - \sigma_v \sqrt{t} \tag{17}$$

### *1.5.2 The CDG and LS Models*

In the CDG model, there is no jumps in the X process and the function $\mu_k$ is given by the following

$$\mu_k(r_t, \ln X_t) = \kappa_\ell \left[\ln X_t - \nu - \phi(r_t - \theta)\right] \tag{18}$$

where $\kappa_\ell$, $\nu$, and $\phi$ are constants. Probabilities of default can be computed using a quasi-analytical formula. See the appendix for details.

The LS model is a special case of CDG. The formulas in this model can be obtained by setting $\kappa_\ell$ to zero in CDG.

### *1.5.3 The Double-Exponential Jump-Diffusion Model*

This model is analyzed by Huang and Huang (2002). In this model, the interest rate is assumed to be constant and the function $\mu_k$ is assumed to be zero. The asset return process does have a jump component. Specifically, $N$ is a Poisson process with a constant intensity $\lambda > 0$, the $Z_i$'s are i.i.d. random variables, and $Y \equiv \ln(Z_1)$ has a double-exponential distribution with a density given by

$$f_Y(y) = p_u \eta_u e^{-\eta_u y} \mathbf{1}_{\{y \geq 0\}} + p_d \eta_d e^{\eta_d y} \mathbf{1}_{\{y < 0\}}. \tag{19}$$

In equation (19), parameters $\eta_u, \eta_d > 0$ and $p_u, p_d \geq 0$ are all constants, with $p_u + p_d = 1$. The mean percentage jump size $\xi$ is given by

$$\xi = \mathbf{E}\left[e^Y - 1\right] = \frac{p_u \eta_u}{\eta_u - 1} + \frac{p_d \eta_d}{\eta_d + 1} - 1. \tag{20}$$

To calculate probabilities of default, consider the Laplace transform of $Q(0,\cdot)$ as defined by

$$\widehat{Q}(s;t_0) = \int_0^\infty e^{-st} Q(0,t)dt \tag{21}$$

An analytic solution for $\widehat{Q}(s;t_0)$ was obtained by Kou and Wang (2002, Theorem 4.1). Let $x_b \equiv \ln(V_0/V^*)$ and $\mu_x \equiv -(\pi_0^v + r - \delta - \sigma_v^2/2)$, we have

$$\widehat{Q}(s;t_0) = \frac{\eta_u - y_{1,s}}{s\eta_u} \frac{y_{2,s}}{y_{2,s} - y_{1,s}} e^{-x_b y_{1,s}} + \frac{y_{2,s} - \eta_u}{s\eta_u} \frac{y_{1,s}}{y_{2,s} - y_{1,s}} e^{-x_b y_{2,s}} \tag{22}$$

where $y_{1,s}$ and $y_{2,s}$ are the only two positive roots for the following equation

$$\mu_x y + \frac{1}{2}\sigma_v^2 y^2 + \lambda\left(\frac{p_u \eta_u}{\eta_u - y} + \frac{p_d \eta_d}{\eta_d + y} - 1\right) - s = 0 \tag{23}$$

Given $\widehat{Q}(s;t_0)\,\forall s > 0$, we then follow Kou and Wang (2002) to calculate numerically $Q(0,\cdot)$ using the Gaver-Stehfest algorithm for Laplace inversion. For brevity, the details of this implementation method are omitted here but can be found in Kou and Wang (2002).

### *1.5.4 The High-Frequency Jump Models*

Like the previous subsection, the interest rate is assumed to be constant and the function $\mu_k$ is assumed to be zero. However, the low-frequency double-exponential jump component in the asset return process will be replaced by a high-frequency jump component. The default boundary is assumed to be the same as in the extended Merton model. Under this assumption, each defaultable zero coupon bond is like a European option. As a result, we can borrow existing results from the option pricing literature, in particular, from Duffie, Pan, and Singleton (2000) and Carr and Wu (2002).

## 2 Conclusion

In this paper, we consider the structural approach to the valuation of defaultable bonds. In particular, we examine an affine class of models which allow for analytical results for the price of defaultable bonds. These models include as special cases some existing ones such as Longstaff and Schwartz (1995), Collin-Dufresne and Goldstein (2001), and Huang and Huang (2002). The class of models examined here also include new models that allow for stochastic asset return volatility, a high-frequency jump component in the asset return process or a multi-factor term structure model.

# Appendix A. Solution to the Jump-diffusion stochastic volatility models

For completeness, we provide a derivation of the equity value for the affine class of Jump-diffusion stochastic volatility models under the framework of time-changed Lev́y processes. We borrow heavily from Huang and Wu (2004) in the discussion that follows.

The log asset return $s_t = \ln(V_t/V_0)$ follows the following Lévy process,

$$s_t = (r-\delta)t + \left(\sigma W_t^v - \frac{1}{2}\sigma_v^2 t\right) + (J_t - \xi t). \tag{A1}$$

Equation (A1) decomposes the log asset return $v_t$ into three components. The first component, $(r-\delta)t$, is from the instantaneous drift, which is determined by no-arbitrage. The second component, $\left(\sigma W_t^v - \frac{1}{2}\sigma_v^2 t\right)$, comes from the diffusion, with $\frac{1}{2}\sigma_v^2 t$ as the concavity adjustment. The last term, $(J_t - \xi t)$, represents the contribution from the jump component, with $\xi$ as the analogous concavity adjustment for $J_t$. The generalized Fourier transform for $v_t$ under equation (A1) is given by

$$\phi_v(u) \equiv E^{\mathbb{Q}}\left[e^{iuv_t}\right] = \exp\left(iu(r-\delta)t - t\psi_d - t\psi_j\right), \quad u \in \mathcal{D} \in \mathbb{C}, \tag{A2}$$

where $E^{\mathbb{Q}}[\cdot]$ denotes the expectation operator under the risk-neutral measure $\mathbb{Q}$, $\mathcal{D}$ denotes a subset of the complex domain ($\mathbb{C}$) where the expectation is well-defined, and

$$\psi_d = \frac{1}{2}\sigma^2\left[iu + u^2\right]$$

is the characteristic exponent of the diffusion component.

The characteristic exponent of the jump component, $\psi_j$, depends on the exact specification of the jump structure. Throughout the paper, we use a subscript (or superscript) "$d$" to denote the diffusion component and "$j$" the jump component. As a key feature of Lévy processes, neither $\psi_d$ nor $\psi_j$ depends on the time horizon $t$.[2] We note that $\phi_s(u)$ is essentially the characteristic function of the log return

[2] See Bertoin (1996) and Sato (1999).

when $u$ is real. The extension of $u$ to the admissible complex domain is necessary for the application of the fast Fourier transform algorithm.

Next, we apply the time change through the mapping $t \to T_t$ as defined in equation (**??**). The generalized Fourier transform of the time-changed return process is given by

$$\begin{aligned}\phi_v(u) &= e^{iu(r-q)t} E^{\mathbb{Q}}\left[e^{iu\left(\sigma W_{T_t^d} - \frac{1}{2}\sigma^2 T_t^d\right) + iu\left(J_{T_t^j} - \xi T_t^j\right)}\right] \\ &= e^{iu(r-q)t} E^{\mathbb{M}}\left[e^{-\psi^\top T_t}\right] \equiv e^{iu(r-q)t} \mathcal{L}_T^{\mathbb{M}}(\psi), \qquad (A3)\end{aligned}$$

where $\psi \equiv [\psi_d, \psi_j]^\top$ denotes the vector of the characteristic exponents and $\mathcal{L}_T^{\mathbb{M}}(\psi)$ represents the Laplace transform of the stochastic time $T_t$ under a new measure $\mathbb{M}$. The measure $\mathbb{M}$ is absolutely continuous with respect to the risk-neutral measure $\mathbb{Q}$ and is defined by a complex-valued exponential martingale,

$$\left.\frac{d\mathbb{M}}{d\mathbb{Q}}\right|_t \equiv \exp\left[iu\left(\sigma W_{T_t^d} - \frac{1}{2}\sigma^2 T_t^d\right) + iu\left(J_{T_t^j} - \xi T_t^j\right) + \psi_d T_t^d + \psi_j T_t^j\right]. \qquad (A4)$$

Note that equation (A3) converts the issue of obtaining a generalized Fourier transform into a simpler problem of deriving the Laplace transform of the stochastic time (Carr and Wu (2002)). The solution to this Laplace transform depends on the specification of the instantaneous activity rate $v(t)$ and on the characteristic exponents, the functional form of which is determined by the specification of the jump structure $J_t$.

Depending on the frequency of jump arrivals, Lévy jump processes can be classified into three categories: finite activity, infinite activity with finite variation, and infinite variation (Sato (1999)). Each jump category exhibits distinct behavior and hence results in different option pricing performance.

Formally, the structure of a Lévy jump process is captured by its Lévy measure, $\pi(dx)$, which controls the arrival rate of jumps of size $x \in R^0$ (the real line excluding zero). A finite activity jump process generates a finite number of jumps within any finite interval. Thus, the integral of the Lévy measure is finite:

$$\int_{R^0} \pi(dx) < \infty. \tag{A5}$$

Given the finiteness of this integral, the Lévy measure has the interpretation and property of a probability density function after being normalized by this integral. A prototype example of a finite activity jump process is the compound Poisson jump process of Merton (1976) (MJ), which has been widely adopted by the finance literature. Under this process, the integral in equation (A5) defines the Poisson intensity, $\lambda$. The MJ model assumes that conditional on one jump occurring, the jump magnitude is normally distributed with mean $\alpha$ and variance $\sigma_j^2$. The Lévy measure of the MJ process is given by

$$\pi_{MJ}(dx) = \lambda \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(x-\alpha)^2}{2\sigma_j^2}\right) dx. \tag{A6}$$

For all finite activity jump models, we can decompose the Lévy measure into two components, a normalizing coefficient often labeled as the Poisson intensity, and a probability density function controlling the conditional distribution of the jump size.

Unlike a finite activity jump process, an infinite activity jump process generates an infinite number of jumps within any finite interval. The integral of the Lévy measure for such processes is no longer finite. Examples of this class include the normal inverse Gaussian model of Barndorff-Nielsen (1998), the generalized hyperbolic class of Eberlein, Keller, and Prause (1998), and the variance-gamma (VG) model of Madan and Milne (1991) and Madan, Carr, and Chang (1998). In our empirical studies, we choose the relatively parsimonious VG model as a representative of the infinite activity jump type. The VG process is obtained by subordinating an arithmetic Brownian motion with drift $\alpha/\lambda$ and variance $\sigma_j^2/\lambda$ by an independent gamma process with unit mean rate and variance rate $1/\lambda$. The Lévy measure for the VG process is given by

$$\pi_{VG}(dx) = \frac{\mu_\pm^2}{v_\pm} \frac{\exp\left(-\frac{\mu_\pm}{v_\pm}|x|\right)}{|x|} dx,$$

where

$$\mu_{\pm} = \sqrt{\frac{\alpha^2}{4\lambda^2} + \frac{\sigma_j^2}{2}} \pm \frac{\alpha}{2\lambda}, \quad v_{\pm} = \mu_{\pm}^2/\lambda.$$

The parameters with plus subscripts apply to positive jumps and those with minus subscripts apply to negative jumps. The jump structure is symmetric around zero when we drop the subscripts. Note that as the jump size approaches zero, the arrival rate approaches infinity. Thus, an infinite activity model incorporates infinitely many small jumps. The Lévy measure of an infinite activity jump process is singular at zero jump size.

When the integral in (**??**) is no longer finite, the sample path of the process exhibits *infinite variation*. A typical example is an $\alpha$-stable motion with $\alpha \in (1,2]$.[3] The Lévy measure under the $\alpha$-stable motion is given by

$$\pi(dx) = c_{\pm}|x|^{-\alpha-1}dx. \tag{A7}$$

The process shows finite variation when $\alpha < 1$; but when $\alpha > 1$, the integral in (**??**) is no longer finite and the process is of infinite variation. Nevertheless, for the Lévy measure to be well-defined, the quadratic variation has to be finite:

$$\int_{R^0} (1 \wedge x^2)\pi(dx) < \infty, \tag{A8}$$

which requires that $\alpha \leq 2$.

The three jump processes considered here (MJ, VG, and LS) all have analytical characteristic exponents, which we tabulate in Table 1. We also include the characteristic exponent for the diffusion component for comparison. Given the Lévy measure $\pi$ for a particular jump process, we can derive the corresponding characteristic exponents using the Lévy-Khintchine formula (Bertoin (1996)),

$$\psi_j(u) \equiv -iub + \int_{R^0} \left(1 - e^{iux} + iux1_{|x|<1}\right) \pi(dx),$$

where $b$ denotes a drift adjustment term.

[3] See Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994).

In this subsection, we consider models where the asset return volatility is stochastic. More specifically, we assume that the asset return process follows a time-changed Lévy process and then apply a stochastic time change to the Brownian component of the asset return to generate stochastic (diffusive) volatility. Notice that the arrival rate of jumps remains constant. The stochastic time change we use is the Heston (1993) model.

We need to derive the Laplace transform of the stochastic time $T_t = \int_0^t v(s)ds$ under the measure $\mathbb{M}$. Thus, we rewrite the Laplace transform as

$$\mathcal{L}_T^{\mathbb{M}}(\psi) = E^{\mathbb{M}}\left[e^{-\psi^\top T_t}\right] = E^{\mathbb{M}}\left[e^{-\int_0^t \psi^\top v(s)ds}\right]. \tag{A9}$$

By Girsanov's Theorem, under measure $\mathbb{M}$, the diffusion function of $v(t)$ remains unchanged and the drift function is adjusted to

$$\mu^{\mathbb{M}} = \kappa(1 - v(t)) + iu\sigma\sigma_v\rho\, v(t) \tag{A10}$$

Substituting the Laplace transform in equation (**??**) into the generalized Fourier transforms in Table 2, we can derive analytical results for 3 jump-diffusion with stochastic volatility models.

## Appendix B. Default Probabilities in the CDG and LS Models

Default probabilities $Q(0,\cdot)$ can be calculated using an approach in the spirit of LS. Namely,

$$Q(0,U) = \sum_{i=1}^{n} q(t_i;t_0), \qquad t_i = iU/n,\ \ U \in (0,T], \tag{B11}$$

where for $i = 1, 2, \ldots, n$,

$$
\begin{aligned}
q(t_i;t_0) &= \frac{N(a(t_i;t_0)) - \sum_{j=1}^{i-1} q(t_{j-\frac{1}{2}};t_0)\, N(b(t_i;t_{j-\frac{1}{2}}))}{N(b(t_i;t_{i-\frac{1}{2}}))} && \text{(B12)}\\
a(t_i;t_0) &= -\frac{M(t_i, T|X_0, r_0)}{\sqrt{S(t_i|X_0, r_0)}} && \text{(B13)}\\
b(t_i;t_j) &= -\frac{M(t_i, T|X_{t_j})}{\sqrt{S(t_i|X_{t_j})}} && \text{(B14)}
\end{aligned}
$$

and where $X = V/V^*$, the sum on the RHS of (B12) is defined to be zero when $i = 1$, and

$$
\begin{aligned}
M(t, T|X_0, r_0) &\equiv E_0[\ln X_t]; && \text{(B15)}\\
S(t|X_0, r_0) &\equiv \mathrm{Var}_0[\ln X_t]; && \text{(B16)}\\
M(t, T|X_u) &= M(t, T|X_0, r_0) - M(u, T|X_0, r_0)\frac{\mathrm{Cov}_0[\ln X_t, \ln X_u]}{S(u|X_0, r_0)}, \quad u \in (t_0, t) && \text{(B17)}\\
S(t|X_u) &= S(t|X_0, r_0) - \frac{\mathrm{Cov}_0[\ln X_t, \ln X_u]^2}{S(u|X_0, r_0)}, \quad u \in (t_0, t) && \text{(B18)}
\end{aligned}
$$

Notice that we follow CDG to discretize at $t_{j-\frac{1}{2}}, j = 1, \ldots, i-1$, on the RHS of (B12). One can see that the implementation of this approach to computing $Q(0, \cdot)$ amounts to calculating the mean $M(t, T|X_0, r_0)$ and the covariance $\mathrm{Cov}_0[\ln X_t, \ln X_u], \forall u \leq t \leq T$.

It follows that

$$
\begin{aligned}
e^{\kappa_\ell t} E_0[\ln X_t] &= \ln X_0 + \left[(\pi^\nu + \bar{\nu}\kappa_\ell) + (1 + \kappa_\ell \phi)\frac{\bar{\alpha}}{\beta}\right] \frac{e^{\kappa_\ell t} - 1}{\kappa_\ell}\\
&\quad + (1 + \kappa_\ell \phi)\left(r_0 - \frac{\bar{\alpha}}{\beta}\right) \frac{e^{(\kappa_\ell - \beta)t} - 1}{\kappa_\ell - \beta} && \text{(B19)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Cov}_0[\ln X_t, \ln X_u] e^{\kappa_\ell (t+u)} = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B20)} \\
& \sigma_v^2 \, E_0 \left[ \int_0^t e^{\kappa_\ell v} \, dZ_v \int_0^u e^{\kappa_\ell v} \, dZ_v \right] & (\equiv I_1) \\
& + \sigma_v (1+\phi\kappa_\ell) \, E_0 \left[ \int_0^t e^{\kappa_\ell v} \, dZ_v \int_0^u e^{\kappa_\ell v} \, r_v dv \right] & (\equiv I_2) \\
& + \sigma_v (1+\phi\kappa_\ell) \, E_0 \left[ \int_0^u e^{\kappa_\ell v} \, dZ_v \int_0^t e^{\kappa_\ell v} \, r_v dv \right] & (\equiv I_3) \\
& + (1+\phi\kappa_\ell)^2 \, \mathrm{Cov}_0 \left[ \int_0^t e^{\kappa_\ell v} \, r_v dv, \int_0^u e^{\kappa_\ell v} \, r_v dv \right] & (\equiv I_4)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{\sigma_v^2}{2\kappa_\ell} \left( e^{2\kappa_\ell u} - 1 \right) \\
I_2 &= (1+\phi\kappa_\ell) \frac{\rho_{rV} \sigma_v \sigma_r}{\kappa_\ell + \beta} \left[ \frac{e^{2\kappa_\ell u} - 1}{2\kappa_\ell} - \frac{e^{(\kappa_\ell - \beta)u} - 1}{\kappa_\ell - \beta} \right] \\
I_3 &= (1+\phi\kappa_\ell) \frac{\rho_{rV} \sigma_v \sigma_r}{\kappa_\ell + \beta} \left[ \frac{1 - e^{(\kappa_\ell - \beta)t}}{\kappa_\ell - \beta} + \frac{e^{2\kappa_\ell u} - 1}{2\kappa_\ell} + e^{(\kappa_\ell + \beta)u} \, \frac{e^{(\kappa_\ell - \beta)t} - e^{(\kappa_\ell - \beta)u}}{\kappa_\ell - \beta} \right] \\
I_4 &= (1+\phi\kappa_\ell)^2 \, \frac{\sigma_r^2}{2\beta} \left[ - \frac{\left( e^{(\kappa_\ell - \beta)t} - 1 \right) \left( e^{(\kappa_\ell - \beta)u} - 1 \right)}{(\kappa_\ell - \beta)^2} + \left( e^{(\kappa_\ell + \beta)u} - 1 \right) \frac{e^{(\kappa_\ell - \beta)t} - e^{(\kappa_\ell - \beta)u}}{\kappa_\ell^2 - \beta^2} \right. \\
& \quad \left. - \frac{\beta}{\kappa_\ell^2 - \beta^2} \frac{e^{2\kappa_\ell u} - 1}{\kappa_\ell} + \frac{1}{\kappa_\ell^2 - \beta^2} \left( 1 - 2e^{(\kappa_\ell - \beta)u} + e^{2\kappa_\ell u} \right) \right]
\end{aligned}
$$

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Table 1: Characteristic Exponent of the Lévy Components in the Asset Return Process

| Component | $\psi_d(u)$ or $\psi_j(u)$ |
|---|---|
| Diffusion | $\frac{1}{2}\sigma^2\left[iu-(iu)^2\right)$ |
| Poisson Jump (MJ) | $\lambda\left[iu\left(e^{\alpha+\frac{1}{2}\sigma_j^2}-1\right)-\left(e^{iu\alpha-\frac{1}{2}u^2\sigma_j^2}-1\right)\right]$ |
| Variance Gamma (VG) | $\lambda\left[-iu\ln\left(1-\alpha-\frac{1}{2}\sigma_j^2\right)+\ln\left(1-iu\alpha+\frac{1}{2}\sigma_j^2u^2\right)\right]$ |
| Log Stable (LS) | $\lambda\left(iu-(iu)^{\alpha}\right)$ |

Table 2: Generalized Fourier Transforms of Log Asset Returns

$x_t$ denotes the time changed component and $y_t$ denotes the unchanged component in the log return $s_t = \ln(V_t/V_0)$. $J_t$ denotes a compensated pure jump martingale component, and $\xi$ its concavity adjustment.

| Model | $x_t$ | $y_t$ | $\phi_s(u)$ |
|---|---|---|---|
| SV1 | $\sigma W_t-\frac{1}{2}\sigma^2 t$ | $J_t-\xi t$ | $e^{iu(r-q)t-t\psi_j}\mathcal{L}_T^{\mathbb{M}}(\psi_d)$ |